# Exercise of lecture 11

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**Exercise 11.1.** Let $H$ be a Hilbert space, $\nu, \rho \in \mathbb{R}$ and $u \in \mathrm{L}_{1,\mathrm{loc}}(\mathbb{R}; H)$. Prove the following statements:

(a) If $u \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}(\partial_{t,\rho})$ then $\partial_{t,\nu} u = \partial_{t,\rho} u$.

(b) If $u \in \mathrm{dom}(\partial_{t,\nu})$ such that $u, \partial_{t,\nu} u \in \mathrm{L}_{2,\rho}(\mathbb{R}; H)$ then $u \in \mathrm{dom}(\partial_{t,\rho})$.

*Solution.* (a) Let $u \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}(\partial_{t,\rho})$. Using Proposition 4.1.1. we first obtain

$$-\int_{\mathbb{R}} \phi' u = \int_{\mathbb{R}} \phi\, \partial_{t,\nu} u \text{ and } -\int_{\mathbb{R}} \phi' u = \int_{\mathbb{R}} \phi\, \partial_{t,\rho} u \text{ for all } \phi \in C_c^\infty(\mathbb{R}).$$

This implies

$$\int_{\mathbb{R}} \phi\, (\partial_{t,\nu} - \partial_{t,\rho}) u = 0 \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}). \tag{1}$$

Now consider a compact interval $I := [a, b]$, where $a, b \in \mathbb{R}$ such that $a < b$. By (1) we obtain:

$$\int_I \phi\, (\partial_{t,\nu} - \partial_{t,\rho}) u = 0 \quad \text{for all} \quad \phi \in C_c^\infty(I). \tag{2}$$

Since the exponential function is continuous, we obviously have $\mathrm{L}_{2,\nu}(I, H) = \mathrm{L}_{2,\rho}(I, H) = \mathrm{L}_2(I, H)$ with equivalent norms. Hence $(\partial_{t,\nu} - \partial_{t,\rho}) u \in \mathrm{L}_2(I, H)$ holds and by using (2) we obtain:

$$\begin{aligned} 0 &= \Big\langle x, \int_I \phi\, (\partial_{t,\nu} - \partial_{t,\rho}) u \Big\rangle_H = \int_I \langle \phi(s)\, x, (\partial_{t,\nu} - \partial_{t,\rho}) u(s) \rangle_H \, \mathrm{d}s \\ &= \langle \phi(\cdot)\, x, (\partial_{t,\nu} - \partial_{t,\rho}) u \rangle_{\mathrm{L}_2(I,H)} \quad \text{for all} \quad x \in H, \phi \in C_c^\infty(I). \end{aligned}$$

Since $C_c^\infty(I)$ is dense in $\mathrm{L}_2(I)$, the set $\{\phi(\cdot)x \mid x \in H, \phi \in C_c^\infty(I)\}$ is total in $\mathrm{L}_2(I, H)$ by Lemma 3.1.7. and therefore we obtain by continuity of the scalar product

$$\langle f, (\partial_{t,\nu} - \partial_{t,\rho}) u \rangle_{\mathrm{L}_2(I,H)} = 0 \quad \text{for all} \quad f \in \mathrm{L}_2(I, H).$$

Thus $(\partial_{t,\nu} - \partial_{t,\rho}) u = 0$ in $\mathrm{L}_2(I, H) = \mathrm{L}_{2,\nu}(I, H) = \mathrm{L}_{2,\rho}(I, H)$ and since $I$ was arbitrary $\partial_{t,\nu} u = \partial_{t,\rho} u$ holds everywhere.
(b) Since $u \in \mathrm{dom}(\partial_{t,\nu})$ we obtain by Proposition 4.1.1. that

$$-\int_{\mathbb{R}} \phi' u = \int_{\mathbb{R}} \phi\, \partial_{t,\nu} u \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}).$$

By assumption, we also know that $u, \partial_{t,\nu} u \in \mathrm{L}_{2,\rho}(\mathbb{R}; H)$ and therefore we can make use of the other direction of Proposition 4.1.1. and directly obtain that $u \in \mathrm{dom}(\partial_{t,\rho})$. ❑

