10 Differential Algebraic Equations

Let H be a Hilbert space and $\nu \in \mathbb{R}$. We saw in the previous lecture how initial value problems can be formulated within the framework of evolutionary equations. More precisely, we have studied problems of the form

$$\begin{cases} (\partial_{t,\nu} M_0 + M_1 + A)\, U = 0 & \text{on } (0,\infty)\,, \\ M_0 U(0+) = M_0 U_0 \end{cases} \tag{10.1}$$

for $U_0 \in H$, $M_0, M_1 \in L(H)$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint; that is, we have considered material laws of the form

$$M(z) \coloneqq M_0 + z^{-1} M_1 \quad (z \in \mathbb{C} \setminus \{0\}).$$

Here, the initial value is attained in a weak sense as an equality in the extrapolation space $H^{-1}(A)$. The first line is also meant in a weak sense since the left-hand side turned out to be a functional in $H_\nu^{-1}(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H^{-1}(A))$. In Theorem 9.4.1 it was shown that the latter problem can be rewritten as

$$(\partial_{t,\nu} M_0 + M_1 + A)\, U = \delta_0 M_0 U_0.$$

In this lecture we aim to inspect initial value problems a little closer but in the particularly simple case when $A = 0$. However, we want to impose the initial condition for U and not just $M_0 U$. Thus, we want to deal with the problem

$$\begin{cases} (\partial_{t,\nu} M_0 + M_1)\, U = 0 & \text{on } (0,\infty)\,, \\ U(0+) = U_0 \end{cases} \tag{10.2}$$

for two bounded operators M_0, M_1 and an initial value $U_0 \in H$. This class of differential equations is known as *differential algebraic equations* since the operator M_0 is allowed to have a non-trivial kernel. Thus, (10.2) is a coupled problem of a differential equation (on $(\ker M_0)^\perp$) and an algebraic equation (on $\ker M_0$). We begin by treating these equations in the finite-dimensional case; that is, $H = \mathbb{C}^n$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$.

10.1 The finite-dimensional Case

Throughout this section let $n \in \mathbb{N}$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$.

Definition. We define the *spectrum of the matrix pair* (M_0, M_1) by

$$\sigma(M_0, M_1) \coloneqq \{z \in \mathbb{C}\,;\, \det(zM_0 + M_1) = 0\}\,,$$

and the *resolvent set of the matrix pair* (M_0, M_1) by

$$\rho(M_0, M_1) \coloneqq \mathbb{C} \setminus \sigma(M_0, M_1).$$

Remark 10.1.1. (a) It is immediate that $\sigma(M_0, M_1)$ is closed since the mapping $z \mapsto \det(zM_0 + M_1)$ is continuous.

(b) Note in particular that the spectrum (the set of eigenvalues) of a matrix A corresponds in this setting to the spectrum of the matrix pair $(1, -A)$.

In contrast to the case of the spectrum of one matrix, it may happen that $\sigma(M_0, M_1) = \mathbb{C}$ (for example we can choose $M_0 = 0$ and M_1 singular). More precisely, we have the following result.

Lemma 10.1.2. *The set $\sigma(M_0, M_1)$ is either finite or equals the whole complex plane $\mathbb{C}$. If $\sigma(M_0, M_1)$ is finite then* $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$.

Proof. The function $z \mapsto \det(zM_0 + M_1)$ is a polynomial of order less than or equal to n. If it is constantly zero, then $\sigma(M_0, M_1) = \mathbb{C}$ and otherwise $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$ by the fundamental theorem of algebra. □

Definition. The matrix pair (M_0, M_1) is called *regular* if $\sigma(M_0, M_1) \neq \mathbb{C}$.

The main problem in solving an initial value problem of the form (10.2) is that one cannot expect a solution for each initial value $U_0 \in \mathbb{C}^n$ as the following simple example shows.

Example 10.1.3. Let $M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $U_0 \in \mathbb{C}^2$. We assume that there exists a solution $U\colon \mathbb{R}_{\geqslant 0} \to \mathbb{C}^2$ satisfying (10.2); that is,

$$\begin{aligned} U_1'(t) + U_2'(t) + U_1(t) &= 0 \quad (t > 0), \\ U_2(t) &= 0 \quad (t > 0), \\ U(0+) &= U_0. \end{aligned}$$

The second and third equation yield that the second coordinate of U_0 has to be zero. Then, for $U_0 = (x, 0) \in \mathbb{C}^2$ the unique solution of the above problem is given by

$$U(t) = \bigl(U_1(t), U_2(t)\bigr) = (x\mathrm{e}^{-t}, 0) \quad (t \geqslant 0).$$

Definition. We call an initial value $U_0 \in \mathbb{C}^n$ *consistent* for (10.2), if there exists $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n)$ such that (10.2) holds. We denote the set of all consistent initial values for (10.2) by

$$\mathrm{IV}(M_0, M_1) \coloneqq \{U_0 \in \mathbb{C}^n\,;\, U_0 \text{ consistent}\}\,.$$

Remark 10.1.4. It is obvious that $\mathrm{IV}(M_0, M_1)$ is a subspace of $\mathbb{C}^n$. In particular, $0 \in \mathrm{IV}(M_0, M_1)$.

It is now our goal to determine the space $\mathrm{IV}(M_0, M_1)$. One possibility for doing so uses the so-called *quasi-Weierstraß normal form*.

Proposition 10.1.5 (quasi-Weierstraß normal form). *Assume that (M_0, M_1) is regular. Then there exist invertible matrices $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

where $C \in \mathbb{C}^{k\times k}$ and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ for some $k \in \{0, \ldots, n\}$. Moreover, the matrix N is nilpotent; that is, there exists $\ell \in \mathbb{N}$ such that $N^\ell = 0$.

