Exercises of Chapter 9

Exercise 9.1. *Let H_0 be a Hilbert space, $T \in L(H_0)$ and $H^{-1}(T) \coloneqq \mathrm{dom}(T)'$. (Note that $\mathrm{dom}(T)$ is equipped with the graph norm.) Then $H^{-1}(T^*) = H^{-1}(T) = H_0'$.*

Proof. In *Chapter 9.2* the dual space is defined as $H' := \{\phi : H \to \mathbb{K}; \phi \text{ linear and bounded}\}$. From *Lemma 2.2.8* of *Chapter 2* we get $T^* \in L(H_0)$, hence $\mathrm{dom}(T^*)$ is defined on the whole space H_0. Furthermore, the continuity of T and T^* implies the equivalence of the graph norms and the $\|\cdot\|_{H_0}$-norm on H_0. Therefore, we have

$$H^{-1}(T^*) = H^{-1}(T) = H_0'.$$

□

Exercise 9.2. *Let H_0, H_1 be Hilbert spaces such that $H_0 \hookrightarrow H_1$ is continuous with dense range. Then $H_1' \hookrightarrow H_0'$ is continuous with dense range as well.*

Proof. By our assumption there exists a one-to-one and continuous operator $T : H_0 \to H_1$ with dense range. We will show that $T' : H_1' \to H_0'$, $f \mapsto f \circ T$, defines a one-to-one and continuous operator with dense range. Note that T' is defined, because T is a continuous operator. The fact that T' is a linear operator follows immediately from the equipped linear structure of dual spaces. To prove that T' is one-to-one, let $f, g \in H_1'$ with $T'f = T'g$. Since T has dense range and f, g are continuous, $f \circ T = g \circ T$ implies $f = g$. Next we will show that T' is continuous. It is true that we have $\|f \circ T\| \leq \|f\|\|T\|$, because T and f are continuous operators. Therefore, we have $\|T'\| \leq \|T\| < \infty$.

Finally, we prove that T' has dense range. Let $x'' \in H_0''$ with $x''(h) = 0$ for all $h \in \mathrm{ran}(T')$. The reflexivity of Hilbert spaces gives an $x \in H_0$ with $h(x) = x''(h)$ for all $h \in H_0'$. For every $f \in H_1'$ we have $0 = x''(f \circ T) = f(Tx)$. The theorem of Hahn-Banach implies $Tx = 0$. The operator T is one-to-one. Therefore, $x = 0$. It follows that $x'' = 0$. The theorem of Hahn-Banach implies $\overline{\mathrm{ran}(T')} = H_0'$. □

Exercise 9.3. *Let H_0 be a Hilbert space, $A \in L(H_0)$. Assume that $T : \mathrm{dom}(T) \subseteq H_0 \to H_0$ is densely defined and closed with $0 \in \rho(T)$ and $T^{-1}A = AT^{-1} + T^{-1}BT^{-1}$ for some bounded $B \in L(H_0)$. Then A admits a unique continuous extension, $\overline{A} \in L(H^{-1}(T^*))$.*

Proof. Firstly, we take a look at T^*. Since T is a densely defined and closable linear operator, it follows from *Lemma 2.2.7, 2.2.2* and *Remark 2.2.3* that T^* is a densely defined and closed linear operator. As a result of the observation in the beginning of *Chapter 2.2*, $\mathrm{dom}(T^*)$ is equipped with the graph scalar form. Thus, $\mathrm{dom}(T^*) \hookrightarrow H_0$ is clearly a continuous embedding with dense range. Applying *Exercise 9.2.* and noting $H^{-1}(T^*) = \mathrm{dom}(T^*)'$ (*Ex. 9.1*), it follows that $H_0' \hookrightarrow H^{-1}(T^*)$ is a continuous embedding with dense range as well. This implies that $A : H_0 \cong H_0' \subseteq H^{-1}(T^*) \to H^{-1}(T^*)$ is a densely defined linear operator. So, it suffices to show that A is continuous. For this,