**Exercise 11.2.** Prove Lemma 11.3.3.

**Lemma 11.3.3.** *Let $H$ be a Hilbert space and $V \subseteq H$ a closed subspace. We denote by*

$$\iota_V : \left\{ \begin{array}{ccc} V & \to & H, \\ x & \mapsto & x, \end{array} \right.$$

*the canonical embedding of $V$ into $H$. Then $\iota_V \iota_V^* : H \to H$ is the orthogonal projection on $V$ and $\iota_V^* \iota_V : V \to V$ is the identity on $V$.*

*Solution.* We denote by $P_V : H \to H$ the orthogonal projection on $V$, which exists since $V$ is closed. For the sake of rigorousity we define the mapping $\widehat{P}_V : H \to V, x \mapsto P_V y$. Let $x \in V$ and $y \in H$.

$$\langle \iota_V x, y \rangle_H = \langle x, y \rangle_H = \langle x, P_V y \rangle_H = \langle x, \widehat{P}_V y \rangle_V .$$

Hence, $\iota_V^* = \widehat{P}_V$ and consequently

$$\begin{aligned} \iota_V \iota_V^* &= \iota_V \widehat{P}_V = P_V \quad \text{and} \\ \iota_V^* \iota_V &= \widehat{P}_V \iota_V = \mathrm{id}\,. \end{aligned}$$ ❑

**Exercise 11.3.** Let $H_0$,$H_1$ be Hilbert spaces and $A : \operatorname{dom} A \subseteq H_0 \to H_1$ a densely defined closed linear operator. Moreover, we assume that $A$ has closed range. Show that the adjoint of the operator $\iota_{\operatorname{ran} A}^* A : \operatorname{dom} A \subseteq H_0 \to \operatorname{ran} A$ is given by $A^* \iota_{\operatorname{ran} A}$. If additionally $A$ is one-to-one, show that $\iota_{\operatorname{ran} A}^* A$ is boundedly invertible.

Note that for a densely defined linear operator $B : \operatorname{dom} B \subseteq X \to Y$ we have $(u, v) \in B^*$ is equivalent to

$$\langle u, Bx \rangle_Y = \langle v, x \rangle_X \quad \text{for all} \quad x \in \operatorname{dom} B.$$

*Solution.* Let $(u, v) \in \left(\iota_{\operatorname{ran} A}^* A\right)^*$ and $x \in \operatorname{dom} A$ be arbitrary. Then

$$\left\langle u, \iota_{\operatorname{ran} A}^* A x \right\rangle_{\operatorname{ran} A} = \langle v, x \rangle_{H_0} .$$

Since $\iota_{\operatorname{ran} A}^* A x = Ax$, this yields $(\iota_{\operatorname{ran} A} u, v) \in A^*$. Therefore, $\iota_{\operatorname{ran} A} u \in \operatorname{dom} A^*$ and

$$\left\langle u, \iota_{\operatorname{ran} A}^* A x \right\rangle_{\operatorname{ran} A} = \langle \iota_{\operatorname{ran} A} u, A^* x \rangle_{H_1} = \langle A^* \iota_{\operatorname{ran} A} u, x \rangle_{H_0},$$

which yields $\left(\iota_{\operatorname{ran} A}^* A\right)^* = A^* \iota_{\operatorname{ran} A}$.

If $A$ is additionally injective, then $\iota_{\operatorname{ran} A}^* A$ is a bijective closed linear mapping. Consequently, $(\iota_{\operatorname{ran} A}^* A)^{-1}$ is bounded by closed graph theorem. ❑

**Exercise 11.4.** Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a strip. We consider the heat conduction with memory term given by the equations

$$\begin{aligned} \partial_{t,\nu} \theta + \operatorname{div} q &= f, \\ q &= (1 - k*) \operatorname{grad}_0 \theta, \end{aligned} \tag{3}$$

where $k \in \mathrm{L}_{1,-\rho_1}(\mathbb{R}_{\geq 0}; \mathbb{R})$ for some $\rho_1 > 0$ with

$$\int_0^\infty |k(t)| \, \mathrm{d}t < 1.$$

Write (3) as a suitable evolutionary equation and prove that this equation is exponentially stable.

