8 Causality and a Theorem of Paley and Wiener

In this lecture we turn our focus back to causal operators. In Lecture 5 we found out that material laws provide a class of causal and autonomous bounded operators. In this lecture we will present another proof of this fact, which rests on a result which characterises functions in L_2 with support contained in the non-negative reals; the celebrated Theorem of Paley and Wiener. With the help of this theorem, which is interesting in its own right, the proof of causality for material laws becomes very easy. At a first glance it seems that holomorphy of a material law is a rather strong assumption. In the second part of this lecture, however, we shall see that in designing autonomous and causal solution operators, there is no way of circumventing holomorphy.
In the following, let H be a Hilbert space.

8.1 A Theorem of Paley and Wiener

We start with the following lemma, for which we need the notion of locally integrable functions. We define

$$\begin{aligned} L_{1,\mathrm{loc}}(\mathbb{R};H) &\coloneqq \{f\,;\,\forall\, K\subseteq\mathbb{R}\text{ compact}:\mathbb{1}_K f\in L_1(\mathbb{R};H)\}\\ &= \{f\,;\,\forall\,\varphi\in C_\mathrm{c}^\infty(\mathbb{R}):\varphi f\in L_1(\mathbb{R};H)\}\,. \end{aligned}$$

Lemma 8.1.1. *Let $f\in L_{1,\mathrm{loc}}(\mathbb{R};H)$. Then we have $f\in L_2(\mathbb{R}_{\geqslant 0};H)$ if and only if $f\in\bigcap_{\nu>0}L_{2,\nu}(\mathbb{R};H)$ with $\sup_{\nu>0}\|f\|_{L_{2,\nu}(\mathbb{R};H)}<\infty$. In the latter case we have that*

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}=\lim_{\nu\to 0+}\|f\|_{L_{2,\nu}(\mathbb{R};H)}=\sup_{\nu>0}\|f\|_{L_{2,\nu}(\mathbb{R};H)}.$$

Proof. Let $f\in L_2(\mathbb{R}_{\geqslant 0};H)$ and $\nu>0$. Then we estimate

$$\int_{\mathbb{R}}\|f(t)\|_H^2\mathrm{e}^{-2\nu t}\,\mathrm{d}t=\int_{\mathbb{R}_{\geqslant 0}}\|f(t)\|_H^2\mathrm{e}^{-2\nu t}\,\mathrm{d}t\leqslant\int_{\mathbb{R}_{\geqslant 0}}\|f(t)\|_H^2\,\mathrm{d}t=\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}^2,$$

which proves that $f\in L_{2,\nu}(\mathbb{R};H)$ with $\|f\|_{L_{2,\nu}(\mathbb{R};H)}\leqslant\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ for each $\nu>0$. Moreover, $\|f\|_{L_{2,\nu}(\mathbb{R};H)}\to\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ as $\nu\to 0$ by monotone convergence and since clearly $\|f\|_{L_{2,\nu}(\mathbb{R};H)}\leqslant\|f\|_{L_{2,\mu}(\mathbb{R};H)}$ for $0<\mu\leqslant\nu$ we obtain

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}=\lim_{\nu\to 0+}\|f\|_{L_{2,\nu}(\mathbb{R};H)}=\sup_{\nu>0}\|f\|_{L_{2,\nu}(\mathbb{R};H)}.$$

Assume now that $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R};H)$ with $C := \sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. We first prove that $\operatorname{spt} f \subseteq \mathbb{R}_{\geqslant 0}$. For doing so, let $a < b < 0$. Then we estimate for $\nu > 0$

$$\begin{aligned}\int_a^b \|f(t)\|_H \,\mathrm{d}t &= \int_a^b \|f(t)\|_H \mathrm{e}^{-\nu t}\mathrm{e}^{\nu t}\,\mathrm{d}t \leqslant \|f\|_{L_{2,\nu}(\mathbb{R};H)} \left(\int_a^b \mathrm{e}^{2\nu t}\,\mathrm{d}t\right)^{1/2} \\ &\leqslant \|f\|_{L_{2,\nu}(\mathbb{R};H)} \sqrt{\frac{\mathrm{e}^{2\nu b} - \mathrm{e}^{2\nu a}}{2\nu}} \leqslant C\mathrm{e}^{\nu b}\sqrt{b-a}.\end{aligned}$$