let $\phi \in H_0$ and compute for all $q \in \mathrm{dom}(T^*)$

$$\begin{aligned}
|(A\phi)(q)| = |\langle A\phi, q\rangle_{H_0}| = |\langle A\phi, (T^*)^{-1}T^*q\rangle_{H_0}| &\overset{(\star)}{=} |\langle T^{-1}A\phi, T^*q\rangle_{H_0}| \\
&\overset{\text{asm.}}{=} |\langle (AT^{-1} + T^{-1}BT^{-1})\phi, T^*q\rangle_{H_0}| \overset{\text{CS}}{\leq} (\|AT^{-1}\phi\|_{H_0} + \|T^{-1}BT^{-1}\phi\|_{H_0})\|T^*q\|_{H_0} \\
&\leq (\|A\|\|T^{-1}\phi\|_{H_0} + \|T^{-1}\|\|B\|\|T^{-1}\phi\|_{H_0})\|T^*q\|_{H_0} \\
&\overset{(\star\star)}{\leq} (\|A\| + \|T^{-1}\|\|B\|)\|T^{-1}\phi\|_{H_0}\|q\|_{\mathrm{dom}(T^*)}
\end{aligned}$$

with $(\star)$: $(T^*)^{-1} = (T^{-1})^*$, $T^{-1} \in L(H_0)$ and $(\star\star)$: $\|T^*q\|_{H_0} \leq \|q\|_{\mathrm{dom}(T^*)}$.
By definition of the operator norm, this means:

$$\|A\phi\|_{H^{-1}(T^*)} \leq (\|A\| + \|T^{-1}\|\|B\|)\|T^{-1}\phi\|_{H_0}.$$

Invoking *Lemma 9.2.7*

$$\|T^{-1}\phi\|_{H_0} \leq c\|T_{-1}(T^{-1}\phi)\|_{H^{-1}(T^*)} \quad \text{with} \quad c \geq 0,$$

we get

$$\|A\phi\|_{H^{-1}(T^*)} \leq (\|A\| + \|T^{-1}\|\|B\|)\|T_{-1}(T^{-1}\phi)\|_{H^{-1}(T^*)}.$$

Because $T_{-1}(T^{-1}\phi)$ is simply the embedding of ϕ in $H^{-1}(T^*)$, the statement follows. □

Exercise 9.4. *Let H_0 be a Hilbert space, $N : \mathrm{dom}(N) \subseteq H_0 \to H_0$ be a normal operator; that is, N is densely defined and closed and $N^*N = NN^*$. Then we have $H^{-1}(N) = H^{-1}(N^*)$. In particular, we have $H^{-1}(\partial_{t,\nu}) = H^{-1}(\partial_{t,\nu}^*)$ for every $\nu \in \mathbb{R}$.*

Notation. In the following we denote the graph scalar product of N by $\langle\cdot,\cdot\rangle_N$.

Proof. Firstly, we show that $(\mathrm{dom}(N^*N), \|\cdot\|_N)$ is dense in $(\mathrm{dom}(N), \|\cdot\|_N)$. For this, we show the surjectivity of the operator $I + N^*N$. We remember that $N^* = \{(-y, x) : (x, y) \in N^\perp\}$. One easily sees that $(N^*)^\perp = \{(-y, x) : (x, y) \in N\} =: M$. Hence, we have $H_0 \times H_0 = N^* \oplus M$. Thus for $z \in H_0$ we find $x \in \mathrm{dom}(N)$ and $y \in \mathrm{dom}(N^*)$ such that

$$(0, z) = (y, N^*y) + (-Nx, x) = (y - Nx, N^*y + x)$$

and therefore we obtain $Nx = y \in \mathrm{dom}(N^*)$, $x \in \mathrm{dom}(N^*N)$ and $z = x + N^*y = (I + N^*N)x$. Consequently, $I + N^*N$ is surjective.

