9 Initial Value Problems and Extrapolation Spaces

Up until now we have dealt with evolutionary equations of the form

$$\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right) U = F$$

for some given $F \in L_{2,\nu}(\mathbb{R}; H)$ for some Hilbert space H, a skew-selfadjoint operator A in H and a material law M defined on a suitable half space satisfying an appropriate positive definiteness condition with $\nu \in \mathbb{R}$ chosen suitably large. Under these conditions, we established that the solution operator, $S_\nu := \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H))$, is eventually independent of ν and causal; that is, if $F = 0$ on $(-\infty, a]$ for some $a \in \mathbb{R}$, then so too is U.
Solving for $U \in L_{2,\nu}(\mathbb{R}; H)$ for some non-negative ν penalises U having support on $\mathbb{R}_{\leq 0}$. This might be interpreted as an implicit *initial condition at* $-\infty$. In this lecture, we shall study how to obtain a solution for initial value problems with an initial condition at 0, based on the solution theory developed in the previous lectures.

9.1 What are Initial Values?

This section is devoted to the motivation of the framework to follow in the subsequent section. Let us consider the following, arguably easiest but not entirely trivial, initial value problem: find a 'causal' $u\colon \mathbb{R} \to \mathbb{R}$ such that for $u_0 \in \mathbb{R}$ we have

$$\begin{cases} u'(t) = 0 & (t > 0), \\ u(0) = u_0. \end{cases} \tag{9.1}$$

First of all note that there is no condition for u on $\mathbb{R}_{<0}$. Since, there is no source term or right-hand side supported on $\mathbb{R}_{<0}$, causality would imply that $u = 0$ on $(-\infty, 0)$. Moreover, $u = c$ for some constant $c \in \mathbb{R}$ on $(0, \infty)$. Thus, in order to match with the initial condition,

$$u(t) = u_0 \mathbb{1}_{[0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Notice also that u is not continuous. Hence, by the Sobolev embedding theorem (Theorem 4.1.2), $u \notin \bigcup_{\nu>0} \operatorname{dom}(\partial_{t,\nu})$.

Proposition 9.1.1. *Let H be a Hilbert space, $u_0 \in H$. Define*

$$\begin{aligned} \delta_0 u_0 \colon C_c^\infty(\mathbb{R}; H) &\to \mathbb{K} \\ f &\mapsto \langle u_0, f(0+) \rangle_H . \end{aligned}$$

Then, for all $\nu \in \mathbb{R}_{>0}$, $\delta_0 u_0$ extends to a continuous linear functional on $\operatorname{dom}(\partial_{t,\nu})$. *Re-using the notation for this extension, for all $f \in \operatorname{dom}(\partial_{t,\nu})$ we have*

$$(\delta_0 u_0)(f) = -\left\langle \mathbb{1}_{[0,\infty)} u_0, (\partial_{t,\nu} - 2\nu) f \right\rangle_{L_{2,\nu}(\mathbb{R};H)}. \tag{9.2}$$

Proof. The equality (9.2) is obvious for $f \in C_c^\infty(\mathbb{R};H)$ as it is a direct consequence of the fundamental theorem of calculus (look at the right-hand side first). The continuity of $\delta_0 u_0$ follows from the Cauchy–Schwarz inequality applied to the right-hand side of (9.2). Note that $\mathbb{1}_{[0,\infty)} u_0 \in L_{2,\nu}(\mathbb{R};H)$. □

Recall from Corollary 3.2.6 that

$$\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

Hence, if we *formally* apply this formula to (9.2), we obtain

$$\left\langle \partial_{t,\nu} \mathbb{1}_{[0,\infty)} u_0, f \right\rangle = \left\langle \mathbb{1}_{[0,\infty)} u_0, \partial_{t,\nu}^* f \right\rangle = (\delta_0 u_0)(f).$$

Therefore, in order to use the introduced time derivative operator for the above initial value problem, we need to extend the time derivative to a broader class of functions than just $\operatorname{dom}(\partial_{t,\nu})$. To utilise the adjoint operator in this way will be central to the construction to follow. It will turn out that indeed

$$\partial_{t,\nu} \mathbb{1}_{[0,\infty)} u_0 = \delta_0 u_0.$$

Moreover, we shall show below that

$$\partial_{t,\nu} u = \delta_0 u_0$$

considered on the full time-line $\mathbb{R}$ is one possible replacement of the initial value problem (9.1).

9.2 Extrapolating Operators

Since we are dealing with functionals, let us recall the definition of the dual space. Throughout this section let H, H_0, H_1 be Hilbert spaces.

Definition. The space

$$H' \coloneqq \{\varphi \colon H \to \mathbb{K}\,;\, \varphi \text{ linear and bounded}\}$$

is called the *dual space of* H. We equip H' with the linear structure

$$(\lambda\varphi + \psi)(x) \coloneqq \lambda^* \varphi(x) + \psi(x) \quad (\lambda \in \mathbb{K}, \varphi, \psi \in H', x \in H).$$

Remark 9.2.1. Note that H' is a Hilbert space itself, since by the Riesz representation theorem for each $\varphi \in H'$ we find a unique element $R_H\varphi \in H$ such that

$$\forall x \in H :\ \varphi(x) = \langle R_H\varphi, x\rangle .$$

Due to the linear structure on H', the so induced mapping $R_H\colon H' \to H$ (which is one-to-one and onto) becomes linear and

$$H' \times H' \ni (\varphi, \psi) \mapsto \langle R_H\varphi, R_H\psi\rangle$$

defines an inner product on H', which induces the usual norm on functionals.

Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be linear, densely defined and closed. We recall that in this case $\operatorname{dom}(C)$ endowed with the graph inner product

$$(u, v) \mapsto \langle u, v\rangle_{H_0} + \langle Cu, Cv\rangle_{H_1}$$

becomes a Hilbert space. Clearly, $\operatorname{dom}(C) \hookrightarrow H_0$ is continuous with dense range. Moreover, we see that $\operatorname{dom}(C) \ni x \mapsto Cx \in H_1$ is continuous. We define

$$\begin{aligned} C^\diamond\colon H_1 &\to \operatorname{dom}(C)' =: H^{-1}(C), \\ (C^\diamond\phi)(x) &:= \langle \phi, Cx\rangle_{H_1} \quad (\phi \in H_1, x \in \operatorname{dom}(C)). \end{aligned}$$

Note that $C^\diamond$ is related to the dual operator of C; up to the Riesz isomorphism they agree.