Since the right-hand side tends to zero as $\nu \to \infty$, we infer that $f = 0$ on $[a,b]$ and thus $\operatorname{spt} f \subseteq \mathbb{R}_{\geqslant 0}$. Moreover, for $n \in \mathbb{N}$ and $\nu > 0$ we estimate

$$\int_0^n \|f(t)\|^2 \,\mathrm{d}t = \int_0^n \|f(t)\|^2 \mathrm{e}^{-2\nu t}\mathrm{e}^{2\nu t}\,\mathrm{d}t \leqslant \mathrm{e}^{2\nu n}\|f\|^2_{L_{2,\nu}(\mathbb{R};H)} \leqslant C^2\mathrm{e}^{2\nu n}.$$

Letting $\nu \to 0$ we infer

$$\int_0^n \|f(t)\|^2 \,\mathrm{d}t \leqslant C \quad (n \in \mathbb{N})$$

and hence, by monotone convergence, we derive that $f \in L_2(\mathbb{R}_{\geqslant 0};H)$. □

For the proof of the Paley–Wiener theorem we need a suitable space of holomorphic functions on the right half-plane, the so-called *Hardy space* $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)$, which we introduce in the following.

Definition. For $\nu \in \mathbb{R}$ we define the Hardy space

$$\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H) := \left\{ g\colon \mathbb{C}_{\mathrm{Re}>\nu} \to H \,;\, g \text{ holomorphic}, \sup_{\rho>\nu} \int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|_H^2 \,\mathrm{d}t < \infty \right\}$$

and equip it with the norm $\|\cdot\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)}$ defined by

$$\|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)} := \sup_{\rho>\nu} \left(\int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|_H^2 \,\mathrm{d}t\right)^{\frac{1}{2}}.$$

We are now able to prove the Theorem of Paley–Wiener, [4]. We mainly follow the proof given in [5, 19.2 Theorem].

Theorem 8.1.2 (Paley–Wiener). *Let $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)$. Then there exists a unique $f \in L_2(\mathbb{R}_{\geqslant 0};H)$ such that*

$$\mathcal{L}_\nu f = g(\mathrm{i}\cdot + \nu) \quad (\nu > 0).$$

Moreover, $\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}$.

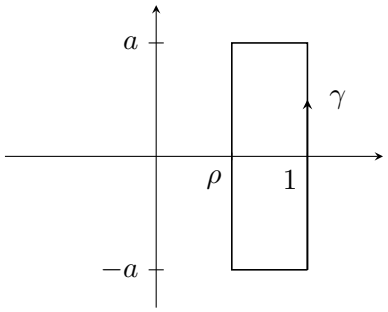

Figure 8.1: Curve γ.

Proof. For $\nu > 0$ we set $g_\nu := g(\mathrm{i}\cdot + \nu) \in L_2(\mathbb{R}; H)$ and $f_\nu := \mathcal{F}^* g_\nu \in L_2(\mathbb{R}; H)$. Moreover, we set $f := \mathrm{e}^{(\cdot)} f_1$. We first prove that $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R}; H)$ with $\sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. For doing so, let $a > 0$, $\rho > 0$ and $x \in \mathbb{R}$. Applying Cauchy's integral theorem to the function $z \mapsto \mathrm{e}^{zx} g(z)$ and the curve γ, as indicated in Figure 8.1, we obtain

$$\begin{aligned} 0 = \mathrm{i} \int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{\rho}^{1} \mathrm{e}^{(\mathrm{i}a+\kappa)x} g(\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \\ - \mathrm{i} \int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t + \int_{\rho}^{1} \mathrm{e}^{(-\mathrm{i}a+\kappa)x} g(-\mathrm{i}a+\kappa)\,\mathrm{d}\kappa. \end{aligned} \tag{8.1}$$

