Solutions to the Exercises of Lecture 14, ISEM 2019-2020 Team Hamburg

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Exercise 14.1

Under the assumptions of Theorem 14.1.1, show that

$$\left\| \left((\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A})^{-1} - (\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A})^{-1} \right) \partial_{t,\nu}^{-1} \right\|_{L(L_{2,\nu}(\mathbb{R},H))} \to 0.$$

Proof. First, observe that

$$\left((\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A})^{-1} - (\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A})^{-1} \right) \partial_{t,\nu}^{-1} \overset{n\to\infty}{\to} 0 \text{ in } L(L_{2,\nu}(\mathbb{R},H))$$

is equivalent to

$$\left((\overline{(i\cdot m+\nu) M_n(i\cdot m+\nu) + A})^{-1} - (\overline{(i\cdot m+\nu) M(i\cdot m+\nu) + A})^{-1} \right) (i\cdot m+\nu)^{-1} \overset{n\to\infty}{\to} 0 \text{ in } L(L_2(\mathbb{R},H)).$$

By Theorem 13.1.1, we have that

$$\forall z \in \mathbb{C}_{\mathrm{Re}>\mu} : (\overline{z M_n(z) + A})^{-1} - (\overline{z M(z) + A})^{-1} \overset{n\to\infty}{\to} 0 \text{ in } L(H)$$

where we assumed bounded pointwise convergence of M_n to M on $\mathbb{C}_{\mathrm{Re}>\mu}$ for some $\mu < \nu$. Now, consider $f \in L_2(\mathbb{R},H)$. Then, we have that

$$\begin{aligned} &\Big\| \underbrace{\left((\overline{(i\cdot m+\nu) M_n(i\cdot m+\nu) + A})^{-1} - (\overline{(i\cdot m+\nu) M(i\cdot m+\nu) + A})^{-1} \right)}_{=:N_n} (i\cdot m+\nu)^{-1} f \Big\|^2_{L_2(\mathbb{R},H)} \\ &= \int_{-\infty}^{+\infty} \left\| \frac{N_n(it+\nu) f(t)}{it+\nu} \right\|_H^2 \mathrm{d}t \le \sup_{t\in\mathbb{R}} \left\| \frac{N_n(it+\nu)}{it+\nu} \right\|^2_{L(H)} \cdot \|f\|^2_{L_2(\mathbb{R},H)}. \end{aligned}$$

So, it remains to show

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left\| \frac{N_n(it+\nu)}{it+\nu} \right\|^2_{L(H)} = 0.$$

Due to $\sup_{t\in\mathbb{R}} \|N_n(it+\nu)\| \le \frac{2}{c}$, it suffices to establish $\lim_{n\to\infty} \sup_{t\in[-R,R]} \|N_n(it+\nu)\|^2_{L(H)} = 0$ for all $R > 0$. This, however, follows from Theorem 13.1.1. More precisely, since

$$(f_n)_n \text{ in } \mathcal{H}^\infty(\mathbb{C}_{\mathrm{Re}>\mu}), \quad \lim_{n\to\infty} f_n(z) = f(z) \ \forall z \in \mathbb{C}_{\mathrm{Re}>\mu}, \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n\| < \infty,$$

we have that $f_n \to f$ uniformly on any compact set $K \subset \mathbb{C}_{\mathrm{Re}>\mu}$. In particular, this also covers the case $K := \nu + i[-R,R]$. Thus, $\lim_{n\to\infty} \sup_{t\in[-R,R]} \|N_n(it+\nu)\|^2_{L(H)} = 0$. With that, the proof is finished. □

Exercise 14.2

Let $\Omega \subset \mathbb{R}^d$ be open and $\varepsilon > 0$. We define the set $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$.
a) Let $\{\Phi_k\}_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ be a δ-sequence and $u \in H^1(\Omega)$. We identify each function on Ω by its extension to $\mathbb{R}^d$ by 0. Prove that for $k \in \mathbb{N}$ large enough $\Phi_k * u \in H(\Omega_\varepsilon)$ with

$$\mathrm{grad}(\Phi_k * u) = \Phi_k * \mathrm{grad}(u) \text{ on } \Omega_\varepsilon.$$