Let now $y \in \mathrm{dom}(N)$ with $\langle x, y\rangle_N = 0$ for all $x \in \mathrm{dom}(N^*N)$. Then we have

$$\langle y, (I + N^*N)x\rangle = \langle y, x\rangle + \langle Ny, Nx\rangle = \langle y, x\rangle_N = 0$$

for all $x \in \mathrm{dom}(N^*N)$. Since $\mathrm{ran}(I + N^*N) = H_0$, we obtain $y = 0$. Therefore, $\mathrm{dom}(N^*N)^\perp = \{0\}$ and thus $(\mathrm{dom}(N^*N), \|\cdot\|_N)$ is dense in $(\mathrm{dom}(N), \|\cdot\|_N)$. By replacing N with N^*, one analogously proves the density of $(\mathrm{dom}(NN^*), \|\cdot\|_{N^*})$ in $(\mathrm{dom}(N^*), \|\cdot\|_{N^*})$.

By assumption we have

$$\|Nx\|^2 = \langle N^*Nx, x\rangle = \langle NN^*x, x\rangle = \|N^*x\|^2$$

and, therefore,

$$\|x\|_N = \|x\|_{N^*} \tag{$*$}$$

for all $x \in \operatorname{dom}(N^*N) = \operatorname{dom}(NN^*)$. By $(*)$ and the density results from above we conclude that $(\operatorname{dom}(N), \|\cdot\|_N) = (\operatorname{dom}(N^*), \|\cdot\|_{N^*})$ and thus

$$H^{-1}(N) = \operatorname{dom}(N)' = \operatorname{dom}(N^*)' = H^{-1}(N^*).$$

Since $\partial_{t,\nu}$ is a normal operator, we particularly have $H^{-1}(\partial_{t,\nu}) = H^{-1}(\partial_{t,\nu}^*)$ for every $\nu \in \mathbb{R}$. □

Exercise 9.5. *Let H be a Hilbert space and $T : \operatorname{dom}(T) \subseteq H \to H$ be densely defined and closed with $0 \in \rho(T)$. Then $T^{-1} : (H, \|T^{-1}\cdot\|) \to H$ has a unique isometric bijective extension*

$$\widetilde{T^{-1}} : \widetilde{(H, \|T^{-1}\cdot\|)} \to H$$

where $\widetilde{(\cdot)}$ denotes the completion. Moreover, $T_{-1} \circ \widetilde{T^{-1}}$ is an isomorphism. In particular,

$$H^{-1}(T^*) \cong \widetilde{(H, \|T^{-1}\cdot\|)}.$$

Proof. Since $T^{-1} : (H, \|T^{-1}\cdot\|) \to H$ is obviously isometric and $\operatorname{ran}(T^{-1}) = \operatorname{dom}(T)$ is dense in H, we can extend T^{-1} uniquely to an isometric isomorphism

$$\widetilde{T^{-1}} : \widetilde{(H, \|T^{-1}\cdot\|)} \to H.$$

By Lemma 9.2.7 $T_{-1} : H \to H^{-1}(T^*)$ is an isomorphism which implies that

$$T_{-1} \circ \widetilde{T^{-1}} : \widetilde{(H, \|T^{-1}\cdot\|)} \to H^{-1}(T^*)$$

is an isomorphism. □

Exercise 9.6. *Let H_0 be a Hilbert space and $T : \operatorname{dom}(T) \subseteq H_0 \to H_0$ be a densely defined, closed linear operator with $0 \in \rho(T)$. For $n \in \mathbb{N}$ we define $H^n(T) := \operatorname{dom}(T^n)$ and $H^{-n}(T) := H^{-1}(T^n)$. Because of $H_0 \cong H_0'$ via the Riesz isomorphism, $H^0(T)$ can mean H_0 or H_0'. Let $\mathcal{F} \subseteq H_0$. For $n \in \mathbb{N}$ we define $\mathcal{F}^* := \{\langle x, \cdot\rangle_{H_0},\, x \in \mathcal{F}\}$.*

