A Hilbert Space Approach to Fractional Differential Equations

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- $\alpha > 0$ (power for the fractional derivative of $\partial_{t,\nu}$),
- $\nu \in \mathbb{R}$,
- $L^2_\nu := L^2_\nu(\mathbb{R}; H)$,
- $H^1_\nu := H^1_\nu(\mathbb{R}; H)$.

Definition

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1. L^2_ν is a Hilbert space.
2. $e^{-\nu m} : L^2_\nu \to L^2, u \mapsto e^{-\nu m}u$ is unitary.

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Define

1. the time derivative

$$\begin{aligned} \partial_{t,\nu} : H^1_\nu \subseteq L^2_\nu &\longrightarrow L^2_\nu \\ u &\longmapsto u', \end{aligned}$$

where $H^1_\nu := \{u \in L^2_\nu; u' \in L^2_\nu\}$,

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2. the Fourier-Laplace transform

$$\begin{array}{lccccccc} \mathcal{L}_\nu : & L^2_\nu & \longrightarrow & L^2 & \longrightarrow & L^2 & & \\ & u & \longmapsto & e^{-\nu m} u & \longmapsto & \mathcal{F} e^{-\nu m} u & =: & \mathcal{L}_\nu u \end{array}$$

   (unitary because composition of unitary maps).

Recall: $\mathcal{L}_\nu = \mathcal{F} e^{-\nu m}$ and $\mathcal{L}_\nu^* = e^{\nu m} \mathcal{F}^*$

Theorem

One has

$$\partial_{t,\nu} = \mathcal{L}_\nu^* (im + \nu) \mathcal{L}_\nu .$$

Proof. We compute

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$$\begin{aligned} H^1_\nu = \operatorname{dom}(\partial_{t,\nu}) &= \operatorname{dom}(\mathcal{L}^*_\nu (im+\nu)\mathcal{L}_\nu) \\ &= \{u \in L^2_\nu : (im+\nu)\mathcal{L}_\nu u \in L^2\}. \end{aligned}$$

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2. $H^1_\nu \subseteq L^2_\nu$ with embedding constant less or equal to $|\nu|^{-1}$.

Indeed,

$$\begin{aligned}\|u\|_{L^2_\nu} &= \|\mathcal{L}_\nu u\|_{L^2} = \|(i\xi+\nu)^{-1}(i\xi+\nu)\mathcal{L}_\nu u\|_{L^2} \\ &\leq \sup_{\xi\in\mathbb{R}} |(i\xi+\nu)^{-1}| \|u\|_{H^1_\nu} \leq |\nu|^{-1}\|u\|_{H^1_\nu}.\end{aligned}$$

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2. $H_\nu^1 \subseteq L_\nu^2$ with embedding constant less or equal to $|\nu|^{-1}$.
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2. $H_\nu^1 \subseteq L_\nu^2$ with embedding constant less or equal to $|\nu|^{-1}$.
3. $\partial_{t,\nu}$ unitarily equivalent to multiplication operator with spectrum $i\mathbb{R}+\nu$.
   $\Rightarrow$ Can define functional calculus of $\partial_{t,\nu}$ via similarity.

Definition

Let $F : \{it + \nu; t \in \mathbb{R}\} \to \mathbb{C}$ measurable.

1. $F(im + \nu)$ is maximal restriction of (measurable) multiplication operator $u \mapsto (\mathbb{R} \ni \xi \mapsto F(i\xi + \nu)u(\xi))$ to L^2.

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Remark: $F(\partial_{t,\nu})$ bounded $\Leftrightarrow$ $F \circ (i\xi + \nu)$ bounded on $\mathbb{R} \Leftrightarrow F$ bounded (on vertical line).

Definition

The fractional derivative of order α is given by

$$\partial_{t,\nu}^{\alpha} := F(\partial_{t,\nu}) = \mathcal{L}_{\nu}^{*}(im+\nu)^{\alpha}\mathcal{L}_{\nu},$$

where $F(z) = z^{\alpha}$.

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Remarks:

1. $\partial_{t,\nu}^{-\alpha}$ is bounded $\Leftrightarrow$ $\nu \neq 0$.
2. $\partial_{t,\nu}^{-\alpha} = (\partial_{t,\nu}^{\alpha})^{-1}$.

Lemma

For all $\nu > 0$ and $\xi \in \mathbb{R}$:

$$\sqrt{2\pi}\mathcal{L}_\nu \left(t \mapsto \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \mathbb{1}_{(0,\infty)}(t) \right)(\xi) = (i\xi + \nu)^{-\alpha}.$$

Proof.