Proposition 9.2.2. *With the notions and definitions from this section, the following statements hold:*

(a) $C^\diamond$ *is continuous and linear.*

(b) *Identifying functionals on H_0 by its Riesz representative, we obtain $C^* \subseteq C^\diamond$; that is,*$(R_{H_0}^{-1}C^*\phi)(x) = \langle C^*\phi, x\rangle_{H_0} = \langle \phi, Cx\rangle_{H_1} = (C^\diamond\phi)(x)$ *for all* $\phi \in \operatorname{dom}(C^*)$ *and* $x \in \operatorname{dom}(C)$.

(c) $\ker(C^*) = \ker(C^\diamond)$.

(d) $C \subseteq (C^*)^\diamond : H_0 \to \operatorname{dom}(C^*)' = H^{-1}(C^*)$.

(e) $H_0 \cong H_0' \hookrightarrow H^{-1}(C)$ *densely and continuously.*

Proof. (a) Let $\phi, \psi \in H_1$, $\lambda \in \mathbb{K}$. Then

$$C^\diamond(\lambda\phi + \psi)(x) = \lambda^*(C^\diamond\phi)(x) + (C^\diamond\psi)(x) = (\lambda C^\diamond\phi + C^\diamond\psi)(x) \quad (x \in \operatorname{dom}(C)).$$

To show continuity, let $\phi \in H_1$ and $x \in \operatorname{dom}(C)$. Then

$$\left|\langle \phi, Cx\rangle_{H_1}\right| \leqslant \|\phi\|_{H_1}\|Cx\|_{H_1} \leqslant \|\phi\|_{H_1}\|x\|_{\operatorname{dom}(C)}.$$

Hence, $\|C^\diamond\| = \sup_{\phi\in H_1, \|\phi\|_{H_1}\leqslant 1} \|C^\diamond\phi\|_{\operatorname{dom}(C)^*} \leqslant 1.$

(b) Let $\phi \in \operatorname{dom}(C^*)$. Then we have for all $x \in \operatorname{dom}(C)$

$$(C^\diamond \phi)(x) = \langle \phi, Cx \rangle_{H_1} = \langle C^* \phi, x \rangle_{H_0} = (C^* \phi)(x).$$

We obtain $C^\diamond \phi = C^* \phi$. (Note that a functional on H_0 is uniquely determined by its values on $\operatorname{dom}(C)$.)
(c) Using (b), we are left with showing $\ker(C^\diamond) \subseteq \ker(C^*)$. So, let $\phi \in \ker(C^\diamond)$. Then for all $x \in \operatorname{dom}(C)$ we have

$$0 = (C^\diamond \phi)(x) = \langle \phi, Cx \rangle_{H_1},$$

which leads to $\phi \in \operatorname{dom}(C^*)$ and $\phi \in \ker(C^*)$.
(d) is a direct consequence of (b) applied to C^*.
(e) Since $\operatorname{dom}(C) \hookrightarrow H_0$ is dense and continuous, we obtain that $H_0' \hookrightarrow \operatorname{dom}(C)'$ is so, too; cf. Exercise 9.2. □

We will also write $C_{-1} \coloneqq (C^*)^\diamond$ for the so-called *extrapolated operator* of C. Then $(C^*)_{-1} = C^\diamond$. We will record the index -1 at the beginning, but in order to avoid too much clutter in the notation we will drop this index again, bearing in mind that $C_{-1} \supseteq C$ and $(C^*)_{-1} \supseteq C^*$.

Example 9.2.3. We have shown that for all $\nu \in \mathbb{R}$ the operator $\partial_{t,\nu}$ is densely defined and closed. Then for $f \in L_{2,\nu}(\mathbb{R})$ we have for all $\phi \in C_c^\infty(\mathbb{R})$

$$((\partial_{t,\nu})_{-1} f)(\phi) = \langle f, \partial_{t,\nu}^* \phi \rangle_{L_{2,\nu}} = \langle f, (-\partial_{t,\nu} + 2\nu)\phi \rangle_{L_{2,\nu}} = -\int_{\mathbb{R}} \langle f, (\mathrm{e}^{-2\nu \cdot} \phi)' \rangle_{\mathbb{C}}.$$

Hence, $(\partial_{t,\nu})_{-1} f$ acts as the 'usual' distributional derivative taking into account the exponential weight in the scalar product.
With this observation we deduce that for $\nu > 0$ we have

$$(\partial_{t,\nu})_{-1} \mathbb{1}_{[0,\infty)} = \partial_{t,\nu} \mathbb{1}_{[0,\infty)} = \delta_0.$$

Hence, the initial value problem from the beginning reads: find u such that

$$(\partial_{t,\nu})_{-1} u = u_0 \delta_0.$$

Example 9.2.4. Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider $\operatorname{grad}_0 \colon H_0^1(\Omega) \subseteq L_2(\Omega) \to L_2(\Omega)^d$. We compute $\operatorname{div}_{-1} \colon L_2(\Omega)^d \to H^{-1}(\Omega)$ with $H^{-1}(\Omega) \coloneqq H_0^1(\Omega)'$. For $q \in L_2(\Omega)^d$ we obtain for all $\phi \in H_0^1(\Omega)$

$$(\operatorname{div}_{-1} q)(\phi) = \langle q, \operatorname{div}^* \phi \rangle_{L_2(\Omega)^d} = -\langle q, \operatorname{grad}_0 \phi \rangle_{L_2(\Omega)^d}.$$

Also, with similar arguments, we see that

$$\big((\operatorname{grad})_{-1} f\big)(q) = -\langle f, \operatorname{div}_0 q \rangle_{L_2(\Omega)}$$

for all $f \in L_2(\Omega)$ and $q \in H_0(\operatorname{div}, \Omega)$.

We consider a case of particular interest within the framework of evolutionary equations.