- *(a) We have that $\mathrm{Id} : H^{k+\ell}(T) \to H^\ell(T)$ and $\mathrm{Id}' : H^{-\ell}(T) \to H^{-\ell-k}(T)$ are continuous embeddings with dense range for all $\ell \geq 0$ and $k \in \mathbb{N}$.*
- *(b) We have that $\mathcal{D} := \bigcap_{n\in\mathbb{N}} \operatorname{dom}(T^n)$ is dense in $H^\ell(T)$ for all $\ell \geq 0$. Let $\mathcal{F} \subseteq H_0$ be dense. We have that $\mathcal{F}^* \subseteq H^{-\ell}(T)$ is dense (Note that $\mathcal{F}^* \subseteq H^{-\ell}(T)$ means $\big\{\langle x, \cdot\rangle_{H_0}|_{\operatorname{dom}(T^\ell)},\, x \in \mathcal{F}\big\} \subseteq H^{-\ell}(T)$.)*

(c) We have that $T : H^{\ell+1}(T) \to H^{\ell}(T)$ is a topological isomorphism and

$$T' : \operatorname{dom}(T)^* \subseteq H^{-\ell}(T^*) \to H^{-\ell-1}(T^*),$$
$$\langle x, \cdot \rangle_{H_0} \mapsto \langle Tx, \cdot \rangle_{H_0},$$

can be extended to a topological isomorphism for all $\ell \geq 0$.

Proof. For part (a) it suffices to prove that $\mathrm{Id} : H^{\ell+1}(T) \to H^{\ell}(T)$ and $\mathrm{Id}' : H^{-\ell}(T) \to H^{-\ell-1}(T)$ are continuous embeddings with dense range for all $\ell \geq 0$. The composition of finitely many continuous embeddings with dense range is a continuous embedding with dense range itself. Since $H^{\ell+1}(T) \subseteq H^{\ell}(T)$, the identity operator $\mathrm{Id} : H^{\ell+1}(T) \to H^{\ell}(T)$ is defined. We note that Id is one-to-one and linear. We show the continuity of Id. In the case $\ell = 0$ we have $\|x\|_{H_0} \leq \|x\|_{H^1(T)}$ for all $x \in H^1(T)$. Additionally, we have

$$\left(H^{\ell}(T), \left\|T^{\ell} \cdot\right\|_{H_0}\right) \cong H^{\ell}(T) \tag{1}$$

for all $\ell \geq 0$, because $0 \in \rho(T)$ implies the equivalence of the involved norms. Furthermore, for $\ell > 0$ we have $\left\|T^{\ell}x\right\|_{H_0} \leq \left\|T^{-1}\right\| \left\|T^{\ell+1}x\right\|_{H_0}$ for all $x \in H^{\ell+1}(T)$. Hence, Id is continuous. To show that Id has dense range, let $\ell \geq 0$ and $x \in H^{\ell}(T)$. Then $T^{\ell}x \in H_0$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \in \operatorname{dom}(T)^{\mathbb{N}}$ with $\lim_{n\to\infty} \left\|x_n - T^{\ell}x\right\|_{H_0} = 0$. Furthermore, $T^{-\ell}x_n \in H^{\ell+1}(T)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left\|T^{\ell}T^{-\ell}x_n - T^{\ell}x\right\|_{H_0} = \lim_{n\to\infty} \left\|x_n - T^{\ell}x\right\|_{H_0} = 0.$$

Hence, Id has dense range. Furthermore, Exercise 9.2 implies that $\mathrm{Id}' : H^{-\ell}(T) \to H^{-\ell-1}(T)$ is a continuous embedding with dense range.

We show that $\mathcal{D}$ is dense in $H^{\ell}(T)$ for all $\ell \geq 0$. Let $m \geq 0$ and $x \in H^m(T)$. We will construct a sequence, which, on the one hand, is a Cauchy sequence in $H^{\ell}(T)$ for all $\ell \geq 0$. On the other hand, its limit is supposed to be arbitrarily close to x in $H^m(T)$. Let $\epsilon > 0$ and $K \coloneqq \max\left\{1, \left\|T^{-1}\right\|\right\}$. We have already proved that $H^{m+1}(T)$ is dense in $H^m(T)$. Using (1), we find an $x_1 \in H^{m+1}(T)$ with $\left\|T^m x - T^m x_1\right\|_{H_0} \leq \frac{\epsilon}{(2K)^{m+1}}$. Recursively, we find an $x_{n+1} \in H^{m+n+1}(T)$ with

