14 Continuous Dependence on the Coefficients II

This lecture is concerned with the study of problems of the form

$$\left(\partial_{t,\nu} M_n(\partial_{t,\nu}) + A\right) U_n = F$$

for a suitable sequence of material laws $(M_n)_n$ when $A \neq 0$. The aim of this section will be to provide the conditions required for convergence of the material law sequence to imply the existence of a limit material law M such that the limit $U = \lim_{n\to\infty} U_n$ exists and satisfies

$$\left(\partial_{t,\nu} M(\partial_{t,\nu}) + A\right) U = F.$$

Additionally, for material laws of the form $M_n(\partial_{t,\nu}) = M_{0,n} + \partial_{t,\nu}^{-1} M_{1,n}$ it will desirable to have the respective limit material law satisfy $M(\partial_{t,\nu}) = M_0 + \partial_{t,\nu}^{-1} M_1$ for some $M_0, M_1 \in L(H)$. This cannot be expected (as we have seen in the guiding example in the previous lecture) if A is a bounded operator, the Hilbert space H is infinite-dimensional, and the material law sequence only converges pointwise in the weak operator topology. It will turn out, however, that if A is "strictly unbounded" then a suitable result can hold, even if we only assume weak convergence of the material law operators.

14.1 A Convergence Theorem

The main convergence theorem of this lecture will be presented next.

Theorem 14.1.1. *Let H be a Hilbert space, $\nu \in \mathbb{R}$, $(M_n)_n$ in $\mathcal{M}(H,\nu)$ and $M \in \mathcal{M}(H,\nu)$. Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\operatorname{Re} z M_n(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu}).$$

Let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint and assume that $\operatorname{dom}(A) \hookrightarrow H$ is compact. If $M_n(z) \to M(z)$ as $n \to \infty$ in the weak operator topology for all $z \in \mathbb{C}_{\operatorname{Re}>\nu}$, then

$$\left(\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A}\right)^{-1} \to \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1}$$

in the strong operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$.

For the proof of this theorem, we need a lemma first.

Lemma 14.1.2. *Let H be a Hilbert space, $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint, $c > 0$, $(T_n)_n$ in $L(H)$ with $\operatorname{Re} T_n \geqslant c$ and all $n \in \mathbb{N}$, and $T \in L(H)$. Assume $\operatorname{dom}(A) \hookrightarrow H$ compactly and $T_n \to T$ in the weak operator topology. Then $0 \in \bigcap_{n\in\mathbb{N}} \rho(T_n+A) \cap \rho(T+A)$ and*

$$(T_n + A)^{-1} \to (T + A)^{-1}$$

in the norm topology of $L(H)$.

Proof. From $\operatorname{Re} T_n \geqslant c$ it follows that $0 \in \rho(T_n + A)$ $(n \in \mathbb{N})$ and $\left((T_n + A)^{-1}\right)_n$ is bounded in $L(H)$. Moreover, since

$$A(T_n + A)^{-1} = 1 - T_n(T_n + A)^{-1}$$

for all $n \in \mathbb{N}$, it follows that $\left((T_n + A)^{-1}\right)_n$ is also bounded in $L(H, \operatorname{dom}(A))$ by the boundedness of $(T_n)_n$ in $L(H)$. Due to the convergence of $(T_n)_n$ to T, it follows that $\operatorname{Re} T \geqslant c$, and thus, $(T+A)^{-1} \in L(H, \operatorname{dom}(A))$. Before we come to a proof of the desired result, we will prove an auxiliary observation.
Claim: for all $(f_n)_n$ in H weakly converging to f, we have $(T_n + A)^{-1} f_n \to (T + A)^{-1} f$ in the norm topology of H.
For proving the claim, let $(f_n)_n$ in H be weakly convergent to some f. Consider $u_n \coloneqq (T_n + A)^{-1}f_n$. Then $(u_n)_n$ is bounded in $\operatorname{dom}(A)$, since $\left((T_n + A)^{-1}\right)_n$ is bounded in $L(H, \operatorname{dom}(A))$ and $(f_n)_n$ is bounded in H. Hence, there exists a subsequence $(u_{n_k})_k$ which weakly converges to some u in $\operatorname{dom}(A)$. Since $\operatorname{dom}(A) \hookrightarrow H$ compactly, we infer $u_{n_k} \to u$ in the norm topology of H. Hence, in the equality

$$T_{n_k}u_{n_k} + Au_{n_k} = f_{n_k},$$

as $T_{n_k} \to T$ in the weak operator topology and $u_{n_k} \to u$ in H, we may let $k \to \infty$ and obtain for the weak limits

$$Tu + Au = f;$$

that is, $u = (T + A)^{-1} f$. Having identified the limit, a contradiction argument (here a so-called 'subsequence argument') concludes that $(u_n)_n$ itself converges weakly in $\operatorname{dom}(A)$ and strongly in H to u. Thus, the claim is proved.
Next, assume by contradiction that $\left((T_n + A)^{-1}\right)_n$ does not converge in operator norm to $(T + A)^{-1}$. Then we find an $\varepsilon > 0$ and a strictly increasing sequence of integers, $(n_k)_k$, and a sequence of unit vectors $(f_{n_k})_k$ in H such that

$$\left\|(T_{n_k} + A)^{-1}f_{n_k} - (T + A)^{-1} f_{n_k}\right\| \geqslant \varepsilon. \tag{14.1}$$

By possibly taking another subsequence, we may assume without loss of generality that $(f_{n_k})_k$ converges weakly to some $f \in H$. By the claim proved above, we deduce $(T_{n_k} + A)^{-1} f_{n_k} \to (T + A)^{-1} f$ and $(T + A)^{-1} f_{n_k} \to (T + A)^{-1} f$, both in the norm topology of H as $k \to \infty$. Thus, we may let $k \to \infty$ in (14.1), and obtain the desired contradiction. □

Proof of Theorem 14.1.1. By Theorem 13.1.1 it suffices to show that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$

$$(zM_n(z) + A)^{-1} \to (zM(z) + A)^{-1} \quad (n \to \infty)$$

in the strong operator topology. This, however, follows from Lemma 14.1.2 applied to $T_n = zM_n(z)$. □

Remark 14.1.3. Note that we only used convergence in the strong operator topology in the proof of Theorem 14.1.1. However, the assertion in Lemma 14.1.2 is about convergence in the norm topology. The reason that we cannot assert the convergence claimed in Theorem 14.1.1 in the norm topology is that the compact embedding of $\operatorname{dom}(A) \hookrightarrow H$ only works locally for fixed z, and not uniformly in z. This situation can, however, be rectified. We refer to Exercise 14.1 for this.

14.2 The Theorem of Rellich and Kondrachov

In order to apply Theorem 14.1.1, we need to provide a setting where the condition on the compactness of the embedding is satisfied. In fact, it is true that $H^1(\Omega)$ embeds compactly into $L_2(\Omega)$ given $\Omega \subseteq \mathbb{R}^d$ is bounded and has 'continuous boundary' [1, Theorem 7.11]. In this lecture, we restrict ourselves to a proof of a less general statement.
A preparatory result needed to prove the compact embedding theorem is given next.

Proposition 14.2.1. *Let $I \subseteq \mathbb{R}$ be an open, bounded, non-empty interval. Then the mapping $H^1(\mathbb{R}) \ni f \mapsto f|_I \in H^1(I)$ is well-defined, continuous and onto. Moreover, there exists a continuous right inverse $H^1(I) \to H^1(\mathbb{R})$.*

For the proof of this proposition, we need an auxiliary result first.