Proof. Since (M_0, M_1) is regular we find $\lambda \in \mathbb{C}$ such that $\lambda M_0 + M_1$ is invertible. We set $P_1 \coloneqq (\lambda M_0 + M_1)^{-1}$ and obtain

$$\begin{aligned} M_{0,1} &\coloneqq P_1 M_0 = (\lambda M_0 + M_1)^{-1} M_0, \\ M_{1,1} &\coloneqq P_1 M_1 = (\lambda M_0 + M_1)^{-1} M_1 = 1 - \lambda M_{0,1}. \end{aligned}$$

Let now $P_2 \in \mathbb{C}^{n\times n}$ such that

$$M_{0,2} \coloneqq P_2 M_{0,1} P_2^{-1} = \begin{pmatrix} J & 0 \\ 0 & \widetilde{N} \end{pmatrix}$$

for some invertible matrix $J \in \mathbb{C}^{k\times k}$ and a nilpotent matrix $\widetilde{N} \in \mathbb{C}^{(n-k)\times(n-k)}$ (use the Jordan normal form of $M_{0,1}$ here). Then

$$M_{1,2} \coloneqq P_2 M_{1,1} P_2^{-1} = \begin{pmatrix} 1 - \lambda J & 0 \\ 0 & 1 - \lambda\widetilde{N} \end{pmatrix}.$$

Now, by the nilpotency of $\widetilde{N}$, the matrix $(1 - \lambda\widetilde{N})$ is invertible by the Neumann series. We set

$$P_3 \coloneqq \begin{pmatrix} J^{-1} & 0 \\ 0 & (1 - \lambda\widetilde{N})^{-1} \end{pmatrix}$$

and obtain

$$P_3 M_{0,2} = \begin{pmatrix} 1 & 0 \\ 0 & (1 - \lambda\widetilde{N})^{-1}\widetilde{N} \end{pmatrix}, \text{ and } \quad P_3 M_{1,2} = \begin{pmatrix} J^{-1} - \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $(1 - \lambda\widetilde{N})^{-1}\widetilde{N}$ is nilpotent, since the matrices commute and $\widetilde{N}$ is nilpotent. Thus, the assertion follows with $N \coloneqq (1 - \lambda\widetilde{N})^{-1}\widetilde{N}$, $C \coloneqq J^{-1} - \lambda$, $P = P_3P_2P_1$, and $Q = P_2^{-1}$. □

It is clear that the matrices P, Q, C and N in the previous proposition are not uniquely determined by M_0 and M_1. However, the size of N and C as well as the degree of nilpotency of N are determined by M_0 and M_1 as the following proposition shows.

Proposition 10.1.6. *Let $P, Q \in \mathbb{C}^{n\times n}$ be invertible such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

where $C \in \mathbb{C}^{k\times k}$, $N \in \mathbb{C}^{(n-k)\times(n-k)}$ for some $k \in \{0, \ldots, n\}$, and N is nilpotent. Then (M_0, M_1) is regular and

(a) *k is the degree of the polynomial* $z \mapsto \det(zM_0 + M_1)$.

(b) $N^\ell = 0$ *if and only if*

$$\sup_{|z|\geqslant r} \left\| z^{-\ell+1}(zM_0 + M_1)^{-1} \right\| < \infty$$

for one (or equivalently all) $r > 0$ *such that* $B(0,r) \supseteq \sigma(M_0, M_1)$.

Proof. First, note that

$$\det(zM_0+M_1) = \frac{1}{\det P \, \det Q} \det \begin{pmatrix} z+C & 0 \\ 0 & zN+1 \end{pmatrix} = \frac{1}{\det P \, \det Q} \det(z+C) \quad (z \in \mathbb{C}).$$

Hence, (M_0, M_1) is regular and

$$k = \deg \det((\cdot) + C) = \deg \det((\cdot)M_0 + M_1),$$

which shows (a). Moreover, we have $\rho(M_0, M_1) = \rho(-C)$ and

$$(zM_0 + M_1)^{-1} = Q \begin{pmatrix} (z+C)^{-1} & 0 \\ 0 & (zN+1)^{-1} \end{pmatrix} P \quad (z \in \rho(M_0, M_1)),$$

and hence, for $r > 0$ with $B(0,r) \supseteq \sigma(M_0, M_1)$ we have

$$\sup_{|z|\geqslant r} \left\|(zM_0 + M_1)^{-1}\right\| \leqslant K_1 \sup_{|z|\geqslant r} \left\|(zN+1)^{-1}\right\|$$

for some $K_1 \geqslant 0$, since $\sup_{|z|\geqslant r} \left\|(z+C)^{-1}\right\| < \infty$. Now let $\ell \in \mathbb{N}$ such that $N^\ell = 0$. Then

$$\left\|(zN+1)^{-1}\right\| = \left\| \sum_{k=0}^{\ell-1} (-1)^k z^k N^k \right\| \leqslant K_2 |z|^{\ell-1} \quad (|z| \geqslant r)$$

for some constant $K_2 \geqslant 0$ and thus,

$$\left\|(zM_0 + M_1)^{-1}\right\| \leqslant K_1 K_2 |z|^{\ell-1} \quad (|z| \geqslant r).$$

Assume on the other hand that

$$\sup_{|z|\geqslant r} \left\| z^{-\ell+1}(zM_0 + M_1)^{-1} \right\| < \infty$$

for some $\ell \in \mathbb{N}$ and $r > 0$ with $\sigma(M_0, M_1) \subseteq B(0,r)$. Then there exist $\widetilde{K}_1, \widetilde{K}_2 \geqslant 0$ such that

$$\left\|(zN+1)^{-1}\right\| \leqslant \left\| \begin{pmatrix} (z+C)^{-1} & 0 \\ 0 & (zN+1)^{-1} \end{pmatrix} \right\| \leqslant \widetilde{K}_1 \left\|(zM_0 + M_1)^{-1}\right\| \leqslant \widetilde{K_2} |z|^{\ell-1}$$

for all $z \in \mathbb{C}$ with $|z| \geqslant r$. Now, let $p \in \mathbb{N}$ be minimal such that $N^p = 0$. We show that $p \leqslant \ell$ by contradiction. Assume $p > \ell$. Then we compute