Proposition 9.2.5. *Let* $A\colon \operatorname{dom}(C) \times \operatorname{dom}(C^*) \subseteq H_0 \times H_1 \to H_0 \times H_1$ *be given by*

$$A \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} C^*\psi \\ -C\phi \end{pmatrix}.$$

Then $A_{-1}\colon H_0 \times H_1 \to H^{-1}(C) \times H^{-1}(C^*)$ *acts as*

$$A_{-1} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & (C^*)_{-1} \\ -C_{-1} & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} (C^*)_{-1}\psi \\ -C_{-1}\phi \end{pmatrix}.$$

Next, we will look at the solution theory when carried over to distributional right-hand sides.
An immediate consequence of the introduction of extrapolated operators, however, is that we are now in the position to omit the closure bar for the operator sum in an evolutionary equation, which we will see in an abstract version in Theorem 9.2.6 and for evolutionary equations in Theorem 9.3.2. The main advantage is that we can calculate an operator sum much easier than the closure of it. The price we have to pay is that we have to work in a larger space H^{-1} of an operator in $L_{2,\nu}(\mathbb{R}; H)$ rather than in the original Hilbert space $L_{2,\nu}(\mathbb{R}; H)$. Put differently, this provides another notion of "solutions" for evolutionary equations. For this, we need to introduce the set

$$\operatorname{Fun}(H) := \{\phi\colon \operatorname{dom}(\phi) \subseteq H \to \mathbb{K}\,;\, \phi \text{ linear}\}$$

of not necessearily everywhere defined linear functionals on H. Any $u \in H$ is thus identified with an element in $\operatorname{Fun}(H)$ via $\psi \mapsto \langle u, \psi \rangle_H$. Note that we can add and scalarly multiply elements in $\operatorname{Fun}(H)$ with respect to the same addition and multiplication defined on H' and with their natural domains. As usual, we will use the $\subseteq$-sign for extension/restriction of mappings.

Theorem 9.2.6. *Let* $A\colon \operatorname{dom}(A) \subseteq H \to H$, $B\colon \operatorname{dom}(B) \subseteq H \to H$ *be densely defined and closed such that* $A + B$ *is closable, and assume that there exists* $(T_n)_{n\in\mathbb{N}}$ *in* $L(H)$ *such that* $T_n \to 1_H$ *in the strong operator topology with* $\operatorname{ran}(T_n) \subseteq \operatorname{dom}(B)$ *and*

$$T_n A \subseteq A T_n, \quad T_n B \subseteq B T_n \text{ for all } n \in \mathbb{N}.$$

Then $T_n^* A^* \subseteq A^* T_n^*$ *and* $T_n^* B^* \subseteq B^* T_n^*$ *for each* $n \in \mathbb{N}$ *and* $\operatorname{ran}(T_n^*) \subseteq \operatorname{dom}(B^*)$. *Moreover, for* $x, f \in H$ *the following conditions are equivalent:*

(i) $x \in \operatorname{dom}(\overline{A + B})$ *and* $(\overline{A + B})x = f$.

(ii) $A_{-1}x + B_{-1}x \subseteq f \in \operatorname{Fun}(H)$.

Proof. Let $n \in \mathbb{N}$. Taking adjoints in the inclusion $T_n A \subseteq A T_n$, we derive $(A T_n)^* \subseteq (T_n A)^*$. By Theorem 2.3.4 and Remark 2.3.7 we obtain

$$T_n^* A^* \subseteq \overline{T_n^* A^*} = (A T_n)^* \subseteq (T_n A)^* = A^* T_n^*.$$

The same argument shows the claim for B^*. Moreover, since BT_n is a closed linear operator defined on the whole space H, it follows that $BT_n \in L(H)$ by the closed graph theorem. Hence, $(BT_n)^*$ is bounded by Lemma 2.2.8 and since $(BT_n)^* \subseteq (T_nB)^* = B^*T_n^*$, we derive that $\operatorname{dom}(B^*T_n^*) = H$, showing that $\operatorname{ran}(T_n^*) \subseteq \operatorname{dom}(B^*)$.
We now prove the asserted equivalence.
(i)⇒(ii): By definition, there exists $(x_n)_n$ in $\operatorname{dom}(A) \cap \operatorname{dom}(B)$ such that $x_n \to x$ in H and $Ax_n + Bx_n \to f$. By continuity, we obtain $A_{-1}x_n \to A_{-1}x$ and $B_{-1}x_n \to B_{-1}x$ in $H^{-1}(A^*)$ and $H^{-1}(B^*)$, respectively. From

$$(A+B)\,x_n - Bx_n = Ax_n \to A_{-1}x \in H^{-1}(A^*)$$

and

$$(A+B)\,x_n - Ax_n = Bx_n \to B_{-1}x \in H^{-1}(B^*)$$

it follows that

$$A_{-1}x + B_{-1}x \subseteq f \in \operatorname{Fun}(H).$$

(ii)⇒(i): For $n \in \mathbb{N}$ we put $x_n \coloneqq T_nx$. Then $x_n \in \operatorname{dom}(B)$ and for all $y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$, we obtain

$$\begin{aligned}\langle T_nf - Bx_n, y\rangle &= \langle T_nf, y\rangle - \langle T_nx, B^*y\rangle = \langle f, T_n^*y\rangle - \langle x, T_n^*B^*y\rangle \\ &= \langle f, T_n^*y\rangle - \langle x, B^*T_n^*y\rangle = f(T_n^*y) - (B_{-1}x)(T_n^*y) = (A_{-1}x)(T_n^*y) \\ &= \langle x, A^*T_n^*y\rangle = \langle x, T_n^*A^*y\rangle = \langle x_n, A^*y\rangle\,,\end{aligned}$$

where we have used that $T_n^*y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$. Let now $y \in \operatorname{dom}(A^*)$. Then $T_k^*y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$ for each $k \in \mathbb{N}$ and thus, by what we have shown above

$$\langle T_k(T_nf - Bx_n), y\rangle = \langle T_nf - Bx_n, T_k^*y\rangle = \langle x_n, A^*T_k^*y\rangle = \langle x_n, T_k^*A^*y\rangle = \langle T_kx_n, A^*y\rangle$$

for each $k \in \mathbb{N}$. Letting k tend to infinity, we derive

$$\langle T_nf - Bx_n, y\rangle = \langle x_n, A^*y\rangle\,.$$

Since this holds for each $y \in \operatorname{dom}(A^*)$, this implies that $x_n \in \operatorname{dom}(A)$ and $Ax_n + Bx_n = T_nf$. Letting $n \to \infty$, we deduce $x_n \to x$ and $Ax_n + Bx_n \to f$; that is, (i). □

