

Da Prato and Grisvard's approach to define sums of sectorial operators

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June 2020

Abstract

Motivated by applications to linear Cauchy problems Da Prato and Grisvard studied sums of two unbounded commuting sectorial operators on a Banach space, and the related problems of wellposedness of Cauchy problems and the so-called maximal regularity. We review some ideas from a paper of Da Prato and Grisvard from 1975 [1].

1 The sum of unbounded sectorial operators

Let A, B be two closed unbounded operators with domains D_A, D_B on a complex Banach space X . We define the sum:

$$D_L = D_A \cap D_B; \quad Lx := Ax + Bx \quad \forall x \in D_L \quad (1)$$

Assume that the resolvents commute in the following sense:

$$[(A - \lambda)^{-1}, (B - \mu)^{-1}] = 0 \quad \forall \lambda \in \rho(A), \mu \in \rho(B) \quad (2)$$

where $\rho(A), \rho(B)$ denote the resolvents of A and B .

Definition 1.1. Let P be a linear operator on X with domain D_P and let $\phi \in [0, \pi)$. We say that P is *sectorial* of angle ϕ if the following properties hold:

$$(1) \quad \rho(P) \supset \Sigma_P := \{z \in \mathbb{C} : -\pi + \phi < \arg(z) < \pi - \phi\}$$

(2) There exists a map $C : (-\pi + \phi, \pi - \phi) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\|(P - z)^{-1}\| \leq \frac{C_P(\theta)}{|z|} \quad \forall z \in \Sigma_P, \theta = \arg z$$

Definition 1.2. Assume that A and B are sectorial of angle θ_A respectively θ_B . Then we call the problem $Ax + Bx = f$ for some given $f \in X$ parabolic if

$$\theta_A + \theta_B < \pi.$$

Definition 1.3 (Candidate for the inverse of L).

$$S_\lambda := \frac{1}{2\pi i} \int_\gamma (A - z - \lambda)^{-1} (B + z)^{-1} dz \quad \forall \lambda > 0 \quad (3)$$

Lemma 1.1. There exists $N > 0$ such that $\|S_\lambda\| \leq \frac{N}{\lambda}$ for all $\lambda > 0$.

Proof. See [1] pp. 322. □

Lemma 1.2. (1) $S_\lambda(Lx - \lambda x) = x \quad \forall x \in D_L$

(2) If $x \in D_A + D_B$, then $S_\lambda x \in D_L$ and $(L - \lambda)S_\lambda x = x$.

Proof. See [1] pp. 323. □

Theorem 1.1. Let A, B be two closed operators on a Banach space X with commuting resolvents (cf. (2)) such that $D_A + D_B$ is dense in X . Let L be defined as in (1) and assume that $A + B$ is parabolic. Then the closure \bar{L} of L satisfies $\rho_{\bar{L}} \supset (0, +\infty)$ and

$$(\bar{L} - \lambda I)^{-1} = S_\lambda \quad \forall \lambda > 0,$$

where S_λ is defined as in (1.3). I.e. \bar{L} is sectorial.

Proof. See [1] pp. 324. □

2 Maximal regularity problem

Definition 2.1. Let A be closed sectorial operator on Banach space X . For $\theta \in \langle 0, 1 \rangle$ and $p \in [1, +\infty]$ let $D_A(\theta; p)$ denote the set of all $x \in X$ such that $\|t^\theta A(A - t)^{-1}x\| \in L_*^p$. (L_*^p denotes the set of Borel functions defined on $\langle 0, +\infty \rangle$ that are p -integrable with respect to the measure $\frac{dt}{t}$ in case $p < +\infty$ or the set of Borel essentially bounded functions on $\langle 0, +\infty \rangle$ in case $p = +\infty$)

Two facts about interpolation spaces:

- $D_A(\theta'; q) \subseteq D_A(\theta; p)$ if $\theta' > \theta$ or ($\theta' = \theta$ and $q \leq p$)
- $D_A \subseteq D_A(\theta; p)$

Proposition 2.1. S_λ is continuous linear operator from X to $D_A(1; +\infty) \cap D_B(1; +\infty)$.

Proof. Detailed proof can be found in [1], pp 325 – 327. □

Proposition 2.2. Assume that $y \in D_A(\theta; p) + D_B(\theta; p)$ for some $\theta \in \langle 0, 1 \rangle$ and $p \in [1, +\infty]$.

With S_λ defined as above it is true that:

$$x = S_\lambda y \in D_L \quad \text{and} \quad (L - \lambda)x = y.$$

Proof. Detailed proof can be found in [1], pp 327 – 328. □

Theorem 2.1. (Young's convolution inequality) Let G be abelian locally compact topological group endowed with Haar measure μ . Then for all $(p, q, r) \in [1, +\infty]^3$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

and any $(f, g) \in L^p(G, \mu) \times L^q(G, \mu)$, we have $f * g \in L^r(G, \mu)$ and $\|f * g\|_{L^r(G, \mu)} \leq \|f\|_{L^p(G, \mu)} \|g\|_{L^q(G, \mu)}$.

Theorem 2.2. (Da Prato, Grisvard) Let X be a Banach space. Let A and B be two operators on X which are closed and sectorial of angle θ_A and θ_B , respectively, that satisfy:

1. $\theta_A + \theta_B < \frac{\pi}{2}$
2. $[(A - \lambda I)^{-1}; (B - \mu I)^{-1}] = 0, \forall \lambda \in \rho_A, \forall \mu \in \rho_B$

Then, for all $y \in D_B(\theta; p)$ [respectively $D_A(\theta; p)$] for some $\theta \in \langle 0, 1 \rangle$ and $p \in [1, +\infty]$, the solution

$$x = S_\lambda y \text{ of } \begin{cases} x \in D_L \\ Lx - \lambda x = y \end{cases} \text{ satisfies that } Ax, Bx \in D_B(\theta; p) \text{ [respectively } D_A(\theta; p)]$$

Proof. Detailed proof can be found in [1], pp 328 – 329. □

3 Applications to evolution equations

$$\begin{cases} -u'(t) + \Lambda u(t) - \lambda u(t) = f(t), & 0 < t < T \text{ and } \lambda > 0 \\ u(0) = 0 \end{cases} \quad (4)$$

When the sectorial Λ is a closed operator in Banach space E .