Moreover, since

$$\begin{aligned} \int_{\mathbb{R}} \left\| \int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} g(\pm\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \right\|_H^2 \mathrm{d}a &\leqslant \int_{\mathbb{R}} \left| \int_{\rho}^{1} \left| \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} \right|^2 \mathrm{d}\kappa \int_{\rho}^{1} \|g(\pm\mathrm{i}a+\kappa)\|_H^2\,\mathrm{d}\kappa \right| \mathrm{d}a \\ &\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| \left| \int_{\rho}^{1} \int_{\mathbb{R}} \|g(\pm\mathrm{i}a+\kappa)\|_H^2\,\mathrm{d}a\,\mathrm{d}\kappa \right| \\ &\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| |1-\rho| \|g\|^2_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)} < \infty, \end{aligned}$$

we infer that $\left(a \mapsto \int_\rho^1 \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} g(\pm\mathrm{i}a+\kappa)\,\mathrm{d}\kappa\right) \in L_2(\mathbb{R}; H)$ and thus, we find a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $a_n \to \infty$ and

$$\int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a_n+\kappa)x} g(\pm\mathrm{i}a_n+\kappa)\,\mathrm{d}\kappa \to 0$$

as $n \to \infty$. Hence, using (8.1) with a replaced by a_n and letting n tend to infinity, we

derive that

$$\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t \to 0 \quad (n\to\infty).$$

Noting that for each $\mu > 0$ we have

$$\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\mu)x} g(\mathrm{i}t+\mu)\,\mathrm{d}t = \sqrt{2\pi}\mathrm{e}^{\mu x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]}g_\mu)(x) \quad (x\in\mathbb{R})$$

and that $\mathbb{1}_{[-a_n,a_n]}g_\mu \to g_\mu$ in $L_2(\mathbb{R};H)$ as $n\to\infty$, we may choose a subsequence (again denoted by (a_n)) such that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left(\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t\right)\\
&= \lim_{n\to\infty}\left(\sqrt{2\pi}\mathrm{e}^{x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]}g_1)(x) - \sqrt{2\pi}\mathrm{e}^{\rho x}\mathcal{F}^*(1_{[-a_n,a_n]}g_\rho)(x)\right)\\
&= \sqrt{2\pi}\big(\mathrm{e}^{x}f_1(x) - \mathrm{e}^{\rho x}f_\rho(x)\big)
\end{aligned}$$

for almost every $x\in\mathbb{R}$. Hence, $f = \mathrm{e}^{(\cdot)}f_1 = \exp(\rho\mathrm{m})f_\rho$ for each $\rho>0$ and thus,

$$\int_{\mathbb{R}} \|f(t)\|_H^2 \mathrm{e}^{-2\rho t}\,\mathrm{d}t = \int_{\mathbb{R}} \|f_\rho(t)\|_H^2\,\mathrm{d}t < \infty$$

which shows $f\in\bigcap_{\rho>0} L_{2,\rho}(\mathbb{R};H)$ with

$$\sup_{\rho>0}\|f\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>0}\|f_\rho\|_{L_2(\mathbb{R};H)} = \sup_{\rho>0}\|g_\rho\|_{L_2(\mathbb{R};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}.$$

Thus, $f\in L_2(\mathbb{R}_{\geqslant 0};H)$ with $\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}$ by Lemma 8.1.1. Moreover,

$$\mathcal{L}_\nu f = \mathcal{F}\exp(-\nu\mathrm{m})f = \mathcal{F}\exp(-\nu m)\exp(\nu\mathrm{m})f_\nu = \mathcal{F}f_\nu = g_\nu = g(\mathrm{i}\cdot+\nu)$$

for each $\nu>0$, which shows the claim. □

Corollary 8.1.3. *Let $\nu\in\mathbb{R}$. Then the mapping*

$$\begin{aligned}
\mathcal{L}\colon L_{2,\nu}(\mathbb{R}_{\geqslant 0};H) &\to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)\\
f &\mapsto \big(z\mapsto (\mathcal{L}_{\mathrm{Re}\,z}f)(\mathrm{Im}\,z)\big)
\end{aligned}$$

is a surjective isometry. In particular, $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)$ is a Hilbert space.