Lemma 9.2.7. *Let $T\colon \operatorname{dom}(T) \subseteq H \to H$ be densely defined and closed with $0 \in \rho(T)$. Then $T_{-1}\colon H \to H^{-1}(T^*)$ is an isomorphsim. In particular, we have that $\left\|(T_{-1})^{-1}\cdot\right\|_H$ and $\|\cdot\|_{H^{-1}(T^*)}$ are equivalent.*

Proof. Note that since $0 \in \rho(T)$ we obtain $\{0\} = \ker(T) = \ker((T^*)^\diamond) = \ker(T_{-1})$, see Proposition 9.2.2(c). Thus, T_{-1} is one-to-one. Next, let $f \in H^{-1}(T^*)$. Since $0 \in \rho(T)$, we obtain $0 \in \rho(T^*)$ by Exercise 2.4, which implies that $\langle T^*\cdot, T^*\cdot\rangle$ defines an equivalent scalar product on $\operatorname{dom}(T^*)$. Thus, by the Riesz representation theorem, we find $\phi \in \operatorname{dom}(T^*)$ such that for all $\psi \in \operatorname{dom}(T^*)$ we have

$$f(\psi) = \langle T^*\phi, T^*\psi\rangle = \left((T^*)^\diamond\,(T^*\phi)\right)(\psi).$$

Hence, $f \in \operatorname{ran}((T^*)^\diamond) = \operatorname{ran}(T_{-1})$, thus proving that T_{-1} is onto. □

The following alternative description of $H^{-1}(T^*)$ is content of Exercise 9.5.

Proposition 9.2.8. *Let $T\colon \operatorname{dom}(T) \subseteq H \to H$ be densely defined and closed with $0 \in \rho(T)$. Then*

$$H^{-1}(T^*) \cong \big(H, \widetilde{\|T^{-1}\cdot\|_H}\big),$$

where $\cong$ means isomorphic as Banach spaces and $\widetilde{(\cdot)}$ denotes the completion.

Proposition 9.2.9. *Let $B \in L(H)$. Assume that $T\colon \operatorname{dom}(T) \subseteq H \to H$ is densely defined and closed with $0 \in \rho(T)$ and $T^{-1}B = BT^{-1}$. Then B admits a unique continuous extension $\overline{B} \in L(H^{-1}(T^*))$.*

Proof. By Proposition 9.2.2(e), $\operatorname{dom}(B) = H$ is dense in $H^{-1}(T^*)$. Thus, it suffices to show that $B\colon H \subseteq H^{-1}(T^*) \to H^{-1}(T^*)$ is continuous. For this, let $\phi \in H$ and compute for all $q \in \operatorname{dom}(T^*)$

$$\begin{aligned}|(B\phi)(q)| = |\langle B\phi, q\rangle| &= \left|\left\langle B\phi, (T^*)^{-1} T^* q\right\rangle\right| = \left|\langle T^{-1}B\phi, T^*q\rangle\right| = \left|\langle BT^{-1}\phi, T^*q\rangle\right| \\ &\leqslant \|B\|\big\|T^{-1}\phi\big\|\|q\|_{\operatorname{dom}(T^*)}.\end{aligned}$$

The statement now follows upon invoking Lemma 9.2.7. □

The abstract notions and concepts just developed will be applied to evolutionary equations next.

9.3 Evolutionary Equations in Distribution Spaces

In this section, we will specialise the results from the previous section and provide an extension of the solution theory in $L_{2,\nu}(\mathbb{R}; H)$. For this, and throughout this whole section, we let H be a Hilbert space, $\mu \in \mathbb{R}$ and $M\colon \mathbb{C}_{\mathrm{Re}>\mu} \to L(H)$ be a material law. Furthermore, let $\nu > \max\{\mathrm{s_b}\,(M)\,, 0\}$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. Assume that there exists $c > 0$ such that

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

We recall from Theorem 6.2.1 that the operator $\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A}$ is continuously invertible in $L_{2,\nu}(\mathbb{R}; H)$. In order to keep track of the Hilbert spaces involved, we shall put

$$H_\nu^{-1}(\mathbb{R}; H) \coloneqq \operatorname{dom}(\partial_{t,\nu})' \cong \operatorname{dom}(\partial_{t,\nu}^*)'.$$

Proposition 9.3.1. *Let H be a Hilbert space. Let $D\colon \operatorname{dom}(D) \subseteq H \to H$ be densely defined and closed and $B \in L(H)$. Assume that DB is densely defined. Then for all $\phi \in H$, $(DB)_{-1}(\phi) = (D_{-1}B)(\phi)$ on $\operatorname{dom}(D^*)$.*

Proof. First of all, note that $(DB)^* = \overline{B^*D^*}$, by Theorem 2.3.4. Next, let $\phi \in H$ and $x \in \operatorname{dom}(D^*)$. Then

$$\begin{aligned}((DB)_{-1}\phi)(x) &= \langle \phi, (DB)^*x\rangle = \left\langle \phi, \overline{B^*D^*}x\right\rangle \\ &= \langle \phi, B^*D^*x\rangle = \langle B\phi, D^*x\rangle = (D_{-1}B\phi)(x).\end{aligned}$$ □

The first application of the theory developed in the previous section reads as follows.

Theorem 9.3.2. *Let $U, F \in L_{2,\nu}(\mathbb{R};H)$. Then the following statements are equivalent:*

(i) $U \in \operatorname{dom}(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A})$ *and* $\left(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A}\right)U = F$.

(ii) $\partial_{t,\nu}M(\partial_{t,\nu})U + AU \subseteq F$ *where the left-hand side is considered as an element of* $H_\nu^{-1}(\mathbb{R};H) \cap L_{2,\nu}(\mathbb{R};H^{-1}(A)) \subseteq \operatorname{Fun}(L_{2,\nu}(\mathbb{R};H))$.