Proof. Let $K \in \mathbb{N}$ be such that $\mathrm{dist}(0, \partial[-\frac{1}{K}, +\frac{1}{K}]^d) < \varepsilon$ and $\mathbb{N} \ni k > K$. Then $\Phi_k * u \in H^1(\Omega_\varepsilon)$, because we even have $\Phi_k * u \in C^\infty(\Omega_\varepsilon)$ and the function and all its derivatives are in $L^2(\mathbb{R}^d)$. Furthermore, let $x \in \Omega_\varepsilon$ and $i \in \{1, ..., d\}$ be arbitrary. We calculate

$$\begin{aligned} (\Phi_k * \partial_i u)(x) &= \int_{\mathbb{R}^d} \partial_i u(y) \Phi_k(x-y) \mathrm{d}y = \int_\Omega \partial_i u(y) \Phi_k(x-y) \mathrm{d}y \\ &= \int_\Omega u(y) \partial_i \Phi_k(x-y) \mathrm{d}y = (u * \partial_i \Phi_k)(x) = \partial_i (u * \Phi_k)(x). \end{aligned}$$

The second equality follows from the fact that $\partial_i u(\cdot) = 0$ on $\mathbb{R}^d \setminus \Omega$ (by definition) and the third from $y \mapsto \Phi_k(x-y) \in C_c^\infty(\Omega)$ (using the choice of k and that $x \in \Omega_\varepsilon$) combined with the weak differentiability of u. The claim follows, since i was arbitrary. □

b) Let $\Omega \subset \mathbb{R}^d$ be open and connected and $u \in H^1(\Omega)$. If $\operatorname{grad}(u) = 0$, then u is constant.

Proof. Let $\varepsilon > 0$ be arbitrary. Then, for k large enough, it follows from a) that

$$\operatorname{grad}(\Phi_k * u) = \Phi_k * \operatorname{grad}(u) = 0 \text{ on } \Omega_\varepsilon.$$

Since $\Phi_k * u$ is smooth we obtain that $\Phi_k * u$ is locally constant on Ω_ε. Furthermore, $\Phi_k * u$ converges to u in the L^2-norm, which means that u is locally constant on Ω_ε. By letting ε to 0, we infer that u is locally constant on Ω. But Ω is connected and, therefore, it follows that u is constant. □

Exercise 14.3

Let X be a topological space and $(x_n)_n$ a sequence in X. Assume that there exists $x \in X$ such that each subsequence of $(x_n)_n$ has a subsequence converging to x. Then $x_n \to x$ as $n \to \infty$.

Proof. A sequence $(x_n)_n$ in a topological space X converges to a point $x \in X$ if and only if for every open set O in X that contains x one can find $N \in \mathbb{N}$ such that for all $n \geq N$ $x_n \in O$ holds. If we assume that the statement is false, then it implies that there is an open set U in X that contains x such that for all $N \in \mathbb{N}$ there is $n \geq N$ with $x_n \notin U$. Hence, by this one can construct a subsequence of $(x_n)_n$, again denoted by $(x_n)_n$, for which we have that no x_n is in U. Thus, it cannot have a subsequence convergent to x and, therefore, the statement is proven. □

Exercise 14.4

Let H_0, H_1 be Hilbert spaces and $C : \operatorname{dom}(C) \subseteq H_0 \to H_1$ be a closed linear operator such that $\operatorname{dom}(C) \hookrightarrow H_0$ compactly. Let $P_{\ker(C)^\perp} : H_0 \to H_1$ denote the orthogonal projection onto the closed subspace $\ker(C)^\perp$. Prove that there exists $c > 0$ such that

$$\forall u \in \operatorname{dom}(C) : \left\| P_{\ker(C)^\perp} u \right\|_{H_0} \leqslant c \left\| Cu \right\|_{H_1}.$$

Apply this result to prove Proposition 14.3.6.