Proof. We first prove that the mapping is well-defined. For $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ we have that $\big(t \mapsto \mathrm{e}^{-\rho t} f(t)\big) \in L_1(\mathbb{R}_{\geqslant 0}; H) \cap L_2(\mathbb{R}_{\geqslant 0}; H)$ for all $\rho > \nu$ and thus,

$$(\mathcal{L}f)(\mathrm{i}t + \rho) = (\mathcal{L}_\rho f)(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\rho)s} f(s)\,\mathrm{d}s \quad (t \in \mathbb{R}, \rho > \nu).$$

Hence, $\mathcal{L}f \colon \mathbb{C}_{\mathrm{Re}>\nu} \to H$ is holomorphic (cf. Exercise 5.6). Moreover, by Lemma 8.1.1

$$\begin{aligned}\sup_{\rho>\nu} \|\mathcal{L}f(\mathrm{i}\cdot + \rho)\|_{L_2(\mathbb{R};H)} &= \sup_{\rho>\nu} \|\mathcal{L}_\rho f\|_{L_2(\mathbb{R};H)} = \sup_{\rho>\nu} \|f\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>0} \big\|\mathrm{e}^{-\nu\cdot} f\big\|_{L_{2,\rho}(\mathbb{R};H)} \\ &= \big\|\mathrm{e}^{-\nu\cdot} f\big\|_{L_2(\mathbb{R};H)} = \|\exp(-\nu\mathrm{m}) f\|_{L_2(\mathbb{R};H)} = \|f\|_{L_{2,\nu}(\mathbb{R};H)},\end{aligned}$$

which proves that $\mathcal{L}$ is well-defined and isometric. Let now $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ and define $\widetilde{g}(z) := g(z + \nu)$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$. Then $\widetilde{g} \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0}; H)$ and thus, Theorem 8.1.2 yields the existence of $\widetilde{f} \in L_2(\mathbb{R}_{\geqslant 0}; H)$ with

$$g(\mathrm{i}\cdot + \rho) = \widetilde{g}(\mathrm{i}\cdot + \rho - \nu) = \mathcal{L}_{\rho-\nu}\widetilde{f} = \mathcal{L}_\rho\big(\mathrm{e}^{\nu\cdot}\widetilde{f}\big) \quad (\rho > \nu).$$

Hence, setting $f := \mathrm{e}^{\nu\cdot}\widetilde{f} \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$, we obtain $\mathcal{L}f = g$. □

We can now provide an alternative proof of Theorem 5.3.5 by proving causality with the help of the Theorem of Paley–Wiener.

Proposition 8.1.4. *Let $M \colon \mathrm{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then for $\nu > \mathrm{s_b}(M)$ we have $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$ and $M(\partial_{t,\nu})$ is causal and autonomous (see Exercise 5.7).*

Proof. Let $\nu > \mathrm{s_b}(M)$. Then $M \colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \to L(H)$ is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$. Hence, by unitary equivalence, $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$. Moreover, $M(\partial_{t,\nu})$ is autonomous by Exercise 5.7. Thus, for causality it suffices to check that $\operatorname{spt} M(\partial_{t,\nu}) f \subseteq \mathbb{R}_{\geqslant 0}$ whenever $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. So let $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. Then $\mathcal{L}f \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ by Corollary 8.1.3 and since M is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$, we infer also that

$$\big(z \mapsto M(z)(\mathcal{L}f)(z)\big) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H).$$

Again by Corollary 8.1.3 there exists $g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\mathcal{L}g(z) = M(z)(\mathcal{L}f)(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

Thus, in particular

$$\mathcal{L}_\rho g = M(\mathrm{im} + \rho)\mathcal{L}_\rho f \quad (\rho > \nu).$$