Proof. Consider the inclusion

$$(\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U + A_{-1}U \subseteq F. \tag{9.3}$$

We show next that (9.3) is equivalent to (ii) by applying Proposition 9.3.1 to the case $D = \partial_{t,\nu}, B = M(\partial_{t,\nu})$. For this assume that (9.3) holds. By Proposition 9.3.1, we deduce that

$$((\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U+A_{-1}U)|_{\operatorname{dom}(\partial_{t,\nu}^*)\cap\operatorname{dom}(A)} = ((\partial_{t,\nu})_{-1}M(\partial_{t,\nu})U+A_{-1}U)|_{\operatorname{dom}(\partial_{t,\nu}^*)\cap\operatorname{dom}(A)}$$

Thus, (9.3) implies (ii).
On the other hand, assume that (ii) holds. Let $\phi \in \operatorname{dom}((\partial_{t,\nu}M(\partial_{t,\nu}))^*) \cap L_{2,\nu}(\mathbb{R};\operatorname{dom}(A))$. Then, for $\varepsilon > 0$, $\phi_\varepsilon := (1+\varepsilon\partial_{t,\nu}^*)^{-1}\phi \in \operatorname{dom}(\partial_{t,\nu}^*) \cap L_{2,\nu}(\mathbb{R};\operatorname{dom}(A))$. By Lemma 9.3.3, we deduce $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$ in $L_{2,\nu}(\mathbb{R};\operatorname{dom}(A))$ and

$$(\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi_\varepsilon = (1+\varepsilon\partial_{t,\nu}^*)^{-1}(\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi \to (\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi \quad (\varepsilon \to 0)$$

in $L_{2,\nu}(\mathbb{R};H)$. By (ii) we obtain

$$((\partial_{t,\nu})_{-1}M(\partial_{t,\nu})U + A_{-1}U)(\phi_\varepsilon) = F(\phi_\varepsilon).$$

Using Proposition 9.3.1, we infer

$$((\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U + A_{-1}U)(\phi_\varepsilon) = F(\phi_\varepsilon).$$

Letting $\varepsilon \to 0$, we deduce (9.3).
We are now in the position to apply Theorem 9.2.6 from above to the case $L_{2,\nu}(\mathbb{R};H)$ being the Hilbert space, A the operator in $L_{2,\nu}(\mathbb{R};H)$, $B = \partial_{t,\nu}M(\partial_{t,\nu})$, and $T_n = \left(1 + \frac{1}{n}\partial_{t,\nu}\right)^{-1}$. We need to establish the commutativity properties next. From Hille's theorem in Proposition 3.1.5, we deduce $\partial_{t,\nu}^{-1}A \subseteq A\partial_{t,\nu}^{-1}$. Hence, by induction, we obtain $\partial_{t,\nu}^{-k}A \subseteq A\partial_{t,\nu}^{-k}$ for all $k \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$ we deduce $T_nA \subseteq AT_n$, by Lemma 9.3.3.
Next, we infer $\operatorname{ran}(T_n) \subseteq \operatorname{dom}(\partial_{t,\nu}) \subseteq \operatorname{dom}(B)$ and

$$T_nB \subseteq BT_n$$

for all $n \in \mathbb{N}$ by using the Fourier–Laplace transformation, see also Theorem 5.2.3. The closability of $A + B$ is implied by Theorem 6.2.1. Thus, the assertion is indeed implied by Theorem 9.2.6 once we have proved the following lemma. □

Lemma 9.3.3. *Let $\varepsilon > 0$ and $\nu > 0$. Then with $r \coloneqq 1/(2\nu)$ we have*

$$(1+\varepsilon\partial_{t,\nu})^{-1} = \varepsilon^{-1}\partial_{t,\nu}^{-1}\frac{1}{1+(r/\varepsilon)}\sum_{k=0}^{\infty}\left(\frac{\left(r-\partial_{t,\nu}^{-1}\right)/\varepsilon}{1+(r/\varepsilon)}\right)^k$$

with the series on the right-hand side converging uniformly in $L(L_{2,\nu}(\mathbb{R};H))$. Moreover, we obtain $(1+\varepsilon\partial_{t,\nu})^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ and $(1+\varepsilon\partial_{t,\nu}^*)^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ in the strong operator topology as $\varepsilon \to 0$.

The proof of Lemma 9.3.3 is left as Exercise 9.7.

Theorem 9.3.4. *The operator $S_\nu \coloneqq \left(\overline{\partial_{t,\nu}M(\partial_{t,\nu})+A}\right)^{-1} \in L(L_{2,\nu}(\mathbb{R};H))$ admits a continuous extension to $L(H_\nu^{-1}(\mathbb{R};H))$.*

Proof. We apply Proposition 9.2.9 to $L_{2,\nu}(\mathbb{R};H)$ being the Hilbert space, $T = \partial_{t,\nu}$ and $B = S_\nu$. For this, it remains to prove that $T^{-1}S_\nu = S_\nu T^{-1}$. This however follows from the fact that $z \mapsto S(z) \coloneqq (zM(z)+A)^{-1}$ is a material law and $S(\partial_{t,\nu}) = S_\nu$. □

9.4 Initial Value Problems for Evolutionary Equations

Let H be a Hilbert space, $\mu \in \mathbb{R}$, $M\colon \mathbb{C}_{\mathrm{Re}>\mu} \to L(H)$ a material law, $\nu > \max\{\mathrm{s_b}(M), 0\}$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint. Furthermore, assume there exists $c > 0$ such that

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

In this section we shall focus on the implementation of initial value problems for evolutionary equations. A priori there is no explicit initial condition implemented in the theory established in $L_{2,\nu}(\mathbb{R};H)$. Indeed, choosing $\nu > 0$ we have only an implicit exponential decay condition at $-\infty$. For initial values at 0, we would rather solve the following type of equation. In the situation of the previous section, for a given initial value $U_0 \in H$ we seek to solve the initial value problem

$$\begin{cases}(\partial_{t,\nu}M(\partial_{t,\nu})+A)\,U = 0 & \text{on } (0,\infty),\\ U(0+) = U_0.\end{cases} \tag{9.4}$$

In this generality the initial value problem cannot be solved. Indeed, for $U \in L_{2,\nu}(\mathbb{R};H)$ evaluation at 0 is not well-defined. A way to overcome this difficulty is to weaken the attainment of the initial value. For this, we specialise to the case when

$$M(\partial_{t,\nu}) = M_0 + \partial_{t,\nu}^{-1}M_1$$

with $M_0, M_1 \in L(H)$.

Theorem 9.4.1. *Let $U_0 \in \operatorname{dom}(A)$, $U \in L_{2,\nu}(\mathbb{R};H)$. Then the following statements are equivalent:*

(i) $M_0U - \mathbb{1}_{[0,\infty)}M_0U_0 \in H_\nu^1(\mathbb{R}; H^{-1}(A))$, $\operatorname{spt} U \subseteq [0,\infty)$ *and*

$$\begin{cases} \partial_{t,\nu} M_0 U + M_1 U + AU = 0 & \text{on } (0,\infty)\,, \\ M_0U(0+) = M_0U_0 & \text{in } H^{-1}(A), \end{cases}$$

where the first equality is meant in the sense that

$$\forall \varphi \in H_\nu^1(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A)),\ \operatorname{spt}\varphi \subseteq [0,\infty) : \big(\partial_{t,\nu}M_0U + M_1U + AU\big)(\varphi) = 0.$$

(ii) $\big(\overline{\partial_{t,\nu}M_0 + M_1 + A}\big)(U - \mathbb{1}_{[0,\infty)}U_0) = -(M_1 + A)U_0\mathbb{1}_{[0,\infty)}$.