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{1}{n^\ell}(nN+1)^{-1}N^{p-\ell-1} = \lim_{n\to\infty} \sum_{k=0}^{p-1} (-1)^k n^{k-\ell} N^{k+p-\ell-1} \\
&= \lim_{n\to\infty} \sum_{k=0}^{\ell-1} (-1)^k n^{k-\ell} N^{k+p-\ell-1} + (-1)^\ell N^{p-1} + \lim_{n\to\infty} \sum_{k=\ell+1}^{p-1} (-1)^k n^{k-\ell} N^{k+p-\ell-1} \\
&= (-1)^\ell N^{p-1}
\end{aligned}
$$

which contradicts the minimality of p. $\square$

Theorem 10.1.7. *Let (M_0, M_1) be regular and $P, Q \in \mathbb{C}^{n\times n}$ be chosen according to Proposition 10.1.5. Let $k = \deg\det((\cdot)M_0 + M_1)$. Then*

$$
\mathrm{IV}(M_0, M_1) = \left\{ U_0 \in \mathbb{C}^n \,;\, Q^{-1}U_0 \in \mathbb{C}^k \times \{0\} \right\}.
$$

Moreover, for each $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution U of (10.2) is unique and satisfies $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap C^1(\mathbb{R}_{>0}; \mathbb{C}^n)$ as well as

$$
\begin{aligned}
M_0 U'(t) + M_1 U(t) &= 0 \quad (t > 0), \\
U(0+) &= U_0.
\end{aligned}
$$

Proof. Let $C \in \mathbb{C}^{k\times k}$ and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be nilpotent as in Proposition 10.1.5. Obviously U is a solution of (10.2) if and only if $V := Q^{-1}U$ is continuous on $\mathbb{R}_{\geqslant 0}$ and solves

$$
\begin{aligned}
\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \right) V &= 0 \quad \text{on } (0, \infty), \\
V(0+) &= Q^{-1}U_0 =: V_0.
\end{aligned} \tag{10.3}
$$

Clearly, if $Q^{-1}U_0 = (x, 0) \in \mathbb{C}^k \times \{0\}$ then V by $V(t) := (\mathrm{e}^{-tC}x, 0)$ for $t \geqslant 0$ is a solution of (10.3) for $\nu > 0$ large enough. On the other hand, if V given by $V(t) = (V_1(t), V_2(t)) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ $(t \geqslant 0)$ is a solution of (10.3) then we have

$$
\partial_{t,\nu} N V_2 + V_2 = 0 \quad \text{on } (0, \infty).
$$

Since N is nilpotent, there exists $\ell \in \mathbb{N}$ with $N^\ell = 0$. Hence,

$$
N^{\ell-1}V_2(t) = -N^{\ell-1}\partial_{t,\nu} N V_2(t) = \partial_{t,\nu} N^\ell V_2(t) = 0 \quad (t > 0),
$$

which in turn implies $\partial_{t,\nu} N^{\ell-1} V_2 = 0$ on $(0, \infty)$. Using again the differential equation, we infer $N^{\ell-2}V_2(t) = 0$ for $t > 0$. Inductively, we deduce $V_2(t) = 0$ for $t > 0$ and by continuity $V_2(0+) = 0$ which yields $V_0 = Q^{-1}U_0 \in \mathbb{C}^k \times \{0\}$. The uniqueness follows from Proposition 10.2.7 below. $\square$

10.2 The infinite-dimensional Case

Let now $M_0, M_1 \in L(H)$. Again, it is our aim to determine the space of consistent initial values for the problem

$$\begin{cases}(\partial_{t,\nu} M_0 + M_1)\, U = 0 \quad \text{on } (0,\infty)\,, \\ U(0+) = U_0. \end{cases} \tag{10.4}$$

Here, consistent initial values are defined as in the finite-dimensional setting:

Definition. We call an initial value $U_0 \in H$ *consistent* for (10.4), if there exist $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that (10.4) holds. We denote the set of all consistent initial values for (10.4) by

$$\mathrm{IV}(M_0, M_1) \coloneqq \{U_0 \in H\,;\, U_0 \text{ consistent}\}\,.$$

Before we try to determine $\mathrm{IV}(M_0, M_1)$ we prove a regularity result for solutions of (10.4).

Proposition 10.2.1. *Let $\nu > 0$, $U_0 \in H$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ be a solution of (10.4). Then $M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0) \in H^1_\nu(\mathbb{R}; H)$ and*

$$\partial_{t,\nu} M_0 \left(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0\right) + M_1 U = 0.$$

Proof. Let $\varphi \in H^1_\nu(\mathbb{R}; H)$ and $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $C^1(\mathbb{R})$ satisfying $\operatorname{spt} \eta_n \subseteq [1/n, \infty)$, $\eta_n = 1$ on $[2/n, \infty)$ and $\|\eta_n\|_\infty \leq 1$, $\|\eta_n'\|_\infty \leqslant 2n$ for each $n \in \mathbb{N}$. We define $\varphi_n \coloneqq \varphi\eta_n \in H^1_\nu(\mathbb{R}; H)$ and note that $\operatorname{spt}\varphi_n \subseteq \mathbb{R}_{>0}$ for each $n \in \mathbb{N}$. Using $\operatorname{spt} U \subseteq \mathbb{R}_{\geqslant 0}$ and the equality

$$(\partial_{t,\nu}^*\varphi)\eta_n = (-\partial_{t,\nu}\varphi)\eta_n + 2\nu\varphi\eta_n = -\varphi'\eta_n + 2\nu\varphi_n = -(\varphi\eta_n)' + \varphi\eta_n' + 2\nu\varphi_n = \partial_{t,\nu}^*\varphi_n + \varphi\eta_n',$$

we compute

$$\begin{aligned}
\left\langle M_0 U, \partial_{t,\nu}^*\varphi\right\rangle_{L_{2,\nu}} &= \lim_{n\to\infty} \left\langle M_0 U, \left(\partial_{t,\nu}^*\varphi\right)\eta_n\right\rangle_{L_{2,\nu}} = \lim_{n\to\infty} \left\langle M_0 U, \partial_{t,\nu}^*\varphi_n + \varphi\eta_n'\right\rangle_{L_{2,\nu}} \\
&= \lim_{n\to\infty} \Big(\langle \partial_{t,\nu} M_0 U, \varphi_n\rangle_{L_{2,\nu}} + \left\langle M_0 U, \varphi\eta_n'\right\rangle_{L_{2,\nu}} \Big) \\
&= \lim_{n\to\infty} \Big(\langle -M_1 U, \varphi_n\rangle_{L_{2,\nu}} + \left\langle M_0 U, \varphi\eta_n'\right\rangle_{L_{2,\nu}} \Big) \\
&= \langle -M_1 U, \varphi\rangle_{L_{2,\nu}} + \lim_{n\to\infty} \left\langle M_0 U, \varphi\eta_n'\right\rangle_{L_{2,\nu}}.
\end{aligned}$$