Proof. We will argue by contradiction. Assume the statement to be wrong. Then, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\operatorname{dom}(C)$, w.l.o.g. we may even assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is in $\ker(C)^\perp$, such that $\| u_n \|_C = 1$ and

$$\| u_n \|_{H_0} > n \| Cu_n \|_{H_1}. \tag{1}$$

By the assumption on the compact embedding, this sequence contains a subsequence convergent in H_0 towards some element $u \in H_0$. In order to simplify notation let us assume that the original sequence $(u_n)_{n\in\mathbb{N}}$ is already convergent in H_0. This means that the left-hand side of (1) is bounded. From Inequality (1), we conclude that $Cu_n \to 0$ in H_1. Since C is a closed operator we get $u \in \operatorname{dom}(C)$ and $Cu = 0$, i.e., $u \in \ker(C)$. Because we assumed the sequence $(u_n)_{n\in\mathbb{N}}$ to be in $\ker(C)^\perp$, it follows that $u = 0$. Putting all together, we have $u_n \to 0$ in H_0 and $Cu_n \to 0$ in H_1, i.e., $u_n \to 0$ in $\operatorname{dom}(C)$ which finally contradicts $\| u_n \|_C = 1$.
As for the application to Proposition 14.3.6, choose $C := \operatorname{grad}_\sharp$ and $H_0 := L_2(Y)$. By Theorem 14.3.5, the embedding $\operatorname{dom}(C) \hookrightarrow H_0$ is compact and C is closed by definition. So the theorem from above is applicable. Let $u \in \ker(\operatorname{grad}_\sharp)$. Then $u \in \ker(\operatorname{grad})$ since by Remark 14.3.1 it holds that $\operatorname{grad}_\sharp \subseteq \operatorname{grad}$. By Exercise 14.2 b), there is $c \in \mathbb{C}$ with $u = c\mathbb{1}_Y$ or, equivalently, $\ker(\operatorname{grad}_\sharp) = \operatorname{lin}\{\mathbb{1}_Y\}$. Proposition 14.3.6 follows now directly from an application of the abstract result.

□

Exercise 14.5

Let H_0, H_1 be Hilbert spaces. Let $C : \operatorname{dom}(C) \subseteq H_0 \to H_1$ be closed and densely defined. Assume that $\operatorname{dom}(C) \cap \ker(C)^\perp \hookrightarrow H_0$ compactly. Then also $\operatorname{dom}(C^*) \cap \ker(C^*) \hookrightarrow H_1$ compactly holds.

Proof. Consider a bounded sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n \in \operatorname{dom}(C^*) \cap \ker(C^*)^\perp$ for all $n \in \mathbb{N}$. By definition of the graph norm, this implies the boundedness of the sequence $(C^* x_n)_{n\in\mathbb{N}}$ in H_0. Let $\iota : \operatorname{dom}(C) \cap \ker(C)^\perp \hookrightarrow H_0$ be the embedding. Then, by Schauder's theorem, it follows that the dual operator

$$\iota^* : H_0 \to \operatorname{dom}(C) \cap \ker(C)^\perp,$$

is also compact. Here, $\mathrm{dom}(C) \cap \ker(C)^\perp$ is a Hilbert space with the graph scalar product. Thus, there exists a convergent subsequence $(C^* x_{n_k})_{k\in\mathbb{N}}$ in $\mathrm{dom}(C) \cap \ker(C)^\perp$. By applying Exercise 14.4 to the map $C|_{\mathrm{dom}(C)\cap\ker(C)^\perp} : \mathrm{dom}(C) \cap \ker(C)^\perp \subseteq H_0 \to H_1$, there exists a constant $K > 0$ such that

$$\| u \|_{H_0} \leq K \left\| C|_{\mathrm{dom}(C)\cap\ker(C)^\perp} u \right\|_{H_1}$$