(iii) $U = S_\nu \delta_0 M_0 U_0$, *with* $S_\nu \in L(H_\nu^{-1}(\mathbb{R}; H))$ *as in Theorem 9.3.4.*

Proof. (i)⇒(ii): We apply Theorem 9.3.2 for showing the claim. Let $\psi \in H_\nu^1(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H)$ and for $n \in \mathbb{N}$ we define the function $\varphi_n \in H_\nu^1(\mathbb{R})$ by

$$\varphi_n(t) := \begin{cases} 0 & \text{if } t \leqslant 0, \\ nt & \text{if } t \in (0, 1/n)\,, \\ 1 & \text{if } t \geqslant 1/n. \end{cases}$$

Note that $\varphi_n\psi \in H_\nu^1(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H \operatorname{dom}(A))$ and $\operatorname{spt}(\varphi_n\psi) \subseteq [0,\infty)$ for each $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned}
&\big((\partial_{t,\nu}M_0 + M_1 + A)(U - \mathbb{1}_{[0,\infty)}U_0)\big)(\psi) \\
&= \big((\partial_{t,\nu}M_0 + M_1 + A)U\big)(\psi) - \big((\partial_{t,\nu}M_0 + M_1 + A)(\mathbb{1}_{[0,\infty)}U_0)\big)(\psi) \\
&= \big((\partial_{t,\nu}M_0 + M_1 + A)U\big)(\varphi_n\psi) + \big((\partial_{t,\nu}M_0 + M_1 + A)U\big)((1-\varphi_n)\psi) - \\
&\quad - \big((\partial_{t,\nu}M_0 + M_1 + A)(\mathbb{1}_{[0,\infty)}U_0)\big)(\psi) \\
&= \big((\partial_{t,\nu}M_0 + M_1 + A)U\big)((1-\varphi_n)\psi) - (\delta_0M_0U_0)(\psi) - \big((M_1 + A)(\mathbb{1}_{[0,\infty)}U_0)\big)(\psi)
\end{aligned}$$

for each $n \in \mathbb{N}$. Thus, the claim follows if we can show that

$$\big((\partial_{t,\nu}M_0 + M_1 + A)U\big)((1-\varphi_n)\psi) - (\delta_0M_0U_0)(\psi) \to 0 \quad (n \to \infty).$$

For doing so, we first observe that for all $n \in \mathbb{N}$ we have

$$(\delta_0M_0U_0)(\psi) = (\delta_0M_0U_0)((1-\varphi_n)\psi) = (\partial_{t,\nu}M_0\mathbb{1}_{[0,\infty)}U_0)((1-\varphi_n)\psi),$$

since $\varphi_n(0) = 0$. Moreover,

$$\big((M_1 + A)U\big)((1-\varphi_n)\psi) = \langle U, (1-\varphi_n)(M_1^* + A^*)\psi\rangle_{L_{2,\nu}} \to 0 \quad (n \to \infty),$$

since $1 - \varphi_n(\mathrm{m}) \to \mathbb{1}_{(-\infty,0]}(\mathrm{m})$ stongly in $L_{2,\nu}(\mathbb{R}; H)$ and $\operatorname{spt} U \subseteq [0,\infty)$. Thus, it remains to show that

$$\big(\partial_{t,\nu}M_0(U - \mathbb{1}_{[0,\infty)}U_0)\big)((1-\varphi_n)\psi) \to 0 \quad (n \to \infty).$$

We compute

$$\begin{aligned}
&\big(\partial_{t,\nu}M_0(U-\mathbb{1}_{[0,\infty)}U_0)\big)((1-\varphi_n)\psi)\\
&=\left\langle M_0(U-\mathbb{1}_{[0,\infty)}U_0),\partial_{t,\nu}^*((1-\varphi_n)\psi)\right\rangle_{L_{2,\nu}}\\
&=\left\langle M_0(U-\mathbb{1}_{[0,\infty)}U_0),n\mathbb{1}_{[0,1/n]}\psi\right\rangle_{L_{2,\nu}}-\left\langle M_0(U-\mathbb{1}_{[0,\infty)}U_0),(1-\varphi_n)\partial_{t,\nu}\psi\right\rangle_{L_{2,\nu}}\\
&+2\nu\left\langle M_0(U-\mathbb{1}_{[0,\infty)}U_0),(1-\varphi_n)\psi\right\rangle_{L_{2,\nu}}.
\end{aligned}$$

Note that the last to terms on the right-hand side tend to 0 as $n\to\infty$ since, as above, $1-\varphi_n(\mathrm{m})\to\mathbb{1}_{(-\infty,0]}(\mathrm{m})$ stongly in $L_{2,\nu}(\mathbb{R};H)$ and $\operatorname{spt}U\subseteq[0,\infty)$. For the first term, we observe that

$$\begin{aligned}
&\left|\left\langle M_0(U-\mathbb{1}_{[0,\infty)}U_0),n\mathbb{1}_{[0,1/n]}\psi\right\rangle_{L_{2,\nu}}\right|\\
&\leqslant n\int_0^{1/n}|\langle M_0(U(t)-U_0),\psi(t)\rangle_H|\mathrm{e}^{-2\nu t}\,\mathrm{d}t\\
&\leqslant n\int_0^{1/n}\|M_0(U(t)-U_0)\|_{H^{-1}(A)}\|\psi(t)\|_{\mathrm{dom}(A^*)}\mathrm{e}^{-2\nu t}\,\mathrm{d}t\to 0\quad(n\to\infty),
\end{aligned}$$

since $(M_0U)(t)\to M_0U_0$ in $H^{-1}(A)$ as $t\to 0+$.
(ii)⇒(iii): We have that

$$U-\mathbb{1}_{[0,\infty)}U_0=-S_\nu((M_1+A)\mathbb{1}_{[0,\infty)}U_0).$$

Applying $\partial_{t,\nu}^{-1}$ to both sides of this equality we infer that

$$\begin{aligned}
\partial_{t,\nu}^{-1}(U-\mathbb{1}_{[0,\infty)}U_0)&=-S_\nu((M_1+A)\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0)\\
&=-\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0+S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0),
\end{aligned}$$

which gives

$$\partial_{t,\nu}^{-1}U=S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0)=S_\nu(M_0\mathbb{1}_{[0,\infty)}U_0).$$