$$\left\|T^{m+n}x_n - T^{m+n}x_{n+1}\right\|_{H_0} \leq \frac{\epsilon}{(2K)^{m+n+1}}$$

for all $n \in \mathbb{N}$. For every $\ell \geq 0$ the sequence $(x_n)_{n\geq\max\{m+1,\ell\}}$ is a Cauchy sequence in $\left(H^{\ell}(T), \left\|T^{\ell} \cdot\right\|\right)$ or, equivalently, in $H^{\ell}(T)$, because for all $n \geq \max\{1, \ell - m\}$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}\left\|T^{\ell}x_n - T^{\ell}x_{n+k}\right\|_{H_0} &\leq \sum_{p=n}^{n+k-1} \left\|T^{\ell}x_p - T^{\ell}x_{p+1}\right\|_{H_0} \\ &\leq \sum_{p=n}^{n+k-1} \left\|T^{-1}\right\|^{m+p-\ell} \left\|T^{m+p}x_p - T^{m+p}x_{p+1}\right\|_{H_0} \leq \sum_{p=n}^{n+k-1} \frac{\epsilon}{2^p K^{\ell+1}} \leq \epsilon.\end{aligned}$$

Every $H^\ell(T)$ is a Hilbert space and thus $y_\ell \coloneqq \lim_{n\to\infty} x_n \in H^\ell(T)$. We have proved that the identity operator $\mathrm{Id} : H^{\ell+1} \to H^\ell$ is continuous. In particular, we have

$$y_{\ell+1} = \lim_{n\to\infty} x_n = \mathrm{Id}\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} \mathrm{Id}(x_n) = y_\ell.$$

This implies $y_0 = y_\ell$ for all $\ell \geq 0$ and $y_0 \in \mathcal{D}$. Let $x_0 \coloneqq x$. Then

$$\|T^m x - T^m y\|_{H_0} \leq \sum_{n=0}^{N-1} \|T^m x_n - T^m x_{n+1}\|_{H_0} + \|T^m x_N - T^m y\|_{H_0}.$$

Since $\lim_{N\to\infty} \|T^m x_N - T^m y\|_{H_0} = 0$, we have

$$\|T^m x - T^m y\|_{H_0} \leq \sum_{n=0}^{\infty} \|T^m x_n - T^m x_{n+1}\|_{H_0} \leq \sum_{n=0}^{\infty} \frac{\epsilon}{2^n K^{m+1}} \leq \epsilon.$$

With (1) it follows that y is arbitrary close to x in $H^\ell(T)$.

To prove that $\mathcal{F}^*$ is dense in $H^{-\ell}(T)$ for $\ell \geq 0$, one has to keep part (a) in mind. Since $\mathcal{F}$ is the image of $\mathcal{F}^* \subseteq H_0'$ under the Riesz isomorphism, we see that $\mathcal{F}^* \subseteq H_0'$ is dense. The embedding $\mathrm{Id}' : H_0' \to H^{-\ell}(T)$ is continuous with dense range and thus $\mathcal{F}^* \subseteq H^{-\ell}(T)$ is dense for $\ell > 0$.

We prove that $T : H^{\ell+1}(T) \to H^\ell(T)$ is a topological isomorphism for all $\ell \geq 0$. Since T is an operator with $T^{-1} \in L(H_0)$ by assumption, T is one-to-one. And for $x \in H^\ell(T)$ the pre-image of x under T is $T^{-1}x \in H^{\ell+1}(T)$. Let $x \in H^{\ell+1}(T)$. Then $\big\|T^\ell T x\big\|_{H_0} = \big\|T^{\ell+1}x\big\|_{H_0}$. Therefore,

$$T : \Big(H^{\ell+1}(T), \big\|T^{\ell+1}\cdot\big\|_{H_0}\Big) \to \Big(H^\ell(T), \big\|T^\ell \cdot\big\|_{H_0}\Big)$$

is an isometric isomorphism. With (1) it follows that $T : H^{\ell+1}(T) \to H^\ell(T)$ is a topological isomorphism.