*Solution.* The evolutionary equation is given by

$$\left(\partial_{t,\nu}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&(1-k*)^{-1}\end{pmatrix}+\begin{pmatrix}0&\operatorname{div}\\\operatorname{grad}_0&0\end{pmatrix}\right)\begin{pmatrix}\theta\\q\end{pmatrix}=\begin{pmatrix}f\\0\end{pmatrix}\tag{4}$$

for $\nu > -\rho$ for some $\rho > 0$. Note that at this point we don't know anything about the existence of $(1-k*)^{-1}$ and in order to show that and to prove the exponentially stability of (4) we will find a material law $M_1 : \operatorname{dom}(M_1) \subseteq \mathbb{C} \to L(\mathrm{L}_2(\Omega))$ satisfying $M_1(\partial_{t,\nu}) = (1-k*)^{-1}$, $s_b(M_1) < -\rho$ and $\operatorname{Re} M_1(z) \geq c_1$ for all $z \in \mathbb{C}_{\operatorname{Re}>-\rho}$, where $c_1 > 0$. Since

$$\int_0^\infty \mathrm{e}^{\rho_1 t}|k(t)|\,\mathrm{d}t < \infty$$

and

$$\int_0^\infty \mathrm{e}^{0t}|k(t)|\,\mathrm{d}t = \int_0^\infty |k(t)|\,\mathrm{d}t < 1$$

applies the map $T : [0,\rho_1] \to \mathbb{R}$, $x \mapsto \int_0^\infty \mathrm{e}^{xt}|k(t)|\,\mathrm{d}t$ is continuous on $[0,\rho_1]$. Since $T(0) < 1$ there exists a $\rho > 0$ with $T(x) < 1$ for all $x \in [0,\rho]$. Therefore the mapping

$$\widetilde{M}_1(z) \coloneqq 1 - \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-zt}k(t)\,\mathrm{d}t, \qquad \forall z \in \mathbb{C}_{\operatorname{Re}>-\rho}$$

is a material law with $\widetilde{M}_1(\partial_{t,\nu}) = 1 - k*$ for $\nu > -\rho$ (see Example 5.3.3 (d)). In addition to that for all $\phi \in \mathrm{L}_2(\Omega)$ and for all $z \in \mathbb{C}_{\operatorname{Re}>-\rho}$

$$\begin{aligned}
\operatorname{Re}\langle\phi, \widetilde{M}_1(z)\phi\rangle &= \|\phi\|^2_{\mathrm{L}_2(\Omega)} - \|\phi\|^2_{\mathrm{L}_2(\Omega)} \operatorname{Re}\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-zt}k(t)\,\mathrm{d}t\right)\\
&\geq \|\phi\|^2_{\mathrm{L}_2(\Omega)}\left(1 - \left|\frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-zt}k(t)\,\mathrm{d}t\right|\right)\\
&\geq \|\phi\|^2_{\mathrm{L}_2(\Omega)}\left(1 - \frac{1}{\sqrt{2\pi}}\int_0^\infty |\mathrm{e}^{-zt}k(t)|\,\mathrm{d}t\right)\\
&= \|\phi\|^2_{\mathrm{L}_2(\Omega)}\Bigg(1 - \frac{1}{\sqrt{2\pi}}\underbrace{\int_0^\infty \mathrm{e}^{-\operatorname{Re} zt}\,|k(t)|\,\mathrm{d}t}_{\leq 1}\Bigg)\\
&\geq \|\phi\|^2_{\mathrm{L}_2(\Omega)}\left(1 - \frac{1}{\sqrt{2\pi}}\right)
\end{aligned}$$

applies. Let $c \coloneqq 1 - \frac{1}{\sqrt{2\pi}}$. Note that $\widetilde{M}_1(z)$ is a multiplication operator for all $z \in \mathbb{C}_{\operatorname{Re}>-\rho}$. Therefore we get

$$0 < \|\widetilde{M}_1(z)\| \leq 1 + \frac{1}{\sqrt{2\pi}}\int_0^\infty |\mathrm{e}^{-zt}|\,|k(t)|\,\mathrm{d}t \leq 1 + \int_0^\infty \mathrm{e}^{\rho t}|k(t)|\,\mathrm{d}t < \infty$$

for all $z \in \mathbb{C}_{\operatorname{Re}>-\rho}$. Let $d \coloneqq \left(1 + \int_0^\infty \mathrm{e}^{\rho t}|k(t)|\,\mathrm{d}t\right)^2$. By using Proposition 6.2.2 (b) we get

$$\operatorname{Re}\left(\widetilde{M}_1(z)\right)^{-1} \geq \frac{c}{\|\widetilde{M}_1(z)\|^2} \geq \frac{c}{d}.$$