Lemma 14.2.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected. Moreover, let $u \in H^1(\Omega)$ with $\operatorname{grad} u = 0$. Then u is constant.*

We leave the proof of this lemma as Exercise 14.2.

Proof of Proposition 14.2.1. The mapping $H^1(\mathbb{R}) \to H^1(I), f \mapsto f|_I$ is readily confirmed to be continuous. It remains to prove that it is onto. Let $I = (a, b)$, $u \in H^1(I)$ and define the function v by

$$v(t) := \int_a^t \partial u(s)\,\mathrm{d}s \quad (t \in (a, b)).$$

Clearly, $v \in L_2((a, b))$ and we compute for each $\varphi \in C_c^\infty((a, b))$

$$\begin{aligned}\langle v, \varphi' \rangle_{L_2((a,b))} &= \int_a^b \left(\int_a^t \partial u(s)\,\mathrm{d}s \right)^* \varphi'(t)\,\mathrm{d}t = \int_a^b \int_s^b \varphi'(t)\,\mathrm{d}t\,\partial u(s)^*\,\mathrm{d}s \\ &= -\langle \partial u, \varphi \rangle_{L_2((a,b))}.\end{aligned}$$

This shows $v \in H^1((a,b))$ with $\partial v = \partial u$. Hence, by Lemma 14.2.2 there exists a constant $c \in \mathbb{C}$ with $u = c + v$. We now define f by

$$f(t) \coloneqq \begin{cases} 0 & \text{if } t < a-1 \text{ or } t > b+1, \\ ct + c(1-a) & \text{if } a-1 \leqslant t \leqslant a, \\ u(t) & \text{if } a < t < b, \\ (c+v(b))t + (c+v(b))(1-b) & \text{if } b \leqslant t \leqslant b+1. \end{cases}$$

We then easily see that $f \in H^1(\mathbb{R})$ and clearly $f|_{(a,b)} = u$. In order to see that $u \mapsto f$ is continuous, we need to establish that the value c depends continuously on u. From the equality $u = c + v$ and the continuity of v it follows that $c = u(a+)$. Hence, we need to establish that this point evaluation is continuous. For this, take $\phi \in C^\infty(\mathbb{R})$ with $0 \leqslant \phi \leqslant 1$, $\phi(x) = 1$ for all $x \leqslant a$ and $\operatorname{spt}\phi \in (-\infty, b)$. It is then not difficult to see that $H^1((a,b)) \ni w \mapsto \phi w \in H^1((a,\infty))$ is continuous. Thus, by Theorem 12.1.3, the evaluation at the boundary (i.e. at a) is well-defined and continuous. The assertion follows. □

Theorem 14.2.3. *Let $I \subseteq \mathbb{R}$ be an open bounded interval. Then $H^1(I) \hookrightarrow L_2(I)$ compactly.*

Proof. By Proposition 14.2.1, we find a continuous mapping $E\colon H^1(I) \to H^1(\mathbb{R})$ such that for all $u \in H^1(I)$, $E(u)|_I = u$. Moreover, by Exercise 4.3 the mapping $H^1(\mathbb{R}) \hookrightarrow C^{1/2}(\mathbb{R})$ is continuous. Thus,

$$H^1(I) \xrightarrow{E} H^1(\mathbb{R}) \hookrightarrow C^{1/2}(\mathbb{R}) \to C^{1/2}(I),$$

is a composition of continuous mappings, where the last mapping is the restriction to I. Since $C^{1/2}(I) \hookrightarrow C(I)$ compactly by the Arzelà–Ascoli Theorem, and $C(I) \hookrightarrow L_2(I)$ continuously, we infer $H^1(I) \hookrightarrow L_2(I)$ compactly. □

We now have the opportunity to study the limit behaviour of a periodic mixed type problem.

Example 14.2.4 (Highly oscillatory problems)**.** Let $s_1, s_2\colon \mathbb{R} \to [0,1]$ be 1-periodic, measurable functions. Then for $\nu > 0$, we set

$$S^{(n)} \coloneqq \left(\overline{\partial_{t,\nu} \begin{pmatrix} s_1(n\mathrm{m}) & 0 \\ 0 & s_2(n\mathrm{m}) \end{pmatrix} + \begin{pmatrix} 1 - s_1(n\mathrm{m}) & 0 \\ 0 & 1 - s_2(n\mathrm{m}) \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}}\right)^{-1},$$

where $\partial = \operatorname{div}$ and $\partial_0 = \operatorname{grad}_0$ are regarded as operators in $L_2((0,1))$ with respective domains $H^1((0,1))$ and $H_0^1((0,1))$. Then, by Theorem 14.2.3, the operator $A \coloneqq \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}$ satisfies the assumptions of Theorem 14.1.1. Moreover, Theorem 13.2.3 implies that the remaining assumtions of Theorem 14.1.1 are satisfied. Hence, we deduce that $\big(S^{(n)}\big)_n$ converges in the strong operator topology of $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ to the limit

$$\left(\overline{\partial_{t,\nu} \begin{pmatrix} \int_0^1 s_1 & 0 \\ 0 & \int_0^1 s_2 \end{pmatrix} + \begin{pmatrix} 1 - \int_0^1 s_1 & 0 \\ 0 & 1 - \int_0^1 s_2 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}}\right)^{-1}.$$

Next, we aim to provide an application to more than one spatial dimension. For this, we will also need a corresponding compactness statement. This is the subject of the rest of this section.

Theorem 14.2.5 (Rellich–Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ compactly.*

Proof. Without loss of generality (by shifting and shrinking of Ω and extending by 0), we may assume that $\Omega = (0,1)^d$. We carry out the proof by induction on the spatial dimension d. The case $d = 1$ has been dealt with in Theorem 14.2.3. Assume the statement is true for some $d-1$. Using that $C_c^\infty((0,1)^d)$ is dense in $H_0^1((0,1)^d)$, we infer the continuity of the injection

$$\begin{aligned} R\colon H_0^1((0,1)^d) &\to H^1\big(\mathbb{R}; L_2((0,1)^{d-1})\big) \cap L_2\big(\mathbb{R}; H_0^1((0,1)^{d-1})\big) \\ \phi &\mapsto \big(t \mapsto \big(\omega \mapsto \phi(t,\omega)\big)\big), \end{aligned}$$

where we identify ϕ with its extension to $\mathbb{R}^d$ by 0. The range space is endowed with the usual sum scalar product.
Let $(\phi_n)_n$ be a weakly convergent null-sequence in $H_0^1((0,1)^d)$. In particular, $(R\phi_n)_n$ is bounded in $H^1\big(\mathbb{R}; L_2((0,1)^{d-1})\big)$ and hence, it is also bounded in $C_0\big(\mathbb{R}; L_2\,(0,1)^{d-1}\big)$ by Theorem 4.1.2 (and Corollary 4.1.3); that is,

$$\sup_{t\in[0,1], n\in\mathbb{N}} \|\phi_n(t,\cdot)\|_{L_2((0,1)^{d-1})} < \infty. \tag{14.2}$$