Since $f, g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ we infer that $\mathcal{L}_\rho g \to \mathcal{L}_\nu g$ and $\mathcal{L}_\rho f \to \mathcal{L}_\nu f$ in $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ by dominated convergence. Moreover, $M(\mathrm{im} + \rho) \to M(\mathrm{im} + \nu)$ strongly on $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ (cf. Exercise 8.2). Hence, we derive

$$\mathcal{L}_\nu g = M(\mathrm{im} + \nu)\mathcal{L}_\nu f,$$

and thus, $g = M(\partial_{t,\nu}) f$ which shows the causality. □

8.2 A Representation Result

In this section we argue that our solution theory needs holomorphy as a central property for the material law. The main theorem of this section reads as follows:

Theorem 8.2.1. *Let $\nu_0 \in \mathbb{R}$ and let $T \in L(L_{2,\nu_0}(\mathbb{R};H))$ be causal and autonomous. Then $T|_{L_{2,\nu_0}\cap L_{2,\nu}}$ has a unique extension $T_\nu \in L(L_{2,\nu}(\mathbb{R};H))$ for each $\nu > \nu_0$ and there exists a unique $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ holomorphic and bounded such that $T_\nu = M(\partial_{t,\nu})$ for each $\nu > \nu_0$.*

We start with the following lemma.

Lemma 8.2.2. *Let $f \in L_2(\mathbb{R}_{\geqslant 0};H)$ and $z \in \mathbb{C}_{\mathrm{Re}>0}$ such that for each $h \geqslant 0$*

$$f(t+h) = \mathrm{e}^{-zh} f(t) \quad (\text{a.e. } t \in \mathbb{R}_{\geqslant 0}).$$

Then $f \in L_1(\mathbb{R}_{\geqslant 0};H)$ and

$$f(t) = \mathrm{e}^{-zt} z \int_0^\infty f(s)\,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}_{\geqslant 0}).$$

Proof. We compute

$$\begin{aligned}\int_0^\infty \|f(t)\|_H\,\mathrm{d}t &= \sum_{n=0}^\infty \int_n^{n+1} \|f(t)\|_H\,\mathrm{d}t = \sum_{n=0}^\infty \int_0^1 \|f(t+n)\|_H\,\mathrm{d}t \\ &= \sum_{n=0}^\infty \int_0^1 \left|\mathrm{e}^{-zn}\right| \|f(t)\|_H\,\mathrm{d}t \leqslant \|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} \sum_{n=0}^\infty \mathrm{e}^{-\mathrm{Re}\,z\cdot n} < \infty,\end{aligned}$$

and thus, $f \in L_1(\mathbb{R}_{\geqslant 0};H)$. We define the function $F\colon \mathbb{R}_{\geqslant 0} \to H$ by $F(t) := \int_t^\infty f(r)\,\mathrm{d}r$. Then F is continuous and

$$F(t) = \int_t^\infty f(r)\,\mathrm{d}r = \int_0^\infty f(t+r)\,\mathrm{d}r = \mathrm{e}^{-zt} F(0) \quad (t \in \mathbb{R}_{\geqslant 0}).$$

Hence, F is continuously differentiable with $F'(t) = -z\mathrm{e}^{-zt}F(0)$. We will prove that $F'(t) = -f(t)$ for almost every $t \in \mathbb{R}_{\geqslant 0}$, which then yields the assertion. For doing so, for $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}_{>0};H)$ we compute

$$\begin{aligned}\int_0^\infty \big\langle F'(t), \varphi(t)\big\rangle_H\,\mathrm{d}t &= -\int_0^\infty \big\langle F(t), \varphi'(t)\big\rangle_H\,\mathrm{d}t = -\int_0^\infty \left\langle \int_t^\infty f(r)\,\mathrm{d}r, \varphi'(t)\right\rangle_H \mathrm{d}t \\ &= -\int_0^\infty \left\langle f(r), \int_0^r \varphi'(t)\,\mathrm{d}t\right\rangle_H \mathrm{d}r = \int_0^\infty \big\langle -f(r), \varphi(r)\big\rangle_H\,\mathrm{d}r.\end{aligned}$$