- Put:

$$h(\xi) := \int_0^{\infty} e^{-(i\xi+\nu)s} s^{\alpha-1} ds.$$

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Theorem

For all $\nu > 0$, $u \in L^2_\nu$:

$$\partial_{t,\nu}^{-\alpha} u(t) = \int_{-\infty}^{t} \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} u(s) ds.$$

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Remark: $\partial_{t,\nu}^{-\alpha}$ is consistent in ν.

Proof:

- Set $v := \left(\mathbb{R} \ni t \mapsto \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \mathbb{1}_{(0,\infty)}(t)\right) \in L^1_\nu(\mathbb{R})$.

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- Let $u \in L^2_\nu$.

$$e^{-\nu m}(v * u) = (e^{-\nu m} v) * (e^{-\nu m} u).$$

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$$\partial_{t,\nu}^{-\alpha} u = \mathcal{L}_\nu^* \left(im + \nu\right)^{-\alpha} \mathcal{L}_\nu u = \mathcal{L}_\nu^* \left(\sqrt{2\pi}\mathcal{L}_\nu v \cdot \mathcal{L}_\nu u\right)$$

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- Convolution property: $\sqrt{2\pi}\mathcal{L}_\nu v \cdot \mathcal{L}_\nu u = \mathcal{L}_\nu (v * u)$. Then with previous lemma:

$$\begin{aligned} \partial_{t,\nu}^{-\alpha} u = \mathcal{L}_\nu^* (im + \nu)^{-\alpha} \mathcal{L}_\nu u &= \mathcal{L}_\nu^* \left(\sqrt{2\pi}\mathcal{L}_\nu v \cdot \mathcal{L}_\nu u\right) \\ = \mathcal{L}_\nu^* \mathcal{L}_\nu (v * u) &= \int_{-\infty}^{(\cdot)} \frac{1}{\Gamma(\alpha)} \Big((\cdot) - s\Big)^{\alpha-1} u(s) ds. \end{aligned}$$

Definition

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$$\sup_{\lambda > 0} \left\{ \|\lambda(\lambda I + A)^{-1}\| \right\} < +\infty.$$

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Lemma

Let $A : \mathrm{dom}(A) \subseteq H \to H$ *be a normal operator in H. If* $\sigma(A) \subseteq \{z \in \mathbb{C} \mid |\arg(z)| \leq \omega < \pi\}$, *then A is sectorial.*

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Remark:

1. $\partial_{t,\nu}$ is sectorial and invertible (in particular injective) for $\nu > 0$.

For an injective sectorial operator A one can define negative fractional powers α by the formula

$$A^{-\alpha} := \frac{\sin(\pi\alpha)}{\pi} \int\limits_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1} d\lambda, \quad \alpha > 0.$$

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Remarks:

1. The operators $(A^{-\alpha})_{\alpha>0}$ satisfy:

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$$A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)} \qquad \alpha, \beta > 0,$$

2. $A^{-1} \in \mathcal{L}(H)$ iff $A^{-\alpha} \in \mathcal{L}(H)$ for some/all $\alpha > 0$.

Theorem

The relationship between the fractional integral and negative fractional power of the sectorial operator $\partial_{t,\nu}$ is as follows:

$$\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{(\cdot)} (\cdot - s)^{\alpha-1} u(s)\, ds = \frac{\sin(\pi\alpha)}{\pi} \int_{0}^{\infty} \lambda^{-\alpha}(\lambda I + \partial_{t,\nu})^{-1} u \, d\lambda.$$

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Need the following auxiliary result for the proof.

Lemma

$$\Big((\lambda I + \partial_{t,\nu})^{-1} u \Big)(\cdot) = \int_{-\infty}^{(\cdot)} e^{-\lambda((\cdot)-s)} u(s)\, ds, \quad \lambda > 0.$$

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Proof of Theorem:

$$\begin{aligned}
&\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \lambda^{-\alpha}\big((\lambda I+\partial_{t,\nu})^{-1}u\big)(t)\,d\lambda \\
&=\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \lambda^{-\alpha}\int_{-\infty}^{t} e^{-\lambda(t-s)}u(s)\,ds d\lambda \\
&=\frac{\sin(\pi\alpha)}{\pi}\int_{-\infty}^{t}\left\{\int_0^\infty \lambda^{-\alpha}e^{-\lambda(t-s)}d\lambda\right\}u(s)\,ds \\
&=(\star)
\end{aligned}$$

Substitute $\lambda(t-s)=\tau$, then

$$\int_0^\infty \lambda^{-\alpha} e^{-\lambda(t-s)} d\lambda = \frac{1}{(t-s)^{1-\alpha}} \int_0^\infty \tau^{-\alpha} e^{-\tau} d\tau$$