Let $f \in L_2((0,1)^{d-1})$. Then $(\phi_{n,f})_n$ given by

$$\phi_{n,f}\colon t \mapsto \langle \phi_n(t,\cdot), f\rangle_{L_2((0,1)^{d-1})}$$

is a weakly convergent null-sequence in $H^1((0,1))$. We obtain by Theorem 14.2.3 that $\phi_{n,f} \to 0$ in $L_2((0,1))$ as $n \to \infty$. By separability of $L_2((0,1)^{d-1})$ we find $D \subseteq L_2((0,1)^{d-1})$ countable and dense, a subsequence (again labeled by n) and a null-set $N \subseteq \mathbb{R}$ such that $\phi_{n,f}(t) \to 0$ for all $t \in \mathbb{R}\setminus N$ and $f \in D$ as $n \to \infty$. By (14.2), we deduce $\phi_{n,f}(t) \to 0$ for all $t \in \mathbb{R}\setminus N$ and $f \in L_2((0,1)^{d-1})$ as $n \to \infty$, or, in other words, $\phi_n(t,\cdot) \to 0$ weakly in $L_2((0,1)^{d-1})$ for each $t \in \mathbb{R}\setminus N$ as $n \to \infty$.
Next, we show that there exists a null set $N \subseteq N_1 \subseteq \mathbb{R}$ such $\phi_n(t,\cdot) \to 0$ in $L_2((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$. For this, since $(R\phi_n)_n$ in $L_2\big(\mathbb{R}; H_0^1((0,1)^{d-1})\big)$ is bounded, we find a null set $N \subseteq N_1 \subseteq \mathbb{R}$ such that $(\phi_n(t,\cdot))_n$ is bounded in $H_0^1((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$. Let $t \in \mathbb{R}\setminus N_1$. Then there exists a further subsequence $(\phi_{n_k}(t,\cdot))_k$ which converges weakly in $H_0^1((0,1)^{d-1})$. By the induction hypothesis, $(\phi_{n_k}(t,\cdot))_{n_k}$ converges strongly in $L_2((0,1)^{d-1})$, and since we have already seen that it is a weak null-sequence in $L_2((0,1)^{d-1})$, we derive $\phi_{n_k}(t,\cdot) \to 0$ in $L_2((0,1)^{d-1})$. By a subsequence argument we derive that

$$\phi_n(t,\cdot) \to 0$$

in $L_2((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$.

Now, for $n \in \mathbb{N}$ we deduce

$$\|\phi_n\|^2_{L_2((0,1)^d)} = \int_0^1 \|\phi_n(t,\cdot)\|^2_{L_2((0,1)^{d-1})}\,\mathrm{d}t \to 0,$$

where we have used dominated convergence, which is possible due to (14.2). □

14.3 The Periodic Gradient

In this section we investigate the gradient on periodic functions on $\mathbb{R}^d$. Throughout, we set $Y := [0,1)^d$.

Definition (Periodic Gradient)**.** We define

$$C_\sharp^\infty(Y) := \left\{\phi|_Y\,;\, \phi \in C^\infty(\mathbb{R}^d),\ \phi(\cdot + k) = \phi\ (k \in \mathbb{Z}^d)\right\}$$

and

$$\begin{aligned} \operatorname{grad}_{\sharp,\infty}\colon C_\sharp^\infty(Y) \subseteq L_2(Y) &\to L_2(Y)^d \\ \phi &\mapsto \operatorname{grad}\phi. \end{aligned}$$

Moreover, we set $\operatorname{div}_\sharp := -\operatorname{grad}_{\sharp,\infty}^*$ and $\operatorname{grad}_\sharp := -\operatorname{div}_\sharp^* = \overline{\operatorname{grad}_{\sharp,\infty}}$.

Remark 14.3.1. The operators just introduced can easily be shown to lie between the operator realisations we have introduced in earlier lectures. Indeed, it is easy to see that

$$\operatorname{div}_0 \subseteq \operatorname{div}_\sharp \text{ and } \operatorname{grad}_0 \subseteq \operatorname{grad}_\sharp$$

and, consequently, we also have

$$\operatorname{grad}_\sharp \subseteq \operatorname{grad} \text{ and } \operatorname{div}_\sharp \subseteq \operatorname{div}.$$

The corresponding domains for the operators $\operatorname{grad}_\sharp$ and $\operatorname{div}_\sharp$ will be denoted by $H_\sharp^1(Y)$ and $H_\sharp(\operatorname{div}, Y)$, respectively.
For the next results, we define the periodic extension operator. For $\phi \in L_2(Y)^m$ we put

$$\phi_{\mathrm{pe}}(x+k) := \phi(x)$$

for almost every $x \in Y$ and all $k \in \mathbb{Z}^d$.

We start with the following two observations.

Lemma 14.3.2. *Let $f \in L_2(Y)$ and $(\rho_k)_k$ be a δ-sequence in $C_\mathrm{c}^\infty(\mathbb{R}^d)$ (cf. Exercise 3.1). Define*

$$f_k := (\rho_k * f_{\mathrm{pe}})|_Y \quad (k \in \mathbb{N}).$$

Then $f_k \in C_\sharp^\infty(Y)$ for each $k \in \mathbb{N}$ and $f_k \to f$ in $L_2(Y)$ as $k \to \infty$.

Proof. It follows as in Exercise 3.2 that $\rho_k * f_{\mathrm{pe}}$ is in C^∞. Moreover, one easily sees that $\rho_k * f_{\mathrm{pe}}$ is $[0,1)^d$-periodic, and hence, $f_k \in C_\sharp^\infty(Y)$ for each $k \in \mathbb{N}$. For the convergence we observe

$$\big(\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})\big)(x) = f_k(x) \quad (x \in Y, k \in \mathbb{N}).$$

Moreover, by Exercise 3.2 we have $\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}}) \to \mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}}$ in $L_2(\mathbb{R}^d)$ as $k \to \infty$, and thus,

$$f_k = \big(\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})\big)|_Y \to (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})|_Y = f \quad (k \to \infty) \quad \text{in } L_2(Y). \qquad \square$$

Lemma 14.3.3. *$C_\sharp^\infty(Y)^d$ is a core for $\operatorname{div}_\sharp$ and $\operatorname{div}_\sharp \Psi = \operatorname{div} \Psi$ for each $\Psi \in C_\sharp^\infty(Y)^d$.*

Proof. First we note that $C_\sharp^\infty(Y)^d \subseteq \operatorname{dom}(\operatorname{div}_\sharp)$ and $\operatorname{div}_\sharp \Psi = \operatorname{div} \Psi$ for $\Psi \in C_\sharp^\infty(Y)^d$. To see this, for $\phi \in C_\sharp^\infty(Y), \Psi \in C_\sharp^\infty(Y)^d$ we compute

$$\begin{aligned}\langle \operatorname{grad} \phi, \Psi\rangle_{L_2(Y)^d} &= \int_Y \langle \operatorname{grad} \phi(x), \Psi(x)\rangle_{\mathbb{K}^d} \,\mathrm{d}x = -\int_Y \phi(x)^* \operatorname{div} \Psi(x) \,\mathrm{d}x \\ &= \langle \phi, -\operatorname{div} \Psi\rangle_{L_2(Y)}\end{aligned}$$

by integration by parts (note that the boundary values cancel out due to the periodicity of ϕ and Ψ). Now, let $q \in \operatorname{dom}(\operatorname{div}_\sharp)$ and $(\rho_k)_k$ be a δ-sequence in $C_\mathrm{c}^\infty(\mathbb{R}^d)$. For $k \in \mathbb{N}$ we define