In order to determine the last limit, we compute

$$\begin{aligned}
\lim_{n\to\infty} \left|\left\langle M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0), \varphi\eta_n'\right\rangle_{L_{2,\nu}}\right| &= \lim_{n\to\infty} \left| \int_0^{\frac{2}{n}} \langle M_0(U(t) - U_0), \varphi(t)\rangle_{L_{2,\nu}} \eta_n'(t) \mathrm{e}^{-2\nu t} \,\mathrm{d}t \right| \\
&\leqslant \lim_{n\to\infty} 2n \int_0^{\frac{2}{n}} \left| \langle M_0(U(t) - U_0), \varphi(t)\rangle_{L_{2,\nu}} \right| \mathrm{e}^{-2\nu t} \,\mathrm{d}t \\
&= 0,
\end{aligned}$$

since the integrand is continuous on $\mathbb{R}_{\geqslant 0}$ and vanishes at 0. Thus

$$\begin{aligned}\lim_{n\to\infty}\left\langle M_0U,\varphi\eta_n'\right\rangle_{L_{2,\nu}} &= \lim_{n\to\infty}\left\langle \mathbb{1}_{\mathbb{R}_{\geqslant 0}}M_0U_0,\varphi\eta_n'\right\rangle_{L_{2,\nu}}\\ &= \lim_{n\to\infty}\int_0^\infty \left\langle M_0U_0,\varphi(t)\eta_n'(t)\right\rangle_{L_{2,\nu}} \mathrm{e}^{-2\nu t}\,\mathrm{d}t\\ &= -\lim_{n\to\infty}\int_0^\infty \left\langle M_0U_0,\varphi'(t)\eta_n(t)-2\nu\varphi(t)\eta_n(t)\right\rangle_{L_{2,\nu}} \mathrm{e}^{-2\nu t}\,\mathrm{d}t\\ &= \left\langle \mathbb{1}_{\mathbb{R}_{\geqslant 0}}M_0U_0,\partial_{t,\nu}^*\varphi\right\rangle_{L_{2,\nu}},\end{aligned}$$

where we used integration by parts and Corollary 4.1.3. Summarising, we have shown

$$\left\langle M_0U,\partial_{t,\nu}^*\varphi\right\rangle_{L_{2,\nu}} = \left\langle -M_1U,\varphi\right\rangle_{L_{2,\nu}} + \left\langle \mathbb{1}_{\mathbb{R}_{\geqslant 0}}M_0U_0,\partial_{t,\nu}^*\varphi\right\rangle_{L_{2,\nu}} \quad (\varphi\in H_\nu^1(\mathbb{R};H)),$$

and hence, $M_0(U-\mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0)\in H_\nu^1(\mathbb{R};H)$ with

$$\partial_{t,\nu}M_0(U-\mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0)+M_1U=0. \qquad \square$$

We now come back to the space $\mathrm{IV}(M_0,M_1)$. Since we are now dealing with an infinite-dimensional setting, we cannot use normal forms to determine $\mathrm{IV}(M_0,M_1)$ without dramatically restricting the class of operators. Thus, we follow a different approach using so-called Wong sequences.

Definition. We set

$$\mathrm{IV}_0 \coloneqq H$$

and for $k\in\mathbb{N}_0$ we set

$$\mathrm{IV}_{k+1} \coloneqq M_1^{-1}[M_0[\mathrm{IV}_k]].$$

The sequence $(\mathrm{IV}_k)_{k\in\mathbb{N}_0}$ is called the *Wong sequence* associated with (M_0,M_1).

Remark 10.2.2. By induction, we infer $\mathrm{IV}_{k+1}\subseteq\mathrm{IV}_k$ for each $k\in\mathbb{N}_0$.

As in the matrix case, we denote by

$$\rho(M_0,M_1)\coloneqq\left\{z\in\mathbb{C}\,;\,(zM_0+M_1)^{-1}\in L(H)\right\}$$

the *resolvent set of* (M_0,M_1).

Lemma 10.2.3. *Let $k\in\mathbb{N}_0$. Then:*

(a) $M_1(zM_0+M_1)^{-1}M_0 = M_0(zM_0+M_1)^{-1}M_1$ *for each $z\in\rho(M_0,M_1)$.*

(b) $(zM_0+M_1)^{-1}M_0[\mathrm{IV}_k]\subseteq\mathrm{IV}_{k+1}$ *for each $z\in\rho(M_0,M_1)$.*

(c) *If $x\in\mathrm{IV}_k$ we find $x_1,\ldots,x_{k+1}\in H$ such that for each $z\in\rho(M_0,M_1)\setminus\{0\}$*

$$(zM_0+M_1)^{-1}M_0x = \frac{1}{z}x+\sum_{\ell=1}^k\frac{1}{z^{\ell+1}}x_\ell+\frac{1}{z^{k+1}}(zM_0+M_1)^{-1}x_{k+1}.$$

(d) *If* $\rho(M_0, M_1) \neq \varnothing$ *then* $M_1^{-1}[M_0[\overline{\mathrm{IV}_k}]] \subseteq \overline{\mathrm{IV}_{k+1}}$.