holds for all $u \in \mathrm{dom}(C) \cap \ker(C)^\perp$ since the assumptions of Exercise 14.4 are valid here. Therefore, we can define a new norm on $\mathrm{dom}(C) \cap \ker(C)^\perp$

$$\| u \|_* := \| Cu \|_{H_1}$$

for all $u \in \mathrm{dom}(C) \cap \ker(C)^\perp$ which is equivalent to the graph norm. This norm is indeed equivalent to the graph norm since

$$\| Cu \|_{H_1} \leq \sqrt{\| u \|_{H_0}^2 + \| Cu \|_{H_1}^2} \leq \sqrt{K \| Cu \|_{H_1}^2 + \| Cu \|_{H_1}^2} = \sqrt{(1+K)} \| Cu \|_{H_1}$$

holds for all $u \in \mathrm{dom}(C) \cap \ker(C)^\perp$. By taking this into account together with the convergence of $(C^* x_{n_k})_{k\in\mathbb{N}}$ in $\mathrm{dom}(C) \cap \ker(C)^\perp$, we obtain for fixed $\varepsilon > 0$ that

$$\begin{aligned}
\sup_{\substack{y\in\overline{\mathrm{ran}(C)}=\ker(C^*)^\perp \\ \| y \|_{H_1}\leq 1}} |\langle x_{n_k} - x_{n_l}, y\rangle| &= \sup_{\substack{y\in\mathrm{ran}(C) \\ \| y \|_{H_1}\leq 1}} |\langle x_{n_k} - x_{n_l}, y\rangle| = \sup_{\substack{u\in\mathrm{dom}(C)\cap\ker(C)^\perp \\ \| Cu \|_{H_1}\leq 1}} |\langle x_{n_k} - x_{n_l}, Cu\rangle| && (2)\\
&= \sup_{\substack{u\in\mathrm{dom}(C)\cap\ker(C)^\perp \\ \| Cu \|_{H_1}\leq 1}} |\langle C^*(x_{n_k} - x_{n_l}), u\rangle| \\
&= \sup_{\substack{u\in\mathrm{dom}(C)\cap\ker(C)^\perp \\ \| u \|_{*}\leq 1}} |\langle C^*(x_{n_k} - x_{n_l}), u\rangle| && (3)\\
&< \varepsilon && (4)
\end{aligned}$$

holds for $k \geq l$ large enough. To see the second equality in (2), it is enough to show that

$$\mathrm{ran}(C) \subset \{Cu | u \in \mathrm{dom}(C) \cap \ker(C)^\perp\} \tag{5}$$

since $\{Cu | u \in \mathrm{dom}(C) \cap \ker(C)\} \subset \mathrm{ran}(C)$ clearly holds. Let $u \in \mathrm{ran}(C)$. By definition, there exists $v \in \mathrm{dom}(C)$ such that $Cv = u$ holds. By Corollary 2.2.6, it follows that $v = v_1 + v_2$ with $v_1 \in \ker(C), v_2 \in \ker(C)^\perp$. Furthermore since $\ker(C) \subset \mathrm{dom}(C)$ and $v \in \mathrm{dom}(C)$ we finally get that $u = C(v_1 + v_2) = C(v_1) + C(v_2) = C(v_1)$ holds by linearity of C and thus (5) is shown. Moreover, since $x_{n_k} - x_{n_l} \in \ker(C^*)^\perp$ holds, we get

$$\begin{aligned}
\| x_{n_k} - x_{n_l} \|_{H_1} &= \sup_{\substack{y\in H_1 \\ \| y \|_{H_1}=1}} |\langle x_{n_k} - x_{n_l}, y\rangle = \sup_{\substack{y_1\in\ker(C^*)^\perp, y_2\in\ker(C^*) \\ \| y_1 \|_{H_1}+\| y_2 \|_{H_2}\leq 1}} |\langle x_{n_k} - x_{n_l}, y_1 + y_2\rangle| \\
&= \sup_{\substack{y_1\in\ker(C^*)^\perp, y_2\in\ker(C^*) \\ \| y_1 \|_{H_1}+\| y_2 \|_{H_1}\leq 1}} |\langle x_{n_k} - x_{n_l}, y_1\rangle| = \sup_{\substack{y\in\ker(C^*)^\perp \\ \| y \|_{H_1}\leq 1}} |\langle x_{n_k} - x_{n_l}, y\rangle|
\end{aligned}$$