$$q_k := (\rho_k * q_{\mathrm{pe}})|_Y,$$

and obtain $q_k \in C_\sharp^\infty(Y)$ and $q_k \to q$ in $L_2(Y)^d$ as $k \to \infty$ by Lemma 14.3.2. It is left to show that $\operatorname{div} q_k \to \operatorname{div}_\sharp q$ in $L_2(Y)$ as $k \to \infty$. For doing so, we show that $\operatorname{div} q_k = \big(\rho_k * (\operatorname{div}_\sharp q)_{\mathrm{pe}}\big)|_Y$, which would then yield the assertion again by Lemma 14.3.2. So, let $k \in \mathbb{N}$ and $\phi \in C_\sharp^\infty(Y)$. We compute

$$\begin{aligned}\langle q_k, \operatorname{grad} \phi\rangle_{L_2(Y)^d} &= \int_Y \left\langle \int_{\mathbb{R}^d} \rho_k(y) q_{\mathrm{pe}}(x-y) \,\mathrm{d}y, \operatorname{grad} \phi(x)\right\rangle_{\mathbb{K}^d} \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q_{\mathrm{pe}}(x-y), \operatorname{grad} \phi(x)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_{Y-y} \langle q_{\mathrm{pe}}(x), (\operatorname{grad} \phi)_{\mathrm{pe}}(x+y)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q(x), (\operatorname{grad} \phi)_{\mathrm{pe}}(x+y)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q(x), (\operatorname{grad} \phi_{\mathrm{pe}}(\cdot + y))(x)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle \operatorname{div}_\sharp q(x), \phi_{\mathrm{pe}}(x+y)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} \rho_k(y) \int_{Y+y} \langle (\operatorname{div}_\sharp q)_{\mathrm{pe}}(x-y), \phi(x)\rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\left\langle \big(\rho_k * (\operatorname{div}_\sharp q)_{\mathrm{pe}}\big)|_Y, \phi\right\rangle_{L_2(Y)},\end{aligned}$$

where we have used perodicity as well as $\phi_{\mathrm{pe}}(\cdot + y) \in C_\sharp^\infty(Y)$. $\square$

Proposition 14.3.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, $u \in H^1_\sharp(Y)$ and $q \in H_\sharp(\operatorname{div}, Y)$. Then $u_{\mathrm{pe}}|_\Omega \in H^1(\Omega)$, $q_{\mathrm{pe}}|_\Omega \in H(\operatorname{div}, \Omega)$ and*

$$\operatorname{grad}(u_{\mathrm{pe}}|_\Omega) = \big(\operatorname{grad}_\sharp u\big)_{\mathrm{pe}}|_\Omega \text{ and } \operatorname{div}(q_{\mathrm{pe}}|_\Omega) = (\operatorname{div}_\sharp q)_{\mathrm{pe}}|_\Omega.$$

Proof. Let first $\phi \in C^\infty_\sharp(Y)$. Then by definition $\phi_{\mathrm{pe}} \in C^\infty(\mathbb{R}^d)$ and we easily see

$$\operatorname{grad}\phi_{\mathrm{pe}} = (\operatorname{grad}\phi)_{\mathrm{pe}} = (\operatorname{grad}_\sharp \phi)_{\mathrm{pe}}.$$

Moreover, since Ω is bounded, we infer $\phi_{\mathrm{pe}} \in H^1(\Omega)$. By definition of $H^1_\sharp(Y)$ we find a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C^\infty_\sharp(Y)$ such that $\phi_k \to u$ in $L_2(Y)$ and $\operatorname{grad}_\sharp \phi_k \to \operatorname{grad}_\sharp u$ in $L_2(Y)^d$ as $k \to \infty$. Since

$$L_2(Y) \to L_2(\Omega), \quad f \mapsto f_{\mathrm{pe}}$$

is bounded due to the boundedness of Ω, we also derive $\phi_{k,\mathrm{pe}} \to u_{\mathrm{pe}}$ in $L_2(\Omega)$ and $(\operatorname{grad}_\sharp \phi_k)_{\mathrm{pe}} \to (\operatorname{grad}_\sharp u)_{\mathrm{pe}}$ in $L_2(\Omega)^d$ as $k \to \infty$. By what we have shown above, we infer

$$\operatorname{grad}\phi_{k,\mathrm{pe}} = (\operatorname{grad}_\sharp \phi_k)_{\mathrm{pe}} \to (\operatorname{grad}_\sharp u)_{\mathrm{pe}} \quad (k \to \infty)$$

in $L_2(\Omega)^d$, and thus, $u_{\mathrm{pe}} \in H^1(\Omega)$ with $\operatorname{grad} u_{\mathrm{pe}} = (\operatorname{grad}_\sharp u)_{\mathrm{pe}}$ by the closedness of grad. The proof for q follows by the same argument with Lemma 14.3.3 as an additional resource. □

The extension result just established yields the following compactness statement.

Theorem 14.3.5 (Rellich–Kondrachov II)**.** *The embedding $H^1_\sharp(Y) \hookrightarrow L_2(Y)$ is compact.*

Proof. Let $(\phi_n)_n$ be a bounded sequence in $H^1_\sharp(Y)$. Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded such that $\overline{Y} \subseteq \Omega$. By Proposition 14.3.4, we deduce that $(\phi_{n,\mathrm{pe}}|_\Omega)_n$ is bounded in $H^1(\Omega)$. Let $\psi \in C^\infty_{\mathrm{c}}(\Omega)$ with $\psi = 1$ on $\overline{Y}$. Then $(\psi\phi_{n,\mathrm{pe}})_n$ is bounded in $H^1_0(\Omega)$. By Theorem 14.2.5, we find an $L_2(\Omega)$-convergent subsequence. This sequence also converges in $L_2(Y)$. Since $\psi = 1$ on Y, we obtain the assertion. □

Next, we provide a Poincaré-type inequality for the periodic gradient.

Proposition 14.3.6. *There exists $c > 0$ such that for all $u \in H^1_\sharp(Y)$*

$$\Big\| u - \int_Y u \Big\|_{L_2(Y)} \leqslant \big\|\operatorname{grad}_\sharp u\big\|_{L_2(Y)^d}.$$

In particular, $\operatorname{ran}(\operatorname{grad}_\sharp) \subseteq L_2(Y)^d$ *is closed,* $\ker(\operatorname{grad}_\sharp) = \operatorname{lin}\{\mathbb{1}_Y\}$ *and the operator*

$$\operatorname{grad}_\sharp : H^1_\sharp(Y) \cap \{\mathbb{1}_Y\}^\perp \to \operatorname{ran}(\operatorname{grad}_\sharp)$$

is an isomorphism.

Proof. The proof is left as Exercise 14.4. □

We are now in a position to formulate the particular example we have in mind. Problems of this type with highly oscillatory coefficients are also referred to as so-called *homogenisation problems.* We refer to the comments section for more details on this.