Applying $\partial_{t,\nu}$ to both sides and taking into account Theorem 9.3.4, we derive the claim.
(iii)⇒(ii): We do the argument in the proof of (ii)⇒(iii) backwards. First, we apply $\partial_{t,\nu}^{-1}$ to $U=S_\nu(\delta_0M_0U_0)$, which yields

$$\partial_{t,\nu}^{-1}U=\partial_{t,\nu}^{-1}S_\nu(\delta_0M_0U_0)=S_\nu(M_0\mathbb{1}_{[0,\infty)}U_0)=S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0).$$

Thus,

$$\begin{aligned}
\partial_{t,\nu}^{-1}(U-\mathbb{1}_{[0,\infty)}U_0)&=S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0)-\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0\\
&=-S_\nu((M_1+A)\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0).
\end{aligned}$$

An application of $\partial_{t,\nu}$ yields the claim.
(ii),(iii)⇒(i): Since $U=S_\nu(\delta_0M_0U_0)$, we derive that

$$(\partial_{t,\nu}M_0+M_1+A)U\subseteq\delta_0M_0U_0,$$

which in particular yields $(\partial_{t,\nu} M_0 + M_1 + A)U = 0$ on $(0,\infty)$. By (ii) we infer

$$U - \mathbb{1}_{[0,\infty)} U_0 = -S_\nu((M_1 + A)\mathbb{1}_{[0,\infty)} U_0),$$

which shows that $\operatorname{spt}(U - \mathbb{1}_{[0,\infty)} U_0) \subseteq [0,\infty)$ due to causality and hence, $\operatorname{spt} U \subseteq [0,\infty)$. It remains to show that $M_0(U - \mathbb{1}_{[0,\infty)} U_0) \in H^1_\nu(\mathbb{R}; H^{-1}(A))$, since this would imply the continuity of $M_0(U - \mathbb{1}_{[0,\infty)} U_0)$ with values in $H^{-1}(A)$ by Theorem 4.1.2 and thus,

$$M_0(U - \mathbb{1}_{[0,\infty)} U_0)(0+) = M_0(U - \mathbb{1}_{[0,\infty)} U_0)(0-) = 0 \text{ in } H^{-1}(A)$$

since the function is supported on $[0,\infty)$ only. We compute

$$\begin{aligned} M_0(U - \mathbb{1}_{[0,\infty)} U_0) &= -M_0 S_\nu((M_1 + A)\mathbb{1}_{[0,\infty)} U_0) \\ &= \partial_{t,\nu} M_0 S_\nu(\partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)} U_0) \\ &= \partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)} U_0 - (M_1 + A) S_\nu(\partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)} U_0), \end{aligned}$$

and since the right-hand side belongs to $H^1_\nu(\mathbb{R}; H^{-1}(A))$, the assertion follows. □

Remark 9.4.2. By Theorem 9.3.4, we always have $U = S_\nu \delta_0 M_0 U_0 \in H_\nu^{-1}(\mathbb{R}; H)$. This then serves as our generalisation for the initial value problem even if $U_0 \notin \operatorname{dom}(A)$.

The upshot of Theorem 9.4.1 is that we can reformulate initial value problems with the help of our theory only for $L_{2,\nu}$-right-hand sides given $U_0 \in \operatorname{dom}(A)$. Thus, we do not need the detour to extrapolation spaces for being able to solve the initial value problem (9.4) (with an adapted initial condition as in (i)).
Also note that it may seem that U does depend on the 'full information' of U_0 as it is indicated in (ii). In fact, U only depends on the values of U_0 orthogonal to the kernel of M_0 as it is seen in (iii). We conclude this lecture with two examples; the first one is the heat equation, the second example considers Maxwell's equations.

Example 9.4.3 (Initial Value Problems for the Heat Equation)**.** We recall the setting for the heat equation outlined in Theorem 6.2.3. This time, we will use homogeneous Dirichlet boundary conditions for the heat distribution θ. Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $a \in L_\infty(\Omega)^{d\times d}$ with $\operatorname{Re} a(x) \geqslant c > 0$ for a.e. $x \in \Omega$ for some $c > 0$. In this case, we have

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}.$$

For the unknown heat distribution, θ, we ask it to have the initial value $\theta_0 \in \operatorname{dom}(\operatorname{grad}_0)$. Let $\nu > 0$ and $V \in L_{2,\nu}\big(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\big)$ be the unique solution of

$$\big(\overline{\partial_{t,\nu} M_0 + M_1 + A}\big)V = -\,(M_1 + A)\,\mathbb{1}_{[0,\infty)} \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix} = -\mathbb{1}_{[0,\infty)} \begin{pmatrix} 0 \\ \operatorname{grad}_0 \theta_0 \end{pmatrix}.$$

Then $(\theta, q) \coloneqq U \coloneqq V + \mathbb{1}_{[0,\infty)} \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix} \in L_{2,\nu}(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d)$ satisfies (ii) from Theorem 9.4.1. Hence, on $(0,\infty)$ we have

$$\begin{pmatrix} \partial_{t,\nu}\theta \\ a^{-1}q \end{pmatrix} + \begin{pmatrix} \operatorname{div} q \\ \operatorname{grad}_0 \theta \end{pmatrix} = 0$$

and the initial value is attained in the sense that

$$(M_0(\theta, q))(0+) = \begin{pmatrix} \theta(0+) \\ 0 \end{pmatrix} \to \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix} \quad \text{in} \quad H^{-1}(A) = H^{-1}(\operatorname{grad}_0) \times H^{-1}(\operatorname{div}),$$

which follows from Proposition 9.2.5 where we computed $H^{-1}(A)$. Let us have a closer look at the attainment of the initial value. As a particular consequence of strong convergence in $H^{-1}(\operatorname{grad}_0)$, we obtain for all $\phi \in \operatorname{dom}(\operatorname{div})$

$$\langle \theta(t), \operatorname{div}\phi \rangle \to \langle \theta_0, \operatorname{div}\phi \rangle$$

as $t \to 0+$. Since grad_0 is one-to-one and has closed range, we see that div has dense and closed range. Hence div is onto. This implies that for all $\psi \in L_2(\Omega)$

$$\langle \theta(t), \psi \rangle \to \langle \theta_0, \psi \rangle \quad (t \to 0+).$$

We deduce that the initial value is attained weakly. This might seem a bit unsatisfactory, however, we shall see stronger assertions for more particular cases in the next lecture. Next, we have a look at Maxwell's equations.