Since $\widetilde{M_1}(z)$ is a multiplication operator and $\|\widetilde{M_1}(z)\| \neq 0$ for all $z \in \mathbb{C}_{Re>\rho}$ its inverse is given by the multiplicative inverse. Hence, $(\widetilde{M_1}(\,\cdot\,))^{-1}$ is holomorphic and $M_1(\,\cdot\,) \coloneqq (\widetilde{M_1}(\,\cdot\,))^{-1}$ is a material law with the following property

$$\begin{aligned}(1-k*)^{-1} &= (\widetilde{M_1}(\partial_{t,\nu}))^{-1} = (\mathcal{L}_\nu^* \widetilde{M_1}(im+\nu)\mathcal{L}_\nu)^{-1} = \mathcal{L}_\nu^* (\widetilde{M_1}(im+\nu))^{-1}\mathcal{L}_\nu \\ &= \mathcal{L}_\nu^* M_1(im+\nu)\mathcal{L}_\nu = M_1(\partial_{t,\nu})\end{aligned}$$

for all $\nu > -\rho$. By using Lemma 11.2.1 the evolutionary equation (4) is exponentially stable for all $\nu > 0$. ❑

**Exercise 11.5.** Let $A \in \mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$ and consider the evolutionary equation

$$(\partial_{t,\nu} + A)U = F.$$

Prove that the solution operators associtated with this problem are exponentially stable if and only if $A$ has only eigenvalues with strictly positive real part.

*Solution.* We will show the two implications of the equivalence seperately:

$\Leftarrow$ We will show that problem is equivalent to

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & A^{-1}\end{pmatrix} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\begin{pmatrix}U\\ V\end{pmatrix} = \begin{pmatrix}F\\ 0\end{pmatrix}. \tag{5}$$

The second row of the previous equation is $A^{-1}V + U = 0$, which gives $V = -AU$ and the first row is $\partial_{t,\nu}U - V = F$. These two equations combined yield the original problem $\partial_{t,\nu}U + AU = F$.

Hence, if $\left(\begin{smallmatrix}U\\ V\end{smallmatrix}\right)$ is a solution for (5), then $U$ is a solution to the original problem.

On the other hand, if $U$ is a solution to the original problem then $\left(\begin{smallmatrix}U\\ -AU\end{smallmatrix}\right)$ is a solution to (5).

By Theorem 11.2.1 problem (5) is exponentially stable, which yields the exponential stability of the original problem.

$\Rightarrow$ Assume that $A$ has at least one eigenvalue $\lambda$ such that $\operatorname{Re}\lambda \leq 0$. Let $x_0 \in \mathbb{C}^n$ denote the eigenvector corresponding to $\lambda$. Then we choose

$$U_0(t) = \begin{cases}0, & \text{if } t \leq 0,\\ tx_0, & \text{if } 0 < t < 1,\\ \mathrm{e}^{-\lambda t}x_0, & \text{if } t \geq 1.\end{cases}$$

It is easy to check that $U_0 \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathbb{C}^n)$ for every $\nu > -\operatorname{Re}\lambda$. However, for $\rho \geq 0$ we have

$$\int_1^{+\infty} \|\mathrm{e}^{-\lambda t}x_0\|^2 \mathrm{e}^{2\rho t}\,\mathrm{d}t = \|x_0\| \int_1^{+\infty} \mathrm{e}^{2\overbrace{(\rho-\operatorname{Re}\lambda)}^{\geq 0}t}\,\mathrm{d}t = +\infty,$$

which implies $U_0 \notin \mathrm{L}_{2,-\rho}(\mathbb{R},\mathbb{C}^n)$. We define $F$ as $(\partial_{t,\nu}+A)U_0$, which is

$$F(t) = \begin{cases}0, & \text{if } t \leq 0,\\ x_0 + \lambda t x_0, & \text{if } 0 < t < 1,\\ \underbrace{-\lambda \mathrm{e}^{-\rho t}x_0 + \lambda \mathrm{e}^{-\rho t}x_0}_{=0}, & \text{if } t \geq 1.\end{cases}$$