Example 14.3.7 (Homogenisation problem for the wave equation)**.** Let $c > 0$, $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be bounded, measurable, $a(x) = a(x)^* \geqslant c$ for all $x \in \mathbb{R}^d$. Furthermore, assume that a is $[0,1)^d$-periodic. Let $\nu > 0$, $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ and for $n \in \mathbb{N}$ consider the problem of finding $u_n \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ such that

$$\partial_{t,\nu}^2 u_n - \operatorname{div}_\sharp a(n\mathrm{m}) \operatorname{grad}_\sharp u_n = f. \tag{14.3}$$

We have already established that there exists a uniquely determined solution, u_n. Employing the same trick as in Section 11.3, we rewrite (14.3) using $v_n := \partial_{t,\nu} u_n$, the canonical embedding $\iota_\sharp\colon \operatorname{ran}(\operatorname{grad}_\sharp) \hookrightarrow L_2(Y)^d$ as well as $q_n := -\iota_\sharp^* a(n\mathrm{m})\iota_\sharp \iota_\sharp^* \operatorname{grad}_\sharp u_n$ to obtain

$$\left(\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}\right)\begin{pmatrix} v_n \\ q_n\end{pmatrix} = \begin{pmatrix} f \\ 0\end{pmatrix}.$$

Note that we have used that $\left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)\colon \operatorname{ran}(\operatorname{grad}_\sharp) \to \operatorname{ran}(\operatorname{grad}_\sharp)$ is continuously invertible and strictly positive definite (uniformly in n); see Proposition 11.3.5. Also note that $\iota_\sharp^* a(n\mathrm{m})\iota_\sharp$ is self-adjoint. As in Exercise 11.3 we see that $\left(\iota_\sharp^* \operatorname{grad}_\sharp\right)^* = -\operatorname{div}_\sharp \iota_\sharp$. Thus, the operator

$$S^{(n)} := \left(\overline{\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}}\right)^{-1}$$

is well-defined and bounded in $L_{2,\nu}\big(\mathbb{R}; L_2(Y) \times \operatorname{ran}(\operatorname{grad}_\sharp)\big)$. We aim to find the limit of $(S^{(n)})_n$ as $n \to \infty$. For this, we want to apply Theorem 14.1.1. We readily see using Theorem 14.3.5 and Exercise 14.5 that

$$A := \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}$$

satisfies the assumptions in Theorem 14.1.1. Thus, it is left to analyse $\left(\left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\right)_n$. This is the subject of the next section. For this reason, we define

$$\mathfrak{a}_n := \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1} \quad (n \in \mathbb{N}).$$

14.4 The Limit of $(\mathfrak{a}_n)_n$

In this section, we shall apply our earlier findings to higher-dimensional problems. Again, we fix $Y := [0,1)^d$ as well as $\iota_\sharp\colon \operatorname{ran}(\operatorname{grad}_\sharp) \hookrightarrow L_2(Y)^d$, the canonical embedding. Before we are able to present the central result of this section, we need a preliminary result.

Lemma 14.4.1. *Let $c > 0$, $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be measurable, bounded and $[0,1)^d$-periodic such that $\operatorname{Re} a(x) \geqslant c$ for each $x \in \mathbb{R}^d$. Let $\xi \in \mathbb{K}^d$. Then there exists a unique $v \in L_2(Y)^d$ with $v - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and $a(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$.*

Proof. Take $w \in H^1_\sharp(Y)$ such that

$$\operatorname{grad}_\sharp w = -\iota_\sharp \left(\iota_\sharp^* a(\mathrm{m})\iota_\sharp\right)^{-1} \iota_\sharp^* a(\mathrm{m})\xi = -\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi.$$

This is possible, since the right-hand side belongs to $\operatorname{ran}(\operatorname{grad}_\sharp)$ by definition. We put $v := \operatorname{grad}_\sharp w + \xi$. Then $v - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and we have

$$\begin{aligned}\iota_\sharp^* a(\mathrm{m})v &= \iota_\sharp^* a(\mathrm{m}) \left(\operatorname{grad}_\sharp w + \xi\right) = \iota_\sharp^* a(\mathrm{m}) \left(-\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi + \xi\right)\\ &= -\iota_\sharp^* a(\mathrm{m})\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi + \iota_\sharp^* a(\mathrm{m})\xi = 0.\end{aligned}$$

The latter gives $a(\mathrm{m})v \in \operatorname{ran}(\operatorname{grad}_\sharp)^\perp = \ker(\operatorname{div}_\sharp)$. For the uniqueness, we assume $v \in \operatorname{ran}(\operatorname{grad}_\sharp)$ with $a(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$. Then

$$(\iota_\sharp^* a(\mathrm{m})\iota_\sharp)\iota_\sharp^* v = \iota_\sharp^* a(\mathrm{m})v = 0,$$

which implies $\iota_\sharp^* v = 0$ since $\iota_\sharp^* a(\mathrm{m})\iota_\sharp$ is invertible. Thus $v = 0$. □

The previous result induces the linear mapping

$$a_{\mathrm{hom}}\colon \mathbb{K}^d \ni \xi \mapsto \int_Y av \in \mathbb{K}^d,$$

where $v \in L_2(Y)^d$ is the unique vector field from Lemma 14.4.1.

Remark 14.4.2. We gather some elementary facts on a_{hom}.

(a) If $a(x) = a(x)^*$ for each $x \in \mathbb{R}^d$ then $a_{\mathrm{hom}}^* = a_{\mathrm{hom}}$. Indeed, let $\xi, \zeta \in \mathbb{K}^d$ and $v_\xi, v_\zeta \in L_2(Y)^d$ be the corresponding functions according to Lemma 14.4.1. Then there exist $w_\xi, w_\zeta \in \operatorname{dom}(\operatorname{grad}_\sharp)$ with $v_\xi - \xi = \operatorname{grad}_\sharp w_\xi$ and $v_\zeta - \zeta = \operatorname{grad}_\sharp w_\zeta$. We compute

$$\begin{aligned}\langle a_{\mathrm{hom}}\xi, \zeta\rangle_{\mathbb{K}^d} &= \int_Y \left\langle \left(av_\xi\right)(y), v_\zeta(y) - \operatorname{grad}_\sharp w_\zeta(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y\\ &= \int_Y \left\langle \left(av_\xi\right)(y), v_\zeta(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y - \int_Y \left\langle \left(av_\xi\right)(y), \operatorname{grad}_\sharp w_\zeta(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y\\ &= \int_Y \left\langle v_\xi(y), \left(av_\zeta\right)(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y - \left\langle av_\xi, \operatorname{grad}_\sharp w_\zeta\right\rangle_{L_2(Y)^d}\\ &= \int_Y \left\langle v_\xi(y), \left(av_\zeta\right)(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y\\ &= \int_Y \left\langle \operatorname{grad}_\sharp w_\xi(y) + \xi, \left(av_\zeta\right)(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y\\ &= \int_Y \left\langle \xi, \left(av_\zeta\right)(y)\right\rangle_{\mathbb{K}^d}\, \mathrm{d}y = \langle \xi, a_{\mathrm{hom}}\zeta\rangle_{\mathbb{K}^d}.\end{aligned}$$

(b) a_{hom} is strictly positive definite. As above, one shows

$$\operatorname{Re}\langle \xi, a_{\mathrm{hom}}\xi\rangle_{\mathbb{K}^d} = \operatorname{Re}\int_Y \langle v_\xi(y), (av_\xi)(y)\rangle_{\mathbb{K}^d}\,\mathrm{d}y \geqslant c\|v_\xi\|^2_{L_2(Y)^d} \quad (\xi\in\mathbb{K}^d)$$

and since the right-hand side is strictly positive if $\xi \neq 0$ by Lemma 14.4.1, we derive the assertion.

The construction of a_{hom} now allows us to formulate the main result of this section.