Proof. The proof of the statements (a) to (c) are left as Exercise 10.6. We now prove (d). If $k = 0$ there is nothing to show. So assume that the statement holds for some $k \in \mathbb{N}_0$ and let $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_{k+1}}\right]\right]$. Since $\overline{\mathrm{IV}_{k+1}} \subseteq \overline{\mathrm{IV}_k}$, we infer $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_k}\right]\right] \subseteq \overline{\mathrm{IV}_{k+1}}$ by induction hypothesis. Hence, we find a sequence $(w_n)_{n\in\mathbb{N}}$ in IV_{k+1} with $w_n \to x$. Let now $z \in \rho(M_0, M_1)$. Then, by (b), we have $(zM_0 + M_1)^{-1}M_0 w_n \subseteq \mathrm{IV}_{k+2}$ for each $n \in \mathbb{N}$ and hence, $(zM_0 + M_1)^{-1}M_0 x \in \overline{\mathrm{IV}_{k+2}}$. Moreover, since $M_1 x \in M_0\left[\overline{\mathrm{IV}_{k+1}}\right]$, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in IV_{k+1} with $M_0 y_n \to M_1 x$. Setting now $x_n \coloneqq (zM_0+M_1)^{-1}zM_0x + (zM_0 + M_1)^{-1}M_0 y_n \in \overline{\mathrm{IV}_{k+2}}$ (where, again, we have used (b)) for $n \in \mathbb{N}$, we derive

$$
\begin{aligned}
x_n &= (zM_0 + M_1)^{-1}zM_0x + (zM_0 + M_1)^{-1}M_0y_n = x - (zM_0 + M_1)^{-1}\left(M_1x - M_0y_n\right) \\
&\to x
\end{aligned}
$$

as $n \to \infty$ and thus, $x \in \overline{\mathrm{IV}_{k+2}}$. □

The importance of the Wong sequence becomes apparent if we consider solutions of (10.4).

Lemma 10.2.4. *Assume that* $\rho(M_0, M_1) \neq \varnothing$. *Let* $\nu > 0$ *and* $U \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) \cap C(\mathbb{R}_{\geqslant 0}; H)$ *be a solution of (10.4). Then* $U(t) \in \bigcap_{k\in\mathbb{N}_0} \overline{\mathrm{IV}_k}$ *for each* $t \geqslant 0$.

Proof. We prove the claim, $U(t) \in \overline{\mathrm{IV}_k}$ for all $t \geqslant 0$ and $k \in \mathbb{N}_0$, by induction. For $k = 0$ there is nothing to show. Assume now that $U(t) \in \overline{\mathrm{IV}_k}$ for each $t \geqslant 0$ and some $k \in \mathbb{N}_0$. By Proposition 10.2.1 we know that

$$\partial_{t,\nu} M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0) + M_1 U = 0$$

and thus, in particular,

$$M_0U(t) - M_0U_0 + \int_0^t M_1 U(s)\,\mathrm{d}s = 0 \quad (t \geqslant 0).$$

Let now $t \geqslant 0$ and $h > 0$. Then we infer

$$M_0U(t+h) - M_0U(t) + M_1 \int_t^{t+h} U(s)\,\mathrm{d}s = 0$$

and hence,

$$\int_t^{t+h} U(s)\,\mathrm{d}s \in M_1^{-1}M_0[\overline{\mathrm{IV}_k}] \subseteq \overline{\mathrm{IV}_{k+1}}$$

by Lemma 10.2.3(d). Since U is continuous, the fundamental theorem of calculus yields $U(t) \in \overline{\mathrm{IV}_{k+1}}$, which yields the assertion. □

In particular, the space of consistent initial values has to be a subspace of $\bigcap_{k\in\mathbb{N}_0} \overline{\mathrm{IV}_k}$. We now impose an additional constraint on the operator pair (M_0, M_1), which is equivalent to being regular in the finite-dimensional setting (cf. Proposition 10.1.6).

Definition. We call the operator pair (M_0, M_1) *regular* if there exists $\nu_0 \geqslant 0$ such that

(a) $\mathbb{C}_{\mathrm{Re}>\nu_0} \subseteq \rho(M_0, M_1)$, and

(b) there exist $C \geqslant 0$ and $\ell \in \mathbb{N}$ such that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have $\left\|(zM_0+M_1)^{-1}\right\| \leqslant C|z|^{\ell-1}$.

Moreover, we call the smallest $\ell \in \mathbb{N}$ satisfying (b) the *index of* (M_0, M_1), which is denoted by $\operatorname{ind}(M_0, M_1)$.

Remark 10.2.5. Note that for matrices M_0 and M_1 the index equals the degree of nilpotency of N in the Weierstraß normal form by Proposition 10.1.6.

From now on, we will require that (M_0, M_1) is regular. First, we prove an important result on the Wong sequence in this case.

Proposition 10.2.6. *Let (M_0, M_1) be regular, $k \in \mathbb{N}_0$, and $k \geqslant \operatorname{ind}(M_0, M_1)$. Then*

$$\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{\operatorname{ind}(M_0,M_1)}}.$$

Proof. We show that $\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{k+1}}$ for each $k \geqslant \operatorname{ind}(M_0, M_1)$. Since the inclusion "$\supseteq$" holds trivially, it suffices to show $\mathrm{IV}_k \subseteq \overline{\mathrm{IV}_{k+1}}$. For doing so, let $k \geqslant \operatorname{ind}(M_0, M_1)$ and $x \in \mathrm{IV}_k$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$(zM_0 + M_1)^{-1} M_0 x = \frac{1}{z} x + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}} x_\ell + \frac{1}{z^{k+1}} (zM_0 + M_1)^{-1} x_{k+1}$$

for each $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Since $k \geqslant \operatorname{ind}(M_0, M_1)$, we derive

$$z(zM_0 + M_1)^{-1} M_0 x \to x \quad (\operatorname{Re} z \to \infty),$$

and since the elements on the left-hand side belong to IV_{k+1}, by Lemma 10.2.3(b), the assertion immediately follows. □

We now prove that in case of a regular operator pair (M_0, M_1) the solution of (10.4) for a consistent initial value U_0 is uniquely determined.