Contrary, to $U_0$ the function $F$ is in $\mathrm{L}_{2,-\rho}(\mathbb{R},\mathbb{C}^n)$ and by construction $U_0$ is a solution to $(\partial_{t,\nu}+A)U=F$ for $\nu>-\operatorname{Re}\lambda$. Hence, the solution operator to this problem cannot be exponentially stable. ❑

**Exercise 11.6.** Let $\Omega\subseteq\mathbb{R}^d$ be open.

(a) Let $\phi\in C_{\mathrm{c}}^\infty(\Omega)^d$. Prove *Korn's inequality*

$$\|\operatorname{Grad}\phi\|^2_{\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}}\geq\frac{1}{2}\sum_{j=1}^d\|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}.$$

(b) Use Korn's inequality to prove that for $u\in\mathrm{L}_2(\Omega)^d$ we have

$$u\in\operatorname{dom}\operatorname{Grad}_0\quad\Leftrightarrow\quad\forall j\in\{1,\dots,d\}:u_j\in\operatorname{dom}\operatorname{grad}_0.$$

Moreover, show that in either case

$$\frac{1}{2}\sum_{j=1}^d\|\operatorname{grad}_0 u_j\|^2_{\mathrm{L}_2(\Omega)^d}\leq\|\operatorname{Grad}_0 u\|^2_{\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}}\leq\sum_{j=1}^d\|\operatorname{grad}_0 u_j\|^2_{\mathrm{L}_2(\Omega)^d}.$$

(c) Let now $\Omega$ be contained in a strip. Prove that $\operatorname{Grad}_0$ is one-to-one and has closed range.

We will use the short notation $\langle f,g\rangle$ for $\langle f,g\rangle_{\mathrm{L}_2(\Omega)}$ and $\|f\|$ for $\|f\|_{\mathrm{L}_2(\Omega)}$.

*Solution.* We will show all assertions in the listed order.

(a) Note that for $f,g\in C_{\mathrm{c}}^\infty(\Omega)$ we have

$$\langle\partial_i f,\partial_j g\rangle=\langle-\partial_j\partial_i f,g\rangle=\langle-\partial_i\partial_j f,g\rangle=\langle\partial_j f,\partial_i g\rangle.\tag{6}$$

By definition of Grad and the norm in $\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}$ we have for $\phi\in C_{\mathrm{c}}^\infty(\Omega)^d$

$$\begin{aligned}\|\operatorname{Grad}\phi\|^2_{\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}}&=\sum_{j=1}^d\sum_{i=1}^d\frac{1}{4}\|\partial_i\phi_j+\partial_j\phi_i\|^2\\&=\frac{1}{4}\sum_{j=1}^d\sum_{i=1}^d\|\partial_i\phi_j\|^2+\|\partial_j\phi_i\|^2+2\operatorname{Re}\langle\partial_i\phi_j,\partial_j\phi_i\rangle\\&=\frac{1}{2}\sum_{j=1}^d\sum_{i=1}^d\|\partial_i\phi_j\|^2+\frac{1}{2}\sum_{j=1}^d\sum_{i=1}^d\operatorname{Re}\langle\partial_i\phi_j,\partial_j\phi_i\rangle\end{aligned}$$

by the definition of $\|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}$ and (6) we further have

$$\begin{aligned}&=\frac{1}{2}\sum_{j=1}^d\|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}+\frac{1}{2}\operatorname{Re}\underbrace{\sum_{j=1}^d\sum_{i=1}^d\langle\partial_i\phi_i,\partial_j\phi_j\rangle}_{\langle\sum_{i=1}^d\partial_i\phi_i,\sum_{j=1}^d\partial_j\phi_j\rangle}\\&=\frac{1}{2}\sum_{j=1}^d\|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}+\frac{1}{2}\Big\|\sum_{j=1}^d\partial_j\phi_j\Big\|^2\\&\geq\frac{1}{2}\sum_{j=1}^d\|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}.\end{aligned}$$