Theorem 14.4.3. *Let $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be measurable, bounded, and $[0,1)^d$-periodic. Assume there exists $c>0$ such that $\operatorname{Re} a(x) \geqslant c$ for all $x\in\mathbb{R}^d$. Then*

$$\mathfrak{a}_n = \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1} \to \left(\iota_\sharp^* a_{\mathrm{hom}}\iota_\sharp\right)^{-1} =: \mathfrak{a}_{\mathrm{hom}} \quad (n\to\infty)$$

in the weak operator topology of $L(\operatorname{ran}(\operatorname{grad}_\sharp))$.

The proof of Theorem 14.4.3 requires some more preparation. One of the results needed is a variant of Theorem 13.2.3 for $L_2(Y)$. However, it will be beneficial to finish Example 14.3.7 first.

Example 14.4.4 (Example 14.3.7 continued)**.** The operator sequence $(S^{(n)})_n$ converges in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(Y)\times\operatorname{ran}(\operatorname{grad}_\sharp))\big)$ to the following limit

$$\left(\overline{\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & \mathfrak{a}_{\mathrm{hom}}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_\sharp\iota_\sharp\\ \iota_\sharp^*\operatorname{grad}_\sharp & 0\end{pmatrix}}\right)^{-1}.$$

Lemma 14.4.5. *Let $f\colon\mathbb{R}^d\to\mathbb{K}$ be measurable and $[0,1)^d$-periodic. Let $\Omega\subseteq\mathbb{R}^d$ be open, bounded and assume $f|_Y\in L_2(Y)$. Then*

$$f(n\cdot) \to \Big(\int_Y f\Big)\mathbb{1}_\Omega$$

weakly in $L_2(\Omega)$ as $n\to\infty$.

Proof. Due to the boundedness of Ω we find a finite set $F\subseteq\mathbb{Z}^d$ such that $\Omega\subseteq\bigcup_{k\in F}k+Y$. Thus, by periodicity, it suffices to restrict our attention to the case when $\Omega = Y$. First of all we show that $(f(n\cdot))_n$ is a bounded sequence in $L_2(Y)$. For this, we compute

$$\int_Y |f(nx)|^2\,\mathrm{d}x = \frac{1}{n^d}\int_{nY}|f(y)|^2\,\mathrm{d}y = \frac{1}{n^d}n^d\int_Y|f(y)|^2\,\mathrm{d}y = \|f\|^2_{L_2(Y)},$$

where we used periodicity again. This chain of equalities also shows with the help of a density argument, that it suffices to assume $f|_Y$ to be a simple function. Note that in this case $f\in L_\infty(\mathbb{R}^d)$.
Finally, we note that by Theorem 13.2.3

$$\langle f(n\cdot), g\rangle_{L_2(Y)} \to \left\langle\Big(\int_Y f\Big)\mathbb{1}_Y, g\right\rangle_{L_2(Y)} \quad (n\to\infty)$$

for each $g\in L_2(Y)\subseteq L_1(Y)$, which implies the desired assertion. □

Lemma 14.4.6. *Let $(q_n)_n$ and $(r_n)_n$ be weakly convergent sequences in a Hilbert space H with weak limits $q, r \in H$, respectively. Moreover, let $X \subseteq H$ be a closed subspace and $\iota : X \to H$ the canonical embedding. If*

$$q_n \in X \text{ for each } n \in \mathbb{N} \text{ and } (\iota^* r_n)_n \text{ is strongly convergent in } X,$$

then

$$\lim_{n\to\infty} \langle r_n, q_n \rangle_H = \langle r, q \rangle_H .$$

Proof. Since $\iota^* : H \to X$ is continuous it is also weakly continuous, and thus,

$$\iota^* r_n \to \iota^* r \quad (n \to \infty)$$

strongly in X. For $n \in \mathbb{N}$ we compute

$$\langle r_n, q_n \rangle_H = \langle r_n, \iota\iota^* q_n \rangle_H = \langle \iota^* r_n, \iota^* q_n \rangle_X \to \langle \iota^* r, \iota^* q \rangle_X .$$

Since X is a closed subspace, it is also weakly closed and thus $q \in X$ which yields

$$\langle \iota^* r, \iota^* q \rangle_X = \langle r, q \rangle_H . \qquad \square$$

The next theorem is a version of the so-called 'div-curl lemma'.

Theorem 14.4.7. *Let $(q_n)_n$ and $(r_n)_n$ be weakly convergent sequences in $L_2(Y)^d$ to some $q, r \in L_2(Y)^d$, respectively. If*

$$q_n \in \operatorname{ran}(\operatorname{grad}_\sharp) \text{ for each } n \in \mathbb{N} \text{ and } \left(\iota_\sharp^* r_n\right)_n \text{ is strongly convergent in } \operatorname{ran}(\operatorname{grad}_\sharp)$$

then

$$\int_Y \langle r_n(x), q_n(x) \rangle_{\mathbb{K}^d} \phi(x)\,\mathrm{d}x \to \int_Y \langle r(x), q(x) \rangle_{\mathbb{K}^d} \phi(x)\,\mathrm{d}x$$

for all $\phi \in C_c^\infty(Y)$ as $n \to \infty$.

Proof. Let $\phi \in C_c^\infty(Y)$, $n \in \mathbb{N}$. Since $q_n \in \operatorname{ran}(\operatorname{grad}_\sharp)$, we find a unique $w_n \in H_\sharp^1(Y)$ with $w_n \in \{\mathbb{1}_Y\}^\perp = \ker(\operatorname{grad}_\sharp)^\perp$ such that

$$\operatorname{grad}_\sharp w_n = q_n.$$

Moreover, since $\operatorname{grad}_\sharp \colon H_\sharp^1(Y) \cap \{\mathbb{1}_Y\}^\perp \to \operatorname{ran}(\operatorname{grad}_\sharp)$ is an isomorphism by Proposition 14.3.6, we infer that $(w_n)_n$ is a weakly convergent sequence in $H_\sharp^1(Y)$ and denote its weak limit by $w \in H_\sharp^1(Y)$. By Theorem 14.3.5, we deduce $w_n \to w$ strongly in $L_2(Y)^d$. Moreover, note that $(\phi w_n)_n$ weakly converges to ϕw in $H_\sharp^1(Y)$. In particular, $\operatorname{grad}_\sharp(\phi w_n) \to \operatorname{grad}_\sharp(\phi w)$ weakly in $L_2(Y)^d$. For $n \in \mathbb{N}$, we compute

$$\begin{aligned}\int_Y \langle r_n(x), q_n(x) \rangle_{\mathbb{K}^d} \phi(x)\,\mathrm{d}x &= \langle r_n, q_n\phi \rangle_{L(Y)^d} = \left\langle r_n, (\operatorname{grad}_\sharp w_n)\phi \right\rangle_{L(Y)^d} \\ &= \left\langle r_n, \operatorname{grad}_\sharp(\phi w_n) \right\rangle_{L(Y)^d} - \left\langle r_n, w_n \operatorname{grad}_\sharp \phi \right\rangle_{L_2(Y)^d} .\end{aligned}$$

Now, the first term on the right-hand side of this equality tends to $\langle r, \operatorname{grad}_\sharp(\phi w)\rangle_{L_2(Y)^d}$ by Lemma 14.4.6 applied to $X = \operatorname{ran}(\operatorname{grad}_\sharp)$, which is closed by Proposition 14.3.6. The second term tends to $\langle r, w \operatorname{grad}_\sharp \phi\rangle_{L_2(Y)^d}$ by strong convergence of $(w_n)_n$ and weak convergence of $(r_n)_n$ in $L_2(Y)^d$. Thus, we obtain

$$\begin{aligned}\int_Y \langle r_n(x), q_n(x)\rangle_{\mathbb{K}^d} \phi(x)\, \mathrm{d}x &\to \langle r, \operatorname{grad}_\sharp(\phi w)\rangle_{L_2(Y)^d} - \langle r, w \operatorname{grad}_\sharp \phi\rangle_{L_2(Y)^d} \\ &= \int_Y \langle r(x), q(x)\rangle_{\mathbb{K}^d} \phi(x)\, \mathrm{d}x \quad (n \to \infty). \end{aligned}$$ □

We will apply the latter theorem to the concrete case when $r_n = a(n\mathrm{m})q_n$ for a given measurable, bounded and $[0,1)^d$-periodic function $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ with $a(x) = a(x)^*$ for almost every $x \in \mathbb{R}^d$, in order to determine the weak limit of $(a(n\mathrm{m})q_n)_n$.