Proposition 10.2.7. *Let (M_0, M_1) be regular, $U_0 \in \mathrm{IV}(M_0, M_1)$, and $\nu > 0$. Then the solution $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ of (10.4) is unique. In particular*

$$(\mathcal{L}_\rho U)(t) = \big((\mathrm{i}t + \rho)M_0 + M_1\big)^{-1} M_0 U_0 \quad (a.e.\ t \in \mathbb{R})$$

for each $\rho > \max\{\nu, \nu_0\}$.

Proof. By Proposition 10.2.1 we have $M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0) \in H^1_\nu(\mathbb{R}; H)$ and

$$\partial_{t,\nu} M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0) + M_1 U = 0.$$

Applying the Fourier–Laplace transformation, $\mathcal{L}_\rho$, for $\rho > \max\{\nu, \nu_0\}$ the latter yields

$$(\mathrm{i}t+\rho)M_0\big(\mathcal{L}_\rho U(t) - \frac{1}{\mathrm{i}t+\rho}U_0\big) + M_1\mathcal{L}_\rho U(t) = 0 \quad (\text{a.e. } t \in \mathbb{R})$$

which in turn yields

$$\mathcal{L}_\rho U(t) = \big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1} M_0 U_0 \quad (\text{a.e. } t \in \mathbb{R})$$

and, in particular, proves the uniqueness of the solution. □

Remark 10.2.8. The formula in Proposition 10.2.7 shows that $U \in L_{2,\nu_0}(\mathbb{R}; H)$ for all solutions U of (10.4) for a consistent initial value U_0 and hence, we also have $M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}} U_0) \in H^1_{\nu_0}(\mathbb{R}; H)$.

One interesting consequence of the latter proposition is the following.

Corollary 10.2.9. *Let (M_0, M_1) be regular. Then the operator $M_0 \colon \mathrm{IV}(M_0, M_1) \to H$ is injective.*

Proof. Let $U_0 \in \mathrm{IV}(M_0, M_1)$ with $M_0U_0 = 0$. By Proposition 10.2.7, the solution U of (10.4) with $U(0+) = U_0$ satisfies

$$\mathcal{L}_\rho U(t) = \big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1} M_0 U_0 = 0$$

and hence, $U = 0$, which in turn implies $U_0 = U(0+) = 0$. □

We now want to determine the space $\mathrm{IV}(M_0, M_1)$ in terms of the Wong sequence.

Proposition 10.2.10. *Let (M_0, M_1) be regular. Then*

$$\mathrm{IV}_{\mathrm{ind}(M_0,M_1)} \subseteq \mathrm{IV}(M_0, M_1) \subseteq \overline{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}}.$$

Proof. The second inclusion follows from Lemma 10.2.4 and Proposition 10.2.6. Let now $U_0 \in \mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ and set

$$V(z) \coloneqq \frac{1}{\sqrt{2\pi}}(zM_0 + M_1)^{-1} M_0 U_0 \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

Let $k \coloneqq \mathrm{ind}(M_0, M_1)$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$V(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{z}U_0 + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}} x_\ell + \frac{1}{z^{k+1}}\left(zM_0 + M_1\right)^{-1} x_{k+1}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

In particular, we read off that $V \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; H)$ and hence, by the Theorem of Paley–Wiener (more precisely by Corollary 8.1.3) there exists $U \in L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\left(\mathcal{L}_\rho U\right)(t) = V(\mathrm{i}t + \rho) \quad (\text{a.e. } t \in \mathbb{R}, \rho > \nu_0).$$

Moreover,

$$zV(z) - \frac{1}{\sqrt{2\pi}}U_0 = \frac{1}{\sqrt{2\pi}}\left(\sum_{\ell=1}^{k}\frac{1}{z^\ell}x_\ell + \frac{1}{z^k}\left(zM_0+M_1\right)^{-1}x_{k+1}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0})$$

and hence $\left(z \mapsto zV(z) - \frac{1}{\sqrt{2\pi}}U_0\right) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; H)$ as well. Since

$$\begin{aligned}\left(\mathcal{L}_\rho \partial_{t,\rho}(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0)\right)(t) &= (\mathrm{i}t+\rho)\left(\mathcal{L}_\rho U\right)(t) - \frac{1}{\sqrt{2\pi}}U_0 \\ &= (\mathrm{i}t+\rho)V(\mathrm{i}t+\rho) - \frac{1}{\sqrt{2\pi}}U_0 \quad (\text{a.e. } t\in\mathbb{R}, \rho>\nu_0),\end{aligned}$$

we infer $U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0 \in H^1_{\nu_0}(\mathbb{R}; H)$ and, thus, $U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0$ is continuous by Theorem 4.1.2. Hence, $U \in C(\mathbb{R}_{\geqslant 0}; H)$ and since $\operatorname{spt} U \subseteq \mathbb{R}_{\geqslant 0}$ we derive $U(0+) = U_0$. Finally, by the definition of V

$$M_0\left(zV(z) - \frac{1}{\sqrt{2\pi}}U_0\right) = -\frac{1}{\sqrt{2\pi}}M_0(zM_0+M_1)^{-1}M_1U_0 = -M_1V(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}),$$

where we have used Lemma 10.2.3 (a) in the second equality. Hence,

$$\partial_{t,\nu_0}M_0(U - \mathbb{1}_{\mathbb{R}_{\geqslant 0}}U_0) + M_1U = 0,$$

from which we see that U solves (10.4). □

Finally, we treat the case when $\mathrm{IV}(M_0, M_1)$ is closed.

Theorem 10.2.11. *Let (M_0, M_1) be regular and $\mathrm{IV}(M_0, M_1)$ closed. Then the operator $S\colon \mathrm{IV}(M_0, M_1) \to C(\mathbb{R}_{\geqslant 0}; H)$, which assigns to each initial state, $U_0 \in \mathrm{IV}(M_0, M_1)$, its corresponding solution, $U \in C(\mathbb{R}_{\geqslant 0}; H)$, of (10.4) is bounded in the sense that*

$$S_n\colon \mathrm{IV}(M_0, M_1) \to C([0,n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

is bounded for each $n \in \mathbb{N}$.