This together with (4) implies that $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in H_1. This shows the claim. □

Exercise 14.6

Let $\nu > 0$, $\Omega = [0,1)^d$, $s \in L_\infty(\mathbb{R})$ be 1-periodic, $0 \leq s \leq 1$, and a as in Example 14.3.7. Show that $(u_n)_n$ in $L_{2,\nu}(\mathbb{R}; L_2(Y))$ satisfying

$$\partial_{t,\nu}^2 s(nm) u_n + \partial_{t,\nu}(1 - s(nm)) u_n - div_\# a(nm) grad_\# u_n = f \tag{6}$$

for some $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ is convergent to some $u \in L_{2,\nu}(\mathbb{R}; L_2(Y))$. Which limit equation is satisfied by u?

Proof. We will solve this exercise by combining Examples 14.3.7 and 14.2.4. First, we notice that

$$\left[\partial_{t,\nu} \begin{pmatrix} s(nm) & 0 \\ 0 & \left(\iota_\#^* a(nm) \iota_\#\right)^{-1} \end{pmatrix} + \begin{pmatrix} 1 - s(nm) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\# \iota_\# \\ \iota_\#^* \mathrm{grad}_\# & 0 \end{pmatrix}\right] \begin{pmatrix} v_n \\ q_n \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{7}$$

with $v_n := \partial_{t,\nu} u_n$ and $q_n := -\iota_\#^* a(nm) \iota_\# \iota_\#^* \mathrm{grad}_\# u_n$ represents

$$\begin{aligned}
\partial_{t,\nu}^2 s(nm) u_n + \partial_{t,\nu} s(nm)(1 - s(nm)) u_n - \mathrm{div}_\# \iota_\# \iota_\#^* a(nm) \iota_\# \iota_\#^* \mathrm{grad}_\# u_n &= f \\
-\partial_{t,\nu} \left(\iota_\#^* a(nm) \iota_\#\right)^{-1} \iota_\#^* a(nm) \iota_\# \iota_\#^* \mathrm{grad}_\# u_n + \iota_\#^* \mathrm{grad}_\# \partial_{t,\nu} u_n &= 0
\end{aligned}$$

and, therefore, Equation (6). $M_n(z)$, defined by

$$M_n(z) = \begin{pmatrix} s(n\mathrm{m}) & 0 \\ 0 & \left(\iota_\#^* a(n\mathrm{m})\iota_\#\right)^{-1} \end{pmatrix} + z^{-1}\begin{pmatrix} 1-s(n\mathrm{m}) & 0 \\ 0 & 0 \end{pmatrix},$$

is a material law and

$$\mathrm{Re}\; zM_n(z) \geq c$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, since $\iota_\#^* a(n\mathrm{m})\iota_\#$ is boundedly invertible and strictly positive definite by Proposition 11.3.5 and $0 \leq s(n\mathrm{m}) \leq 1$. By Exercise 11.3

$$\begin{pmatrix} 0 & \mathrm{div}_\#\iota_\# \\ \iota_\#^*\mathrm{grad}_\# & 0 \end{pmatrix}$$

is skew self-adjoint. Therefore, the operator

$$S^{(n)} := \left(\overline{\partial_{t,\nu}\begin{pmatrix} s(n\mathrm{m}) & 0 \\ 0 & \left(\iota_\#^* a(n\mathrm{m})\iota_\#\right)^{-1} \end{pmatrix} + \begin{pmatrix} 1-s(n\mathrm{m}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\#\iota_\# \\ \iota_\#^*\mathrm{grad}_\# & 0 \end{pmatrix}}\right)^{-1}$$