Substitute $\lambda(t-s)=\tau$, then

$$\int_0^\infty \lambda^{-\alpha}e^{-\lambda(t-s)}d\lambda = \frac{1}{(t-s)^{1-\alpha}}\int_0^\infty \tau^{-\alpha}e^{-\tau}d\tau = \frac{\Gamma(1-\alpha)}{(t-s)^{1-\alpha}}.$$

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Using this in $(\star)$, we get

$$(\star) = \frac{\sin(\pi\alpha)}{\pi} \int_{-\infty}^{t} \frac{\Gamma(1-\alpha)}{(t-s)^{1-\alpha}} u(s)\, ds$$

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$$A^\alpha x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I + A)^{-1} Ax \, d\lambda,$$
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   where $0 < \alpha < 1$.

For $u \in H^1_\nu$ and $0 < \alpha < 1$ the fractional derivative is given by
$$\partial^\alpha_{t,\nu} u(t) = \partial^{-(1-\alpha)}_{t,\nu} \partial_{t,\nu} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-s)^{-\alpha} u'(s) \, ds.$$

Consider formally the following Cauchy problem of Caputo type for $\alpha \in (0,1)$

$$\begin{aligned} \partial_{t,\nu}^{\alpha} u(t) &= f\big(t, u(t)\big), \quad t > 0, \\ u(0) &= u_0, \end{aligned}$$

with $u_0 \in \mathbb{R}^n$, $f : (0,\infty) \times \mathbb{C}^n \to \mathbb{C}^n$ continuous and satisfying

$$\exists \ell > 0 \; \forall t > 0 \; \forall u, v \in \mathbb{C}^n : \quad |f(t,u) - f(t,v)| \leq \ell |u - v|$$

and moreover

$$\exists \nu_0 > 0 \; \forall \nu \geq \nu_0 : \quad \big(t \mapsto f(t,0)\big) \in L^2_{\nu_0}(\mathbb{R}_+; \mathbb{C}^n) \subseteq L^2_{\nu}(\mathbb{R}_+; \mathbb{C}^n).$$

The initial value problem is equivalent to

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$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds, \quad t > 0.$$

Define $\tilde{f} : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ by

$$\tilde{f}(t,x) := f(t,x)\mathbb{1}_{(0,\infty)}(t) = \begin{cases} f(t,x), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Then the integral equation extends as follows for $t \in \mathbb{R}$:

$$u(t) = u_0 \mathbb{1}_{(0,\infty)}(t) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} \tilde{f}(s, u(s))\, ds.$$

Theorem

The following is a well-defined uniformly Lipschitz continuous mapping for all $\nu \geq \nu_0$:

$$F : L^2_\nu(\mathbb{R}) \ni u \mapsto F(u) := (t \mapsto \tilde{f}(t, u(t))) \in L^2_\nu(\mathbb{R}).$$

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- $F(u) = \big(F(u) - F(0)\big) + F(0) \in L^2_\nu$.

Theorem

Let $\nu > \max\{0, \nu_0\}$. Assume that $u \in L^2_\nu$. Then the following statements are equivalent:

$$(i) \quad u = u_0 \mathbb{1}_{(0,\infty)} + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{(\cdot)} \Big((\cdot) - s\Big)^{\alpha-1} F(u)\, ds,$$

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$$(iii) \quad \partial_{t,\nu}^{\alpha}(u - u_0 \mathbb{1}_{(0,\infty)}) = F(u).$$

Problem. We want to solve the differential equation

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2. *The Titanium operator*

   $$T_\nu := \partial^{-\alpha}_{t,\nu} F : L^2_\nu \to L^2_\nu$$

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Idea: Replace ν by μ and let then $\mu \to \infty$.

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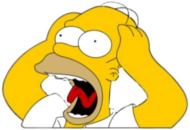
But... How to get rid of this factor?!



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Let $\nu_0 > 0$ *and* $f : (0, \infty) \times \mathbb{C}^n \to \mathbb{C}^n$ *be continuous.*

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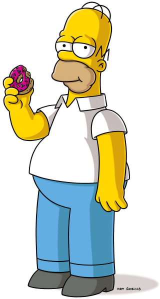

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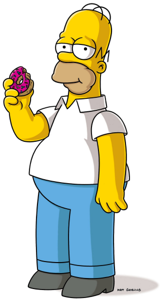

A Hilbert space approach to fractional differential equations

Kai Diethelm, Konrad Kitzing, Rainer Picard,
Stefan Siegmund, Sascha Trostorff, Marcus Waurick

https://arxiv.org/abs/1909.07662