Proof. By Proposition 10.2.10 we infer that $\mathrm{IV}(M_0, M_1) = \overline{\mathrm{IV}_k}$ with $k := \operatorname{ind}(M_0, M_1)$. Let $\nu > \nu_0 \geqslant 0$. By Proposition 10.2.7 and Corollary 8.1.3, there exists $C \geqslant 0$ such that

$$\left\|\partial_{t,\nu}^{-k}Su_0\right\|_{L_{2,\nu}(\mathbb{R}_{\geq 0};H)} = \left\|\left(z \mapsto z^{-k}(zM_0+M_1)^{-1}M_0U_0\right)\right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)} \leqslant C\sqrt{\frac{\pi}{\nu}}\|M_0U_0\|_H$$

for each $U_0 \in \mathrm{IV}(M_0, M_1)$, where we have used the regularity of (M_0, M_1) and

$$\left\|(z \mapsto z^{-1}M_0U_0)\right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)} = \sqrt{\frac{\pi}{\nu}}\|M_0U_0\|_H.$$

In particular, $S\colon \mathrm{IV}(M_0, M_1) \to H^{-1}(\partial_{t,\nu}^k)$ is bounded. Since $L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H) \hookrightarrow H^{-1}(\partial_{t,\nu}^k)$ continuously, we infer that $S\colon \mathrm{IV}(M_0, M_1) \to L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H)$ is bounded by the closed graph theorem. Hence, also

$$S_n\colon \mathrm{IV}(M_0, M_1) \to L_2([0,n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

is bounded for each $n \in \mathbb{N}$ and since $C([0,n]; H) \hookrightarrow L_2([0,n]; H)$ continuously, we infer that S_n is bounded with values in $C([0,n]; H)$ again by the closed graph theorem. □

Remark 10.2.12. The variant of the closed graph theorem used in the proof above is the following: Let X, Y be Banach spaces and Z a Hausdorff topological vector space (e.g. a Banach space) such that $Y \hookrightarrow Z$ continuously. Let $T : X \to Z$ be linear and continuous with $T[X] \subseteq Y$. Then $T \in L(X, Y)$. Indeed, by the closed graph theorem it suffices to show that $T : X \to Y$ is closed. For doing so, let $(x_n)_n$ be a sequence in X with $x_n \to x$ and $Tx_n \to y$ for some $x \in X, y \in Y$. Then $Tx_n \to Tx$ in Z by the continuity of T and $Tx_n \to y$ in Z be the conitnuous embedding. Hence, $y = Tx$ and thus, T is closed.

10.3 Comments

The theory of differential algebraic equations in finite dimensions is a very active field. The main motivation for studying these equations comes from the modelling of electrical circuits and from control theory (see e.g. [2]). The main reference for the statements presented in the first part of this lecture is the book by Kunkel and Mehrmann [3]. Of course, also in the finite dimensional case Wong sequences can be used to determine the consistent initial values. For instance, in [1] the connection between Wong sequences and the quasi-Weierstraß normal form for matrix pairs is studied. Of course, the theory is not restricted to linear and homogeneous problems. Indeed, in the non-homogeneous case it turns out that the set of consistent initial values also depends on the given right-hand side.

The theory of differential algebraic equations in infinite dimensions is less well studied than the finite-dimensional case. We refer to [7], where the theory of C_0-semigroups is used to deal with such equations. Moreover, we refer to [5, 4], where sequences of projectors are used to decouple the system. Moreover, there exist several references in the Russian literature, where the equations are called Sobolev type equations (see e.g. [6]). The results on infinite dimensional problems presented here are based on [9, 10, 8]. In [9] the focus was on systems with index 0 with an emphasis on exponential stability and dichotomy.

We also add the following remark concerning the result in Theorem 10.2.11. By Corollary 10.2.9 we know that $M_0\colon \operatorname{IV}(M_0, M_1) \to H$ is injective. If $\operatorname{IV}(M_0, M_1)$ is closed, it follows that the operator $C\colon \operatorname{dom}(C) \subseteq \operatorname{IV}(M_0, M_1) \to \operatorname{IV}(M_0, M_1)$ given by

$$\begin{aligned} \operatorname{dom}(C) &:= \{U_0 \in \operatorname{IV}(M_0, M_1)\,;\, M_1 U_0 \in M_0\left[\operatorname{IV}(M_0, M_1)\right]\}\,, \\ CU_0 &:= M_0^{-1} M_1 U_0 \quad (U_0 \in \operatorname{dom}(C)) \end{aligned}$$

is well-defined and closed. Using this operator, C, Theorem 10.2.11 states that if $\operatorname{IV}(M_0, M_1)$ is closed then $-C$ generates a C_0-semigroup on $\operatorname{IV}(M_0, M_1)$. The precise statement can be found in [8, Theorem 5.7]. Moreover, C is bounded if $\operatorname{IV}_{\operatorname{ind}(M_0,M_1)}$ is closed (cf. Exercise 10.7).

Exercises

Exercise 10.1. Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ such that (M_0, M_1) is regular and define the Wong sequence $(\operatorname{IV}_j)_{j\in\mathbb{N}_0}$ associated with (M_0, M_1). Moreover, let $P, Q \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{k\times k}$, and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be as in the quasi-Weierstraß normal form for (M_0, M_1) with N nilpotent (cf. Proposition 10.1.5). We decompose a vector $x \in \mathbb{C}^n$ into $\breve{x} \in \mathbb{C}^k$ and $\widehat{x} \in \mathbb{C}^{n-k}$ such that $x = (\breve{x}, \widehat{x})$. Prove that

$$x \in \mathrm{IV}_j \Leftrightarrow \widehat{Q^{-1}x} \in \operatorname{ran} N^j \quad (j \in \mathbb{N}_0).$$

Moreover, show that for each $z \in \rho(M_0, M_1)$ we have

$$\mathrm{IV}_j = \operatorname{ran}\big((zM_0 + M_1)^{-1}M_0\big)^j \quad (j \in \mathbb{N}_0).$$

Exercise 10.2. Let $E \in \mathbb{C}^{n\times n}$. We set $k := \operatorname{ind}(E, 1)$, where 1 denotes the identity matrix in $\mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *Drazin inverse of E* if the following properties hold:

- $EX = XE$,
- $XEX = X$,
- $XE^{k+1} = E^k$.