is well-defined and bounded in $L_{2,\nu}(\mathbb{R}; L_2(Y) \times \mathrm{ran}(\mathrm{grad}_\#))$. As in the Examples 14.3.7 and 14.4.4, $\left(S^{(n)}\right)_n$ converges to

$$\left(\overline{\partial_{t,\nu}\begin{pmatrix} \int_0^1 s & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 1-\int_0^1 s & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\#\iota_\# \\ \iota_\#^*\mathrm{grad}_\# & 0 \end{pmatrix}}\right)^{-1}$$

in the strong operator topology of $L_{2,\nu}(\mathbb{R}; L_2(Y) \times \mathrm{ran}(\mathrm{grad}_\#))$ and, thus, for $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ the sequence $u_n \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ converges to some $u \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ which fulfills

$$\left(\partial_{t,\nu}\begin{pmatrix} \int_0^1 s & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 1-\int_0^1 s & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\#\iota_\# \\ \iota_\#^*\mathrm{grad}_\# & 0 \end{pmatrix}\right)\begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

with $v = \partial_{t,\nu}u$ and $q = -\iota_\#^* a_{\mathrm{hom}}\iota_\#\iota_\#^*\mathrm{grad}_\# u$ by Lemma 14.4.8. Therefore, u solves

$$\partial_{t,\nu}^2\left(\int_0^1 s\right)u + \partial_{t,\nu}\left(1-\int_0^1 s\right)u - \mathrm{div}_\# a_{\mathrm{hom}}\mathrm{grad}_\# u = f.$$

□

Exercise 14.7

Let $(\alpha_n)_n$ be a null-sequence in $[0,1]$ and let a be as in Example 14.3.7. Show

$$\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \partial_{t,\nu}^{\alpha_n}\mathfrak{a}_n \end{pmatrix} + \begin{pmatrix} 0 & div_\sharp\iota_\sharp \\ \iota_\sharp^* grad_\sharp & 0 \end{pmatrix}}\right)^{-1} \to \left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{hom} \end{pmatrix} + \begin{pmatrix} 0 & div_\sharp\iota_\sharp \\ \iota_\sharp^* grad_\sharp & 0 \end{pmatrix}}\right)^{-1}$$

in the strong operator topology. Furthermore, if $f \in L_{2,-\mu}(\mathbb{R}, L_2(Y)_\perp)$, where $L_2(Y)_\perp := \{\phi \in L_2(Y); \int_Y \phi = 0\}$ for some small enough $\mu > 0$, we have

$$\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{hom} \end{pmatrix} + \begin{pmatrix} 0 & div_\sharp\iota_\sharp \\ i_\sharp^* grad_\sharp & 0 \end{pmatrix}}\right)^{-1}\begin{pmatrix} f \\ 0 \end{pmatrix} \in L_{2,-\mu}(\mathbb{R}; L_2(Y)_\perp \times ran(grad_\sharp)).$$

Proof. The first claim is an application of Theorem 14.1.1 which is applicable by the following arguments. By Example 14.3.7,

$$A := \begin{pmatrix} 0 & \mathrm{div}_\sharp\iota_\sharp \\ i_\sharp^*\mathrm{grad}_\sharp & 0 \end{pmatrix}$$

fullfills the conditions of Theorem 14.1.1. Now, consider

$$M_n(z) := \begin{pmatrix} 1 & 0 \\ 0 & z^{\alpha_n - 1}\mathfrak{a}_n \end{pmatrix}.$$

We notice that by Example 5.3.1 (e) the sequence (M_n) is in $\mathcal{M}(H,\nu)$ for $\nu > 0$ where $H := L_2(Y) \times \mathrm{ran}(\mathrm{grad}_\sharp)$ is a Hilbert space since, by Proposition 14.3.6, $\mathrm{ran}(\mathrm{grad}_\sharp)$ is a closed (linear) subspace of the Hilbert space $L_2(Y)^d$ and, thus, is a Hilbert space itself.