Lemma 14.4.8. *Let $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be measurable, bounded and $[0,1)^d$-periodic with $a(x) = a(x)^*$ for almost every $x \in \mathbb{R}^d$. Let $(q_n)_n$ and $(a(n\mathrm{m})q_n)_n$ be weakly convergent in $L_2(Y)^d$ to some q and r, respectively. Moreover, assume*

$q_n \in \operatorname{ran}(\operatorname{grad}_\sharp)$ for each $n \in \mathbb{N}$ and $\left(\iota_\sharp^* a(n\mathrm{m})q_n\right)_n$ is strongly convergent in $\operatorname{ran}(\operatorname{grad}_\sharp)$.

Then $r = a_{\mathrm{hom}} q$.

Proof. Let $\xi \in \mathbb{K}^d$ and choose $v \in L_2(Y)^d$ according to Lemma 14.4.1; that is, $v - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and $a(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$. For $n \in \mathbb{N}$, we define $v_n := v_{\mathrm{pe}}(n\cdot) \in L_2(Y)^d$. Next, let $g \in C_\sharp^\infty(Y)$. Then we compute

$$\begin{aligned}\langle a(n\mathrm{m})v_n, \operatorname{grad}_\sharp g\rangle_{L_2(Y)^d} &= \int_Y \langle a(nx)v_{\mathrm{pe}}(nx), \operatorname{grad}_\sharp g(x)\rangle_{\mathbb{K}^d}\, \mathrm{d}x \\ &= \frac{1}{n^d}\int_{nY} \langle a(y)v_{\mathrm{pe}}(y), \left(\operatorname{grad}_\sharp g\right)(y/n)\rangle_{\mathbb{K}^d}\, \mathrm{d}y \\ &= \frac{1}{n^{d-1}}\int_{nY} \langle a(y)v_{\mathrm{pe}}(y), (\operatorname{grad} g(\cdot/n))(y)\rangle_{\mathbb{K}^d}\, \mathrm{d}y.\end{aligned}$$

In order to compute the last integral, we employ Lemma 14.3.3 to find a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C_\sharp^\infty(Y)^d$ such that $\phi_k \to a(\mathrm{m})v$ and $\operatorname{div}_\sharp \phi_k \to 0$ as $k \to \infty$ in $L_2(Y)^d$ and $L_2(Y)$, respectively. The latter implies $(\phi_k)_{\mathrm{pe}} \to a(\mathrm{m})v_{\mathrm{pe}}$ and $\operatorname{div}(\phi_k)_{\mathrm{pe}} = (\operatorname{div}_\sharp \phi_k)_{\mathrm{pe}} \to 0$ as $k \to \infty$ in $L_2(nY)^d$ and $L_2(nY)$ for each $n \in \mathbb{N}$, respectively, where we have used Proposition 14.3.4. Thus, we obtain with integration by parts (note that the boundary terms vanish due to the periodicity of ϕ_k and g)

$$\begin{aligned}\langle a(n\mathrm{m})v_n, \operatorname{grad}_\sharp g\rangle_{L_2(Y)^d} &= \frac{1}{n^{d-1}} \langle a(\mathrm{m})v_{\mathrm{pe}}, (\operatorname{grad} g(\cdot/n))\rangle_{L_2(nY)^d} \\ &= \frac{1}{n^{d-1}} \lim_{k\to\infty} \langle (\phi_k)_{\mathrm{pe}}, (\operatorname{grad} g(\cdot/n))\rangle_{L_2(nY)^d} \\ &= -\frac{1}{n^{d-1}} \lim_{k\to\infty} \langle \operatorname{div}(\phi_k)_{\mathrm{pe}}, g(\cdot/n)\rangle_{L_2(nY)} = 0.\end{aligned}$$

Since $C_\sharp^\infty(Y)$ is a core for $\mathrm{grad}_\sharp$, we infer that $a(n\mathrm{m})v_n \in \mathrm{ran}(\mathrm{grad}_\sharp)^\perp$ and hence,

$$\iota_\sharp^* a(n\mathrm{m})v_n = 0 \quad (n \in \mathbb{N}).$$

Moreover, we have $a(n\mathrm{m})v_n \to \int_Y av = a_{\mathrm{hom}}\xi$ weakly in $L_2(Y)$ as $n \to \infty$ by Lemma 14.4.5. Thus, by Theorem 14.4.7 applied to q_n and $r_n \coloneqq a(n\mathrm{m})v_n$, we deduce that for all $\phi \in C_{\mathrm{c}}^\infty(Y)$

$$\lim_{n\to\infty} \int_Y \langle a(nx)v_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x = \int_Y \langle a_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x.$$

On the other hand, $v_n \to \left(\int_Y v\right)\mathbb{1}_Y = \xi\mathbb{1}_Y$ weakly in $L_2(Y)$ as $n \to \infty$ by Lemma 14.4.5, where $\int_Y v = \xi$ follows from $v - \xi \in \mathrm{ran}(\mathrm{grad}_\sharp)$. Thus, we can apply Theorem 14.4.7 to $q_n \coloneqq v_n$ and $r_n \coloneqq a(n\mathrm{m})q_n$ and obtain for all $\phi \in C_{\mathrm{c}}^\infty(Y)$

$$\begin{aligned}\lim_{n\to\infty} \int_Y \langle a(nx)v_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_Y \langle v_n(x), a(nx)q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x \\ &= \int_Y \langle \xi, r(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\int_Y \langle a_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x = \int_Y \langle \xi, r(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x$$

for each $\phi \in C_{\mathrm{c}}^\infty(Y)$. Hence, we infer

$$\langle \xi, r(x)\rangle_{\mathbb{K}^d} = \langle a_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d} = \langle \xi, a_{\mathrm{hom}}q(x)\rangle_{\mathbb{K}^d}$$

for almost every $x \in Y$. Since the latter holds for each $\xi \in \mathbb{K}^d$, we deduce $r = a_{\mathrm{hom}}q$. □