Prove that each matrix $E \in \mathbb{C}^{n\times n}$ has a unique Drazin inverse.
Hint: For the existence consider the quasi-Weierstraß form for $(E, 1)$.

Exercise 10.3. Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with (M_0, M_1) regular and $M_0M_1 = M_1M_0$. Denote by M_0^{D} the Drazin inverse of M_0 (see Exercise 10.2). Prove:

(a) $M_0^{\mathrm{D}}M_1 = M_1M_0^{\mathrm{D}}$,

(b) $\operatorname{ran} M_0^{\mathrm{D}}M_0 = \mathrm{IV}(M_0, M_1)$,

(c) For all $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution U of (10.2) is given by

$$U(t) = \mathrm{e}^{-tM_0^{\mathrm{D}}M_1}U_0 \quad (t \geqslant 0).$$

Exercise 10.4. Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with (M_0, M_1) regular. Prove that there exist two matrices $E, A \in \mathbb{C}^{n\times n}$ with (E, A) regular and $EA = AE$ such that

- $\mathrm{IV}(E, A) = \mathrm{IV}(M_0, M_1)$,
- U solves the initial value problem (10.2) for the matrices M_0, M_1 if and only if U solves the initial value problem (10.2) for the matrices E, A with the same initial value $U_0 \in \mathrm{IV}(M_0, M_1)$.

Exercise 10.5. We consider the following electrical circuit (see Figure 10.1) with a resistor with resistance $R > 0$, an inductor with inductance $L > 0$ and a capacitor with capacitance $C > 0$. We denote the respective voltage drops by v_R, v_L and v_C. Moreover, the current is denoted by i. The constitutive relations for resistor, inductor and capacitor are given by

$$Ri = v_R,$$

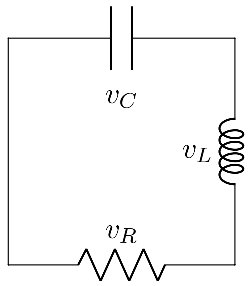

Figure 10.1: Electrical circuit

$$\begin{aligned} Li' &= v_L, \\ Cv_C' &= i, \end{aligned}$$

respectively. Moreover, by Kirchhoff's second law we have

$$v_R + v_C + v_L = 0.$$

Write these equations as a differential algebraic equation and compute the index and the space of consistent initial values. Moreover, compute the solution for each consistent initial value for $R = 2$ and $C = L = 1$.

Exercise 10.6. Prove the assertions (a) to (c) in Lemma 10.2.3.

Exercise 10.7. Let $M_0, M_1 \in L(H)$.

(a) Assume that $\rho(M_0, M_1) \neq \varnothing$. Prove that for each $k \in \mathbb{N}$ the space IV_k is closed if and only if $M_0 [\mathrm{IV}_{k-1}]$ is closed.

(b) Assume that (M_0, M_1) is regular with $\operatorname{ind}(M_0, M_1) \geqslant 1$. Prove that if $\mathrm{IV}_{\operatorname{ind}(M_0,M_1)}$ is closed then the operator

$$M_0|_{\mathrm{IV}_{\operatorname{ind}(M_0,M_1)}} \colon \mathrm{IV}_{\operatorname{ind}(M_0,M_1)} \to M_0 \left[\mathrm{IV}_{\operatorname{ind}(M_0,M_1)-1}\right]$$

is an isomorphism.

References

[1] T. Berger, A. Ilchmann and S. Trenn. 'The quasi-Weierstrass form for regular matrix pencils.' In: *Linear Algebra Appl.* 436.10 (2012), pp. 4052–4069.

[2] L. Dai. *Singular control systems.* Vol. 118. Berlin etc.: Springer-Verlag, 1989, pp. ix + 332.

[3] P. Kunkel and V. Mehrmann. *Differential-algebraic equations.* EMS Textbooks in Mathematics. Analysis and numerical solution. European Mathematical Society (EMS), Zürich, 2006, pp. viii+377.

[4] T. Reis. 'Consistent initialization and perturbation analysis for abstract differential-algebraic equations'. In: *Math. Control Signals Systems* 19.3 (2007), pp. 255–281.

[5] T. Reis and C. Tischendorf. 'Frequency Domain Methods and Decoupling of Linear Infinite Dimensional Differential Algebraic Systems'. In: *Journal of Evolution Equations* 5.3 (Aug. 2005), pp. 357–385.

[6] G. A. Sviridyuk and V. E. Fedorov. *Linear Sobolev type equations and degenerate semigroups of operators.* Inverse and Ill-posed Problems Series. VSP, Utrecht, 2003, pp. viii+216.

[7] B. Thaller and S. Thaller. 'Factorization of degenerate Cauchy problems: The linear case.' In: *J. Oper. Theory* 36.1 (1996), pp. 121–146.

[8] S. Trostorff. *Semigroups associated with differential-algebraic equations.* Tech. rep. arXiv:1905.11197. CAU Kiel, 2019.

[9] S. Trostorff and M. Waurick. 'On differential-algebraic equations in infinite dimensions'. In: *J. Differential Equations* 266.1 (2019), pp. 526–561.

[10] S. Trostorff and M. Waurick. 'On Higher Index Differential-Algebraic-Equations in Infinite Dimensions'. In: *The Diversity and Beauty of Applied Operator Theory.* Ed. by P. S. Albrecht Böttcher Daniel Potts and D. Wenzel. Vol. 268. Operator Theory: Advances and Applications. 2018, pp. 477–486.