Using the assumption on a, there is a constant $c > 0$ such that for $y \in \operatorname{ran}(\operatorname{grad}_\sharp)$ we have

$$\operatorname{Re}\langle y, \mathfrak{a}_n^{-1} y\rangle = \operatorname{Re}\langle y, \iota_\sharp^* a(nm)\iota_\sharp y\rangle = \operatorname{Re}\langle \iota_\sharp y, a(nm)\iota_\sharp y\rangle \geq c\|\iota_\sharp y\|^2_{L^2(Y)^d} = c\|y\|^2_{\operatorname{ran}(\operatorname{grad}_\sharp)}. \tag{$*$}$$

By Theorem 6.2.2 and since $\mathfrak{a}_n^{-1}$ is uniformly bounded in n, we have shown

$$\operatorname{Re}\mathfrak{a}_n \geq \frac{c}{\|\mathfrak{a}_n^{-1}\|^2} \geq \frac{c}{\sup_{n\in\mathbb{N}}\|\mathfrak{a}_n^{-1}\|^2} =: \tilde{c} > 0.$$

We also note that by Example 14.3.7, $\mathfrak{a}_n^{-1}$ is selfadjoint. Hence it is plain to see that also $\mathfrak{a}_n$ is selfadjoint, and we can conclude that $\langle y, \mathfrak{a}_n y\rangle$ is a real number. Thus, we have for $(x, y) \in L_2(Y) \times \operatorname{ran}(\operatorname{grad}_\sharp)$ and $z \in \mathbb{C}_{\operatorname{Re}>\nu}$

$$\begin{aligned}
\operatorname{Re}\,\langle (x,y), zM_n(x,y)\rangle_H &= \operatorname{Re}\left\langle (x,y), z\begin{pmatrix} 1 & 0 \\ 0 & z^{\alpha_n - 1}\mathfrak{a}_n \end{pmatrix}(x,y)\right\rangle_H \\
&= \operatorname{Re}\,\langle (x,y), (zx, z^{\alpha_n}\mathfrak{a}_n y)\rangle_H \\
&= \operatorname{Re}(z)\,\langle x, x\rangle_{L_2(Y)} + \operatorname{Re}(z^{\alpha_n})\langle y, \mathfrak{a}_n y\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)} \\
&\geq \nu\|x\|^2 + \left(\inf_{n\in\mathbb{N}} \operatorname{Re}(z^{\alpha_n})\right)\tilde{c}\,\|y\|^2 \\
&\geq \min\left\{\nu, \left(\inf_{n\in\mathbb{N}} \operatorname{Re}(z^{\alpha_n})\right)\tilde{c}\right\}\|(x,y)\|^2. \\
&\geq \min\left\{\nu, \left(\inf_{n\in\mathbb{N}} \nu^{\alpha_n}\cos\left(\sup_{n\geq n_0} a_n\frac{\pi}{2}\right)\right)\tilde{c}\right\}\|(x,y)\|^2,
\end{aligned}$$

where in the last line we have choosen an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and some $\varepsilon \in (0,1)$ we have $\alpha_n \leq \varepsilon < 1$ and exploited the following calculation:

$$\operatorname{Re}(z^{\alpha_n}) = \operatorname{Re} e^{\alpha_n(\log|z| + i\arg(z))} = |z|^{\alpha_n}\cos(\alpha_n \arg(z)) \geq |\nu|^{\alpha_n}\cos\left(\alpha_n\frac{\pi}{2}\right) \geq |\nu|^{\alpha_n}\cos\left(\sup_{n\geq n_0}\alpha_n\frac{\pi}{2}\right).$$