(b) Since $C_c^\infty(\Omega)^d$ and $C_c^\infty(\Omega)$ are cores of $\operatorname{Grad}_0$ and $\operatorname{grad}_0$ respectively, Korn's inequality can be extended to $\operatorname{dom}\operatorname{Grad}_0$ by continuity. On the other hand we have for $\phi \in C_c^\infty(\Omega)^d$

$$\begin{aligned}\|\operatorname{Grad}\phi\|^2_{\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}} &\leq \frac{1}{4}\sum_{j=1}^d\sum_{i=1}^d \|\partial_i\phi_j\|^2 + 2\|\partial_j\phi_i\|\|\partial_i\phi_j\| + \|\partial_j\phi_i\|^2 \\ &\leq \frac{1}{4}\sum_{j=1}^d\sum_{i=1}^d 4\|\partial_i\phi_j\|^2 \leq \sum_{j=1}^d \|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}.\end{aligned} \tag{7}$$

We used $2ab \leq a^2 + b^2$ and changed the order of summation. (7) can also be extended to $\operatorname{dom}\operatorname{Grad}_0$. Hence,

$$\frac{1}{2}\sum_{j=1}^d \|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d} \leq \|\operatorname{Grad}\phi\|^2_{\mathrm{L}_2(\Omega)^{d\times d}_{\mathrm{sym}}} \leq \sum_{j=1}^d \|\operatorname{grad}\phi_j\|^2_{\mathrm{L}_2(\Omega)^d}.$$

Clearly, this implies the stated equivalence.

(c) Since $\Omega$ is contained in a strip, we have that $\operatorname{grad}_0$ is one-to-one. So the assertion follows immediately from Korn's inequality. ❑

**Exercise 11.7.** Let $\Omega \subseteq \mathbb{R}^d$ be open and $a \in L(\mathrm{L}_2(\Omega)^d)$ with $\operatorname{Re} a \geq c > 0$.

(a) Let $\nu > 0$ and $f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega))$. Moreover, assume that $\Omega$ is contained in a strip and define $\tilde{a} := \iota^*_{\operatorname{ran}\operatorname{grad}_0} a \iota_{\operatorname{ran}\operatorname{grad}_0}$. Let $\theta \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega))$, $q \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega)^d)$ satisfy

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad}_0 & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}$$

and $\tilde{\theta} \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega))$, $\tilde{q} \in \mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}\operatorname{grad}_0)$ satisfy

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \tilde{a}^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}\iota_{\operatorname{ran}\operatorname{grad}_0}\\ \iota^*_{\operatorname{ran}\operatorname{grad}_0}\operatorname{grad}_0 & 0\end{pmatrix}\right)\begin{pmatrix}\tilde{\theta}\\ \tilde{q}\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

Show that $(\theta, \iota^*_{\operatorname{ran}\operatorname{grad}_0} q) = (\tilde{\theta}, \tilde{q})$.

(b) Let $\Omega$ be bounded and consider the evolutionary equation

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

Show that the associated solution operators are not exponentially stable.

*Solution.* We will show all assertions in the listed order.

(a) In the Lecture 11 it is already shown, that the pair $(\tilde{\theta}, \tilde{q}) = (\theta, \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} q)$ is a solution for the evolutionary equation in part a). We will now show that the solution for the evolutionary equation is unique, which follows by Theorem 6.2.1.(Picard) applied to $M : \operatorname{dom} M \subseteq \mathbb{C} \to L(H)$ and the linear operator $A : \operatorname{dom} A \subseteq H \to H$, defined by

$$M(z) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + z^{-1}\begin{pmatrix}0 & 0\\ 0 & \tilde{a}^{-1}\end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & \operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)} \\ \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0 & 0 \end{pmatrix},$$