Finally, we prove the last assertion that is

$$\begin{aligned} T' : \mathrm{dom}(T)^* \subseteq H^{-\ell}(T^*) &\to H^{-\ell-1}(T^*), \\ \langle x, \cdot\rangle_{H_0} &\mapsto \langle Tx, \cdot\rangle_{H_0}, \end{aligned}$$

can be extended to a topological isomorphism for all $\ell \geq 0$. We note that T' is a linear operator. Because T^* is a densely defined, closed operator with $0 \in \rho(T^*)$ and $\mathrm{dom}(T) \subseteq H_0$ is dense, we infer with part (b) that $\mathrm{dom}(T)^* \subseteq H^{-\ell}(T^*)$ is dense. Therefore, T' is densely defined. Let $\langle x, \cdot\rangle_{H_0}, \langle y, \cdot\rangle_{H_0} \in \mathrm{dom}(T)^*$ with $\langle Tx, \cdot\rangle_{H_0} = \langle Ty, \cdot\rangle_{H_0}$ on $\mathrm{dom}\big((T^*)^{\ell+1}\big)$. Then $\langle Tx - Ty, \cdot\rangle_{H_0} = 0$ on $\mathrm{dom}\big((T^*)^{\ell+1}\big)$. The density of $\mathrm{dom}\big((T^*)^{\ell+1}\big) \subseteq H_0$ implies $Tx = Ty$. The operator T is one-to-one and thus $x = y$. In particular, we have $\langle x, \cdot\rangle_{H_0} = \langle y, \cdot\rangle_{H_0}$ on $\mathrm{dom}\big((T^*)^\ell\big)$ and the operator T' is one-to-one. Furthermore, the range of T' is $H_0' \subseteq H^{-\ell-1}(T^*)$ and thus dense. With (1) we obtain

$$\Big(H^\ell(T^*), \big\|(T^*)^\ell \cdot\big\|_H\Big)' \cong H^{-\ell}(T^*) \tag{2}$$

for all $\ell \geq 0$. Let $\langle x, \cdot \rangle_{H_0} \in \mathrm{dom}(T)^*$. We have

$$\sup_{\substack{y \in H^{\ell+1}(T^*), \\ \|(T^*)^{\ell+1} y\| \leq 1}} \big|\langle Tx, y\rangle_{H_0}\big| = \sup_{\substack{y \in H^{\ell+1}(T^*), \\ \|(T^*)^{\ell+1} y\| \leq 1}} \big|\langle x, T^* y\rangle_{H_0}\big| = \sup_{\substack{z \in H^{\ell}(T^*), \\ \|(T^*)^{\ell} z\| \leq 1}} \big|\langle x, T^* z\rangle_{H_0}\big|.$$

This implies that the operator

$$T' : \Big(H^\ell(T^*), \big\|(T^*)^\ell \cdot \big\|_H\Big)' \to \Big(H^{\ell+1}(T^*), \big\|(T^*)^{\ell+1} \cdot \big\|_H\Big)'$$

can be extended to an isometric isomorphism. In particular, with (2) it follows that $T' : H^{-\ell}(T^*) \to H^{-\ell-1}(T^*)$ can be extended to a topological isomorphism. □

In the following we will prove the second assertion of Exercise 9.7 and a slightly different result compared to the first assertion of Exercise 9.7. After proving the alternative version of Exercise 9.7 we will explain why this version also completes the proof of Theorem 9.3.2.

Exercise 9.7. *Let H be a Hilbert space.*

(a) Let $A : \mathrm{dom}(A) \subseteq H \to H$ be closed and $C : \mathrm{dom}(C) \subseteq H \to H$. Assume that $\mu, \lambda \in \rho(C)$ are contained in the same connected component of $\rho(C)$ and

$$(\mu I - C)^{-1} A \subseteq A(\mu I - C)^{-1}.$$

Then we have $(\lambda I - C)^{-1} A \subseteq A(\lambda I - C)^{-1}$.

(b) Let $\nu, \epsilon > 0$. Then we have $(1 + \epsilon \partial_{t,\nu})^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ and $(1 + \epsilon \partial_{t,\nu}^*)^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ in the strong operator topology as $\epsilon \to 0$.