It is left to show that $M_n(z) \to M(z)$ as $n \to \infty$ in the weak operator topology for any $z \in \mathbb{C}_{\operatorname{Re}>\nu}$ where $\nu > 0$. This follows with the following calculation for $x, y \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and $z \in \mathbb{C}_{\operatorname{Re}>\nu}$:

$$\langle x, z^{1-\alpha_n}\mathfrak{a}_n y\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)} = z^{1-\alpha_n}\langle x, \mathfrak{a}_n y\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)} \to z\langle x, \mathfrak{a}_{\hom} y\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)} \quad \text{as} \quad n \to \infty,$$

where we used Theorem 14.4.3 and only showed it for the second component of M_n since the first one is the same is in Example 14.3.7. This completes the proof of the first part.
For the second claim, we will now show that Theorem 11.2.1 is applicable. So, consider H_0 and H_1 to be $L_2(Y)_\perp$ and $\operatorname{ran}(\operatorname{grad}_\sharp)$, respectively. It is not difficult to see that $L_2(Y)_\perp = \operatorname{lin}(1_Y)^\perp$ and, thus, by Proposition 14.3.6, we have $L_2(Y)_\perp = \ker(\operatorname{grad}_\sharp)^\perp$. So H_0 is a Hilbert space. We set $C := \iota_\sharp^* \operatorname{grad}_\sharp : H^1_\sharp(Y) \cap H_0 \to H_1$. By Proposition 14.3.6, we find C to be a densely defined closed linear operator which is boundedly invertible. Furthermore, we set $M_0 := 1$ and $M_1 : \mathbb{C} \to L(H_1)$, $z \mapsto \mathfrak{a}_{\hom}$ which is in $L(H_1)$ by Theorem 14.4.3. M_0 is clearly selfadjoint with $M_0 \geq c_0$ for some $c_0 > 0$. Since for all $z \in \mathbb{C}$ we have $M_1(z) = \mathfrak{a}_{\hom} \in L(H_1)$ it defines a material law by Example 5.3.1 such that $s_b(M_1) = -\infty$ and, thus, for any fixed $\rho_1 > 0$ we have $s_b(M_1) < -\rho_1$. By Remark 14.4.2, we have for all $\xi \in \mathbb{K}^d \setminus \{0\}$

$$\operatorname{Re}\langle \xi, a_{\hom}\xi\rangle_{\mathbb{K}^d} \geq c\|v_\xi\|_{L_2(Y)^d} > 0.$$

Since $\mathbb{K}^d$ is finite dimensional, $\partial B_1(0) \subset \mathbb{K}^d$ is compact and by continuity of $\xi \mapsto \operatorname{Re}\langle \xi, a_{\hom}\xi\rangle$ on $\mathbb{K}^d$, the above inequality yields that there is a (possibly different) constant $c > 0$ such that for $\xi \in \partial B_1(0)$ we have $\operatorname{Re}\langle \xi, a_{\hom}\xi\rangle_{\mathbb{K}^d} \geq c = c\|\xi\|^2$ or, with other words, the continuous function on the left-hand side attains its minimum on the compact set $\partial B_1(0)$ which must be greater than zero. Thus, by linearity of $a_{\hom}$ and a scaling argument in the last inequality, we have for all $u \in \operatorname{ran}(\operatorname{grad}_\sharp)$

$$\begin{aligned}
\operatorname{Re}\langle u, \mathfrak{a}_{\hom}^{-1} u\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)} &= \operatorname{Re}\langle \iota_\sharp u, a_{\hom}\iota_\sharp u\rangle_{L_2(Y)^d} = \int_Y \operatorname{Re}\langle \iota_\sharp u(x), a_{\hom}\iota_\sharp u(x)\rangle_{\mathbb{K}^d} dx \\
&\geq \int_Y c\langle \iota_\sharp u(x), \iota_\sharp u(x)\rangle_{\mathbb{K}^d} dx = c\langle \iota_\sharp u, \iota_\sharp u\rangle_{L_2(Y)^d} = c\langle u, u\rangle_{\operatorname{ran}(\operatorname{grad}_\sharp)}.
\end{aligned}$$

Thus, finally again by Theorem 6.2.2, we arrive at $\operatorname{Re}\mathfrak{a}_{\hom} \geq c\|\mathfrak{a}_{\hom}^{-1}\|^{-2} > 0$. □