Example 9.4.4 (Initial Value Problems for Maxwell's equations)**.** We briefly recall the situation of Maxwell's equations from Theorem 6.2.7. Let $\varepsilon, \mu, \sigma \colon \Omega \to \mathbb{R}^{3\times 3}$ satisfy the assumptions in Theorem 6.2.7 and let $(E_0, H_0) \in \operatorname{dom}(\operatorname{curl}_0) \times \operatorname{dom}(\operatorname{curl})$. Let $(\widehat{E}, \widehat{H}) \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^6)$ satisfy

$$\begin{aligned} &\left(\overline{\partial_{t,\nu} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}}\right) \begin{pmatrix} \widehat{E} \\ \widehat{H} \end{pmatrix} \\ &= -\left(\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \right) \mathbb{1}_{[0,\infty)} \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = \mathbb{1}_{[0,\infty)} \begin{pmatrix} -\sigma E_0 + \operatorname{curl} H_0 \\ -\operatorname{curl}_0 E_0 \end{pmatrix}. \end{aligned}$$

Then, as we have argued for the heat equation,

$$\begin{pmatrix} E \\ H \end{pmatrix} \coloneqq \begin{pmatrix} \widehat{E} \\ \widehat{H} \end{pmatrix} + \mathbb{1}_{[0,\infty)} \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}$$

satisfies a corresponding initial value problem. We note here that although often the second component in the right-hand side is set to 0, as there are 'no magnetic monopoles', in the theory of evolutionary equations the second component of the right-hand side does appear as an initial value in disguise.

9.5 Comments

There are many ways to define spaces generalising the action of an operator to a bigger class of elements; both in a concrete setting and in abstract situations; see e.g. [1, 2]. People have also taken into account simultaneous extrapolation spaces for operators that commute, see e.g. [3, 5].

These spaces are particularly useful for formulating initial value problems as was exemplified above; see also the concluding chapter of [4] for more insight. Yet there is more to it as one can in fact generalise the equation under consideration or even force the attainment of the initial value in a stronger sense. These issues, however, imply that either the initial value is attained in a much weaker sense, or that there are other structural assumptions needed to be imposed on the material law M (as well as on the operator A).

In fact, quite recently, it was established that a particular proper subclass of evolutionary equations can be put into the framework of C_0-semigroups. The conditions required to allow for statements in this direction are, on the other hand, rather hard to check in practice; see [6].

Exercises

Exercise 9.1. Let H_0 be a Hilbert space, $T \in L(H_0)$. Compute $H^{-1}(T)$ and $H^{-1}(T^*)$.

Exercise 9.2. Let H_0, H_1 be Hilbert spaces such that $H_0 \hookrightarrow H_1$ is dense and continuous. Prove that $H_1' \hookrightarrow H_0'$ is dense and continuous as well.

Exercise 9.3. Prove the following statement which generalises Proposition 9.2.9 from above: Let H_0 be a Hilbert space, $A \in L(H_0)$. Assume that $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ is densely defined and closed with $0 \in \rho(T)$ and $T^{-1}A = AT^{-1} + T^{-1}BT^{-1}$ for some bounded $B \in L(H_0)$. Then A admits a unique continuous extension, $\overline{A} \in L(H^{-1}(T^*))$.

Exercise 9.4. Let H_0 be a Hilbert space, $N\colon \operatorname{dom}(N) \subseteq H_0 \to H_0$ be a *normal* operator; that is, N is densely defined and closed and $NN^* = N^*N$. Show that $H^{-1}(N) \cong H^{-1}(N^*)$ and deduce $H^{-1}(\partial_{t,\nu}) \cong H^{-1}(\partial_{t,\nu}^*)$.

Exercise 9.5. Prove Proposition 9.2.8.

Exercise 9.6. Let H_0 be a Hilbert space, $n \in \mathbb{N}$ and $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ be a densely defined, closed linear operator with $0 \in \rho(T)$. We define $H^n(T) := \operatorname{dom}(T^n)$ and $H^{-n}(T) := H^{-1}(T^n)$. Show that for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ we have that $H^{k+\ell}(T) \hookrightarrow H^{\ell}(T)$ continuously. Also show that $\mathcal{D} := \bigcap_{n\in\mathbb{N}} \operatorname{dom}(T^n)$ is dense in $H^{\ell}(T)$ for all $\ell \in \mathbb{Z}$ and that $T|_{\mathcal{D}}$ can be continuously extended to a topological isomorphism $H^{\ell}(T) \to H^{\ell-1}(T^*)$ for all $\ell \in \mathbb{Z}$.

Exercise 9.7. Prove Lemma 9.3.3.
Hint: Prove a similar equality with $\partial_{t,\nu}^{-1}$ formally replaced by $z \in \partial B\,(r,r) \subseteq \mathbb{C}$ and deduce the assertion with the help of Theorem 5.2.3.

References

[1] P. Cojuhari and A. Gheondea. 'Closed embeddings of Hilbert spaces'. In: *J. Math. Anal. Appl.* 369.1 (2010), pp. 60–75.

[2] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations.* Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000, pp. xxii+586.

[3] R. S. Palais. *Seminar on the Atiyah-Singer index theorem.* With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57. Princeton University Press, Princeton, N.J., 1965, pp. x+366.

[4] R. Picard and D. McGhee. *Partial Differential Equations: A unified Hilbert Space Approach.* Vol. 55. Expositions in Mathematics. DeGruyter, Berlin, 2011.

[5] R. Picard. 'Evolution equations as operator equations in lattices of Hilbert spaces'. In: *Glas. Mat. Ser. III* 35(55).1 (2000). Dedicated to the memory of Branko Najman, pp. 111–136.

[6] S. Trostorff. 'Exponential Stability and Initial Value Problems for Evolutionary Equations.' Habilitation Thesis. TU Dresden, 2018.