Proof. (a) Let Z be the connected component of $\rho(C)$ which contains μ. To prove the assertion of (a), we show that $M := \{\eta \in Z : (\eta I - C)^{-1} A \subseteq A(\eta I - C)^{-1}\}$ is open and closed in Z. Firstly, we prove that M is open. For this, let $\eta_0 \in M$ and define $r := \|(\eta_0 I - C)^{-1}\|^{-1}$. Let $\eta \in B(\eta_0, r)$. By the definition of M we have

$$\Big(\sum_{k=0}^{n} (\eta_0 - \eta)^k \big((\eta_0 I - C)^{-1}\big)^{k+1}\Big) Ax = A \sum_{k=0}^{n} (\eta_0 - \eta)^k \big((\eta_0 I - C)^{-1}\big)^{k+1} x$$

for all $n \in \mathbb{N}$ and all $x \in \mathrm{dom}(A)$. Moreover, it follows from the proof of Proposition 2.4.1 that $B(\eta_0, r) \subseteq \rho(C)$ and

$$\sum_{k=0}^{\infty} (\eta_0 - \eta)^k \big((\eta_0 I - C)^{-1}\big)^{k+1}$$

is a power series which converges in the operator norm and equals $(\eta I - C)^{-1}$. Therefore, since A is closed, for every $x \in \mathrm{dom}(A)$ we conclude that $(\eta I - C)^{-1} x \in \mathrm{dom}(A)$ with $A(\eta I - C)^{-1} x = (\eta I - C)^{-1} Ax$. Hence, we have $(\eta I - C)^{-1} A \subseteq A(\eta I - C)^{-1}$. Therefore,

since $\eta \in Z$ due to the connectedness of $B(\eta_0, r)$ and $\eta_0 \in Z$, we obtain $\eta \in M$ and thus $B(\eta_0, r) \subseteq M$. This implies that M is open in Z.

To prove the closedness of M in Z, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in M with $\eta_n \to \eta \in Z$ as $n \to \infty$ and $x \in \mathrm{dom}(A)$. Then we have

$$(\eta_n I - C)^{-1} A x = A(\eta_n I - C)^{-1} x$$

for all $n \in \mathbb{N}$. Since the left-hand side converges to $(\eta I - C)^{-1} A x$ and $(\eta_n I - C)^{-1} x$ converges to $(\eta I - C)^{-1} x$ due to the continuity of the mapping $\rho(C) \ni \nu \mapsto (\nu I - C)^{-1} \in L(H)$ (cf. Proposition 2.4.1), it follows from the closedness of A that $(\eta I - C)^{-1} x \in \mathrm{dom}(A)$ with

$$A(\eta I - C)^{-1} x = (\eta I - C)^{-1} A x.$$

Hence, we have $(\eta I - C)^{-1} A \subseteq A(\eta I - C)^{-1}$ and, therefore, $\eta \in M$. This implies the closedness of M in Z.

Since Z is a connected space and M is open and closed in Z, we have $M \in \{\emptyset, Z\}$. By assumption we have $\mu \in M$ and thus $M = Z$. Therefore, we obtain

$$(\lambda I - C)^{-1} A \subseteq A(\lambda I - C)^{-1}.$$

(b) Firstly, we show that $(1 + \epsilon \partial_{t,\nu})^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ in the strong operator topology as $\epsilon \to 0$. For this, we observe that

$$(1 + \epsilon \partial_{t,\nu})^{-1} = (\mathcal{L}_\nu^* (1 + \epsilon(i\mathrm{m} + \nu)) \mathcal{L}_\nu)^{-1} = \mathcal{L}_\nu^* \left(\frac{1}{1 + \epsilon(i\mathrm{m} + \nu)} \right) \mathcal{L}_\nu \qquad (*)$$

by Theorem 5.2.3. Let $f \in L_{2,\nu}(\mathbb{R}; H)$. For $t \in \mathbb{R}$ we have