$f \in L^p([0, T]; E)$ or $f \in C([0, T]; E)$

Notations:

- $W_0^{1,p}([0, T]; E)$ is the E -valued Sobolev space of order 1 in $[0, T]$ with initial condition $u(0) = 0$.
- $W^{\theta,p}([0, T]; E)$ is the Sobolev space of order (θ) , that is the set of $u \in L^p(0, T; E)$ such that:

$$\int_0^T \int_0^T \|u(t) - u(s)\|_E^p \frac{dt ds}{|t-s|^{1+\theta p}} < +\infty \quad \text{when } \theta < \frac{1}{p}$$

- $W_0^{\theta,p}([0, T]; E)$ is the subspace of $W^{\theta,p}([0, T]; E)$ such that $u(0) = 0$ when $\theta > 1/p$.
- $W_{0,0}^{\theta,p}([0, T]; E)$ is the subspace of $W^{\theta,p}([0, T]; E)$ such that

$$\int_0^T \|u(t)\|_E^p \frac{dt}{t} < +\infty$$

when $\theta = 1/p$.

- $C_0^\theta([0, T]; E)$ is the space of Hölderfunctions of order θ in $[0, T]$ with values in E such that $u(0) = 0$.

$$\begin{cases} Au + Bu - \lambda u = f, \\ u(0) = 0 \end{cases}$$

$$\begin{cases} D_A = \{u \in X; u' \in X, u(0) = 0\} \\ Au = -u' \end{cases}$$

$$\theta_A = \frac{\pi}{2}.$$

- If $X = L^p([0, T], E)$ then, $D_A = W_0^{1,p}([0, T], E)$.
- if $X = C([0, T]; E)$ then $D_A = C_0^1([0, T]; E)$
- When $X = C_0([0, T]; E)$, then D_A is the subspace of $C^1([0, T]; E)$ containing functions which further verify $u'(0) = 0$.
- When $X = L^p([0, T]; E)$ we obtain $D_A(\theta; p) =$

- $W^{\theta,p}([0, T]; E)$, when $\theta < \frac{1}{p}$
- $W_0^{\theta,p}([0, T]; E)$ when $\theta = \frac{1}{p}$
- $W_{0,0}^{\theta,p}([0, T]; E)$ when $\theta > \frac{1}{p}$.

- Finally, when $X = C([0, T]; E)$ or $C_0([0, T]; E)$ we have

$$D_A(\theta; +\infty) = C_0^\theta([0, T]; E).$$

Λ be the closed operator in E , having D_Λ for a domain and ρ_Λ as a resolvent.

We have $(Bu)(t) = (\Lambda u)(t)$, we assume,

$$\begin{aligned} D_B &= L^p([0, T]; D_\Lambda) && \text{when } X = L^p([0, T]; E) \\ D_B &= C([0, T]; D_\Lambda) && \text{when } X = C([0, T]; E) \\ D_B &= C_0([0, T]; D_\Lambda) && \text{when } X = C_0([0, T]; E) \\ D_B(\theta, p) &= L^p([0, T]; D_\Lambda(\theta, p)) && \text{when } X = L^p([0, T]; E) \\ D_B(\theta, \infty) &= C([0, T], D_\Lambda(\theta, \infty)) && \text{when } X = C([0, T], E) \end{aligned}$$

We consider L defined by $Lu = Au + Bu$ for $u \in D_L = D_A \cap D_B$.

The equation (4) is equivalent to the equation :

$$\begin{cases} Lu - \lambda u = f, \\ u(0) = 0. \end{cases} \quad (5)$$

- It is said that u is a strong solution of (4), meaning that $u \in D_B \cap D_A$ so that the functions $t \rightarrow u(t), u'(t), \Lambda u(t)$ are in X

Theorem 3.1. *We assume that the Λ verifies $H(\theta_\Lambda)$ with $\theta_\Lambda < \frac{\pi}{2}$.*

- (i) *If $f \in W^{\theta,p}([0, T]; E)$ $0 < \theta < \frac{1}{p}$ then the problem (4) has a unique strong solution u such that $u', \Lambda u \in W^{\theta,p}([0, T]; E)$*
- (i') *If $f \in L^p([0, T], D_\Lambda(\theta, p))$, $0 < \theta < 1$ then the problem (4) has a unique strong solution u such that $u', \Lambda u \in L^p([0, T], D_\Lambda(\theta, p))$*
- (ii) *If $f \in C_0^\theta([0, T]; E)$, $0 < \theta < 1$ then the problem (4) has a unique strong solution u such that $u', \Lambda u \in C_0^\theta([0, T]; E)$,*
- (ii') *If $f \in C([0, T]; D_\Lambda(\theta, \infty))$, $0 < \theta < 1$ then the problem (4) has a unique strong solution u such that $u', \Lambda u \in C([0, T]; D_\Lambda(\theta, \infty))$*

Proof. $\theta_B = \theta_\Lambda < \frac{\pi}{2}$ and $\theta_A = \frac{\pi}{2}$.

□

References

- [1] Giuseppe Da Prato; Pierre Grisvard. "Sommes d'opérateurs linéaires et équations différentielles opérationnelles". In: *Journal de mathématiques pures et appliquée* (1975).