where $H := \mathrm{L}_{2,\nu}(\mathbb{R}; \mathrm{L}_2(\Omega) \times \mathrm{L}_2(\Omega)^d)$. First consider that

$$\begin{aligned} \operatorname{Re}\langle \phi, \tilde{a}\phi \rangle_{\operatorname{ran}(\operatorname{grad}_0)} &= \operatorname{Re}\langle \phi, \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} a \iota_{\operatorname{ran}(\operatorname{grad}_0)} \phi \rangle_{\operatorname{ran}(\operatorname{grad}_0)} \\ &= \operatorname{Re}\langle \iota_{\operatorname{ran}(\operatorname{grad}_0)} \phi, a \iota_{\operatorname{ran}(\operatorname{grad}_0)} \phi \rangle_{\mathrm{L}_2(\Omega)^d} \geq c \|\phi\|^2_{\operatorname{ran}(\operatorname{grad}_0)}, \end{aligned}$$

for all $\phi \in \operatorname{ran}(\operatorname{grad}_0)$. Hence with Proposition 6.2.2 b) we obtain $\tilde{a}^{-1} \in L(\mathrm{L}_2(\Omega)^d)$ and $\operatorname{Re} \tilde{a}^{-1} \geq c\|a\|^{-2}$. This implies that $M$ is a material law with $s_b(M) = 0$ by Example 5.3.1. a) and provides the required real part condition in Theorem 6.2.1.(Picard). Indeed, there exists a constant $\tilde{c} > 0$, such that

$$\begin{aligned} \operatorname{Re}\langle \phi, zM(z)\phi \rangle_H &= \operatorname{Re}(\langle \phi_1, z\phi_1 \rangle_{\mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega))} + \langle \phi_2, \tilde{a}^{-1}\phi_2 \rangle_{\mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{L}_2(\Omega)^d)}) \\ &= \operatorname{Re} z \|\phi_1\|^2 + \operatorname{Re} \tilde{a}^{-1} \|\phi_2\|^2 \geq \tilde{c}\|a\|^{-2}\|\phi\|^2 > 0, \end{aligned}$$

for all $\phi = (\phi_1, \phi_2) \in H \setminus \{0\}$ and $z \in \mathbb{C}_{\operatorname{Re} \geq \nu_0}$ for some $\nu_0 > 0$ fixed. The matrix $A$ is skew-selfadjoint, as

$$\begin{aligned} A^* &= \begin{pmatrix} 0 & (\iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0)^* \\ (\operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)})^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)} \\ -\iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0 & \end{pmatrix} \end{aligned}$$

(by Proposition 6.2.2. a)) and Exercise 11.3. or the fact that

$$\begin{aligned} \operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)} &= (\operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)})^{**} \\ &= (\overline{-\iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0})^* = (-\iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0)^*, \end{aligned}$$

using that div and $\operatorname{grad}_0$ are closed linear relations and $\iota_{\operatorname{ran}(\operatorname{grad}_0)}$ is a linear operator on $\operatorname{ran}(\operatorname{grad}_0)$.

Now Theorem 6.2.1.(Picard) provides for $\nu > \nu_0$ a solution operator $S_\nu = (\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A})^{-1} \in L(\mathrm{L}_{2,\nu}(\mathbb{R}, H))$, which satisfies

$$S_\nu((f, 0)) = (\theta, \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} q),$$

which yields $(\theta, \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} q) = (\tilde{\theta}, \tilde{q})$.

(b) First consider the mapping $f : \mathbb{R} \to \mathrm{L}_2(\Omega)$ defined by

$$f(t) = \begin{cases} 0, & \text{if } t < -1, \\ \frac{3}{4}(-t^2 + 1)u, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{if } t > 1, \end{cases}$$

where $u \in \mathrm{L}_2(\Omega)$ with $\|u\|_{\mathrm{L}_2(\Omega)} = 1$ and $u \in \ker(\operatorname{grad})$. One could use $u = \frac{1}{\sqrt{\lambda_d(\Omega)}}$, with $\lambda_d$ the Lebesgue-measure on $\mathbb{R}^d$. As $\Omega$ is bounded it is $\lambda_d(\Omega) < \infty$ and

$$\left( \int_\Omega |u|^2 d\lambda_d \right)^{\frac{1}{2}} = \left( \frac{1}{\lambda_d(\Omega)} \int_\Omega d\lambda_d \right)^{\frac{1}{2}} = 1$$

and $\operatorname{grad} u = u' = 0$. The mapping $f$ is well-defined, as for all $t \in \mathbb{R}$ it is

$$\int_\Omega |f(t)(x)|^2 d\lambda_d(x) \leq \int_\Omega |u(x)|^2 d\lambda_d(x) = 1 < \infty.$$