Proof of Theorem 14.4.3. Let $n \in \mathbb{N}$ and for $u \in \mathrm{ran}(\mathrm{grad}_\sharp)$ we put $q_n \coloneqq \mathfrak{a}_n u$. We need to show that $(q_n)_n$ weakly converges to $\mathfrak{a}_{\mathrm{hom}}u$. For this, we choose subsequences (without relabeling) such that both $(q_n)_n$ and $(a(n\mathrm{m})q_n)_n$ weakly converge to some q and r, respectively. By definition, we have $q_n \in \mathrm{ran}(\mathrm{grad}_\sharp)$ and $\iota_\sharp^* a(n\mathrm{m})q_n = u$ for each $n \in \mathbb{N}$. Hence, by Lemma 14.4.8, we deduce $a_{\mathrm{hom}}q = r$. As $\mathrm{ran}(\mathrm{grad}_\sharp)$ is closed, it is also weakly closed, and hence, $q \in \mathrm{ran}(\mathrm{grad}_\sharp)$. Thus, we have

$$\iota_\sharp^* a_{\mathrm{hom}} \iota_\sharp q = \iota_\sharp^* r,$$

or equivalently

$$q = \mathfrak{a}_{\mathrm{hom}} \iota_\sharp^* r.$$

Now, since $u = \iota_\sharp^* a(n\mathrm{m})q_n \to \iota_\sharp^* r$ weakly, we infer

$$q = \mathfrak{a}_{\mathrm{hom}} u.$$

A subsequence argument now yields the claim. □

14.5 Comments

The theory of finding partial differential equations as appropriate limit problems of partial differential equations with highly oscillatory coefficients is commonly referred to as 'homogenisation'. The mathematical theory of homogenisation goes back to the late 1960s and early 70s. We refer to [2] as an early monograph wrapping up the available theory to that date.
The usual way of addressing homogenisation problems is to look at static (i.e. time-independent) problems first. The corresponding elliptic equation is then intensively studied. Even though it might be hidden in the derivations above, the 'study of the elliptic problem' essentially boils down to addressing the limit behaviour of $\mathfrak{a}_n$ as $n \to \infty$; see [6, 15]. Consequently, generalisations of the periodic case have been introduced. The periodic case (and beyond) is covered in [2, 4]; non-periodic cases and corresponding notions have been introduced in [12, 11] and, independently, in [9, 8].
An important technical tool to obtain results in this direction is the div-curl lemma or the notion of 'compensated compactness'. In the above presented material, this is Theorem 14.4.7; the main difficulty to overcome is that of finding a limit of a product $(\langle q_n, r_n\rangle)_n$ of weakly convergent sequences $(q_n)_n, (r_n)_n$ in $L_2(\Omega)^3$ for some open $\Omega \subseteq \mathbb{R}^3$. It turns out that if $(\operatorname{curl} q_n)_n$ and $(\operatorname{div} r_n)_n$ converge strongly in an appropriate sense, then $\int_\Omega \langle q_n, r_n\rangle \phi$ converges to the desired limit for all $\phi \in C_{\mathrm{c}}^\infty(\Omega)$. In Theorem 14.4.7 the curl-condition is strengthened in as much as we ask q_n to be a gradient, which results in $\operatorname{curl} q_n = 0$. The div-condition is replaced by the condition involving $\iota_\sharp^*$, which can in fact be shown to be equivalent. The restriction to periodic boundary value problems is a mere convenience. It can be shown that the arguments work similarly for non-periodic boundary conditions, and even with the same limit.
There are many generalisations of the div-curl lemma. For this, we refer to [3] (and the references given there) and to the rather recently found operator-theoretic perspective, with plenty of applications not solely restricted to the operators div and curl, see [14, 10].
The way of deriving the homogenised equation (i.e. the limit of $\mathfrak{a}_n$) is akin to some derivations in [13, 4]. Further reading on homogenisation problems can also be found in these references. The first step of combining homogenisation processes and evolutionary equations has been made in [16] and has had some profound developments for both quantitative and qualitative results; see [18, 7, 5, 17].

Exercises

Exercise 14.1. Under the same assumptions of Theorem 14.1.1 show

$$\left\| \left(\left(\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A}\right)^{-1} - \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} \right) \partial_{t,\nu}^{-1} \right\|_{L(L_{2,\nu}(\mathbb{R};H))} \to 0.$$

Exercise 14.2. Let $\Omega \subseteq \mathbb{R}^d$ be open and $\varepsilon > 0$. We define the set

$$\Omega_\varepsilon := \{x \in \Omega\,;\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

(a) Let $(\phi_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ be a δ-sequence (cf. Exercise 3.1) and $u \in H^1(\Omega)$. We identify each function on Ω by its extension to $\mathbb{R}^d$ by 0. Prove that for $k \in \mathbb{N}$ large enough, $\phi_k * u \in H^1(\Omega_\varepsilon)$ with

$$\operatorname{grad}(\phi_k * u) = \phi_k * \operatorname{grad} u \text{ on } \Omega_\varepsilon.$$

(b) Use (a) to prove Lemma 14.2.2.

Exercise 14.3. Prove the 'subsequence argument': Let X be a topological space and $(x_n)_n$ a sequence in X. Assume that there exists $x \in X$ such that each subsequence of $(x_n)_n$ has a subsequence converging to x. Show that $x_n \to x$ as $n \to \infty$.

Exercise 14.4. Let H_0, H_1 be Hilbert spaces and $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be a closed linear operator such that $\operatorname{dom}(C) \hookrightarrow H_0$ compactly. Let $P_{\ker(C)^\perp} : H_0 \to H_0$ denote the orthogonal projection onto the closed subspace $\ker(C)^\perp$. Prove that there exists $c > 0$ such that

$$\forall u \in \operatorname{dom}(C) : \left\| P_{\ker(C)^\perp} u \right\|_{H_0} \leqslant c \|Cu\|_{H_1}.$$

Apply this result to prove Proposition 14.3.6.

Exercise 14.5. Let H_0, H_1 be Hilbert spaces. Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be closed and densely defined. Assume that $\operatorname{dom}(C) \cap \ker(C)^\perp \hookrightarrow H_0$ compactly. Show that, then, $\operatorname{dom}(C^*) \cap \ker(C^*)^\perp \hookrightarrow H_1$ compactly.

Exercise 14.6. Let $\nu > 0$, $\Omega = [0,1)^d$, $s \in L_\infty(\mathbb{R})$ be 1-periodic, $0 \leqslant s \leqslant 1$, and a as in Example 14.3.7. Show that $(u_n)_n$ in $L_{2,\nu}(\mathbb{R}; L_2(Y))$ satisfying

$$\partial_{t,\nu}^2 s(n\mathrm{m})u_n + \partial_{t,\nu}(1 - s(n\mathrm{m}))u_n - \operatorname{div}_\sharp a(n\mathrm{m}) \operatorname{grad}_\sharp u_n = f$$

for some $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ is convergent to some $u \in L_{2,\nu}(\mathbb{R}; L_2(Y))$. Which limit equation is satisfied by u?

Exercise 14.7. Let $(\alpha_n)_n$ be a null-sequence in $[0,1]$ and let a be as in Example 14.3.7. Show

$$\begin{aligned}&\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \partial_{t,\nu}^{\alpha_n} \mathfrak{a}_n \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1} \\ &\to \left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1}\end{aligned}$$

in the strong operator topology. Show that if $f \in L_{2,-\mu}(\mathbb{R}; L_2(Y)_\perp)$, where $L_2(Y)_\perp := \left\{\phi \in L_2(Y)\,;\, \int_Y \phi = 0\right\}$ for some small enough $\mu > 0$, we have

$$\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_\sharp \iota_\sharp \\ \iota_\sharp^* \operatorname{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \in L_{2,-\mu}\big(\mathbb{R}; L_2(\Omega) \times \operatorname{ran}(\operatorname{grad}_\sharp)\big).$$

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