$$\left(\frac{1}{1 + \epsilon(i\mathrm{m} + \nu)} \right) \mathcal{L}_\nu f(t) = \frac{1}{1 + \epsilon(it + \nu)} \mathcal{L}_\nu f(t) \to \mathcal{L}_\nu f(t)$$

as $\epsilon \to 0$. Furthermore, we have

$$\begin{aligned}
&\left\| \left(\frac{1}{1 + \epsilon(i\mathrm{m} + \nu)} \right) \mathcal{L}_\nu f(t) - \mathcal{L}_\nu f(t) \right\|^2 \\
&\leq 2 \left| \frac{1}{1 + \epsilon(it + \nu)} \right|^2 \|\mathcal{L}_\nu f(t)\|^2 + 2 \|\mathcal{L}_\nu f(t)\|^2 \\
&= 2 \left(\frac{1}{(1 + \epsilon\nu)^2 + t^2 \epsilon^2} + 1 \right) \|\mathcal{L}_\nu f(t)\|^2 \\
&\leq 4 \|\mathcal{L}_\nu f(t)\|^2
\end{aligned}$$

for all $t \in \mathbb{R}$. Since $\mathcal{L}_\nu f \in L_2(\mathbb{R}; H)$, it follows by the dominated convergence theorem that

$$\left(\frac{1}{1 + \epsilon(i\mathrm{m} + \nu)} \right) \mathcal{L}_\nu f \to \mathcal{L}_\nu f$$

in $L_2(\mathbb{R};H)$ as $\epsilon \to 0$. By $(*)$ and the boundedness of $\mathcal{L}_\nu^*$ we then have

$$(1+\epsilon\partial_{t,\nu})^{-1} f \to \mathcal{L}_\nu^* \mathcal{L}_\nu f = f$$

in $L_{2,\nu}(\mathbb{R};H)$ as $\epsilon \to 0$. Therefore, it follows that $(1+\epsilon\partial_{t,\nu})^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ in the strong operator topology as $\epsilon \to 0$.

By similar arguments one proves that we have $(1+\epsilon\partial_{t,\nu}^*)^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ in the strong operator topology as $\epsilon \to 0$. For this, note that

$$(1+\epsilon\partial_{t,\nu}^*)^{-1} = \mathcal{L}_\nu^* \left(\frac{1}{1+\epsilon(-i\mathrm{m}+\nu)} \right) \mathcal{L}_\nu$$

by Corollary 3.2.6 and Theorem 5.2.3. □

We will now discuss why this alternative version of Exercise 9.7 completes the proof of Theorem 9.3.2.

Remark. The version of Exercise 9.7 on page 118 is about proving Lemma 9.3.3. The second statement of Lemma 9.3.3 equals part (b) of from above and is applied in lines 10 and 11 of the proof of Theorem 9.3.2. For $n \in \mathbb{N}$ let T_n, A and ν as in the proof of Theorem 9.3.2. The first statement of Lemma 9.3.3 is used to show that $T_n A \subseteq A T_n$ for all $n \in \mathbb{N}$. By applying part (a) from above we can also conclude that $T_n A \subseteq A T_n$ as follows:

Let $n \in \mathbb{N}$. We consider the connected set $M := \{z \in \mathbb{C} : \mathrm{Re}\, z < \nu\}$. Note that $-n \in M$ and $0 \in M$. Therefore, since $\sigma(\partial_{t,\nu}) = \{it+\nu : t \in \mathbb{R}\}$ by Theorem 5.2.3 and thus $M \subseteq \rho(\partial_{t,\nu})$, the elements 0 and $-n$ are contained in the same connected component of $\rho(\partial_{t,\nu})$. Since A is closed by Exercise 6.1 and we have

$$\partial_{t,\nu}^{-1} A \subseteq A \partial_{t,\nu}^{-1}$$

by Proposition 3.1.5(c), we obtain

$$(-n-\partial_{t,\nu})^{-1} A \subseteq A(-n-\partial_{t,\nu})^{-1}$$

by applying Exercise 9.7(a). This implies

$$T_n A = -n(-n-\partial_{t,\nu})^{-1} A \subseteq -nA(-n-\partial_{t,\nu})^{-1} = A T_n.$$