Let now $\mu \in \mathbb{R}$, then we have $f \in \mathrm{L}_{2,\mu}(\mathbb{R}; \mathrm{L}_2(\Omega))$ because

$$\int_{\mathbb{R}} \|f(t)\|^2_{\mathrm{L}_2(\Omega)} \mathrm{e}^{-2\mu t} dt \leq \int_{-1}^{1} \mathrm{e}^{-2\mu t} dt < \infty.$$

Furthermore, consider the functions $q = 0 \in \mathrm{L}_{2,\mu}(\mathbb{R}, \mathrm{L}_2(\Omega)^d)$ for all $\mu \in \mathbb{R}$ and $\theta : \mathbb{R} \to \mathrm{L}_2(\Omega)$ defined by

$$\theta(t) = \begin{cases} 0, & \text{if } t < -1, \\ \frac{1}{4}(-t^3 + 3t + 2)u, & \text{if } -1 \leq t \leq 1, \\ u, & \text{if } t > 1. \end{cases}$$

For all $t \in \mathbb{R}$ and since $\Omega$ is bounded it is

$$\int_\Omega |\theta(t)(x)|^2 d\lambda_d(x) \leq \int_\Omega |u(x)|^2 d\lambda_d(x) = \|u\|^2_{\mathrm{L}_2(\Omega)} < \infty.$$

Assume $\mu > 0$, then it is $\theta \in \mathrm{L}_{2,\mu}(\mathbb{R}; \mathrm{L}_2(\Omega))$ as

$$\int_{\mathbb{R}} \|\theta(t)\|^2_{\mathrm{L}_2(\Omega)} \mathrm{e}^{-2\mu t} dt = \int_{-1}^{\infty} \|\theta(t)\|^2_{\mathrm{L}_2(\Omega)} \mathrm{e}^{-2\mu t} dt \leq \int_{-1}^{\infty} \mathrm{e}^{-2\mu t} dt < \infty,$$

whereas for $\mu < 0$ we derive

$$\int_{\mathbb{R}} \|\theta(t)\|^2_{\mathrm{L}_2(\Omega)} \mathrm{e}^{-2\mu t} dt \geq \int_{1}^{\infty} \|u\|^2_{\mathrm{L}_2(\Omega)} \mathrm{e}^{-2\mu t} dt = \int_{1}^{\infty} \mathrm{e}^{-2\mu t} dt$$

and hence $\theta \notin \mathrm{L}_{2,\mu}(\mathbb{R}; \mathrm{L}_2(\Omega))$.

Let $\nu > 0$ . As the following two equations hold, the pair $(\theta, q)$ is a solution for the evolutionary equation in part b) with our $f$ at the right hand side

$$\begin{aligned} \partial_{t,\nu}\theta(t) + \operatorname{div}_0 q(t) = \theta'(t) + 0 &= \begin{cases} 0, & \text{if } t < -1, \\ \frac{3}{4}(-t^2 + 1)u, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{if } t > 1, \end{cases} \\ &= f(t) \end{aligned}$$

and

$$a^{-1} q(t) + \operatorname{grad} \theta(t) = 0.$$

And with our previous calculations we derive for all $\rho > 0$ and $\nu > 0$ that $f \in \mathrm{L}_{2,-\rho}(\mathbb{R}; \mathrm{L}_2(\Omega)) \cap \mathrm{L}_{2,\nu}(\mathbb{R}, \mathrm{L}_2(\Omega))$ but $\theta \notin \mathrm{L}_{2,-\rho}(\mathbb{R}; \mathrm{L}_2(\Omega))$, hence the evolutionary equation is not exponentially stable. ❑