Since $F', f \in L_2(\mathbb{R}_{\geqslant 0};H)$ and $C_\mathrm{c}^\infty(\mathbb{R}_{>0};H)$ is dense in $L_2(\mathbb{R}_{\geqslant 0};H)$ we derive $F' = -f$, which completes the proof. □

Remark 8.2.3. The formula $F' = -f$ also follows from Lebesgue's differentiation theorem (cf. [5, Theorem 7.7] or [1, Corollary 2.13]), which, however, we do not want to prove here.

With this lemma at hand, we are able to prove the following theorem, which will be the key argument in the proof of Theorem 8.2.1.

Theorem 8.2.4. *Let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. Then there exists $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that*

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Proof. For $z \in \mathbb{C}_{\mathrm{Re}>0}$ and $y \in H$ we define

$$g_{z,y}(t) := \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(t)\mathrm{e}^{-zt}y \quad (t \in \mathbb{R}).$$

Then $g_{z,y} \in L_2(\mathbb{R}_{\geqslant 0}; H)$. We have that

$$\left(\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})\tau_h g_{z,y}\right)(t) = \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(t)g_{z,y}(t+h) = \mathrm{e}^{-zh}g_{z,y}(t)$$

for each $h \geqslant 0$. Moreover, since T is assumed to be causal, we have that $\mathbb{1}_{\mathbb{R}_{\leqslant 0}}(\mathrm{m})T = \mathbb{1}_{\mathbb{R}_{\leqslant 0}}(\mathrm{m})T\mathbb{1}_{\mathbb{R}_{\leqslant 0}}(\mathrm{m})$ and thus,

$$\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) = T\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) - \mathbb{1}_{\mathbb{R}_{\leqslant 0}}(\mathrm{m})T\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) = T\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}).$$

Taking adjoints, we derive that $\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) = \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*$. Using that T^* is also autonomous, for each $h \geqslant 0$ we compute

$$\begin{aligned}\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})\tau_h\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*g_{z,y} &= \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})\mathbb{1}_{\mathbb{R}_{\geqslant -h}}(\mathrm{m})T^*\tau_h g_{z,y} = \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*\tau_h g_{z,y}\\ &= \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})\tau_h g_{z,y} = \mathrm{e}^{-zh}\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*g_{z,y}.\end{aligned}$$

In other words, for each $h \geqslant 0$

$$\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(t+h)(T^*g_{z,y})(t+h) = \mathrm{e}^{-zh}\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(t)(T^*g_{z,y})(t) \quad (\text{a.e. } t \in \mathbb{R}_{\geqslant 0}),$$

that is, $\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m})T^*g_{z,y}$ satisfies the assumption of Lemma 8.2.2 and thus,

$$(T^*g_{z,y})(t) = \mathrm{e}^{-zt}z\int_0^\infty (T^*g_{z,y})(s)\,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}_{\geqslant 0}). \tag{8.2}$$

For $z \in \mathbb{C}_{\mathrm{Re}>0}$ we define the mapping

$$\begin{aligned}L(z)\colon H &\to H\\ y &\mapsto z\int_0^\infty (T^*g_{z,y})(s)\,\mathrm{d}s.\end{aligned}$$

Then $L(z)$ is linear, and by (8.2) for $y \in H$ we obtain

$$T^* g_{z,y}(t) = \mathrm{e}^{-zt} L(z)y \quad (\text{a.e. } t \in \mathbb{R}).$$

Therefore,

$$\begin{aligned}\left\|\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) T^* g_{z,y}\right\|^2_{L_2(\mathbb{R};H)} &= \int_0^\infty \left\|\left(T^* g_{z,y}\right)(t)\right\|_H^2 \,\mathrm{d}t = \int_0^\infty \left|\mathrm{e}^{-zt}\right|^2 \|L(z)y\|_H^2 \,\mathrm{d}t \\ &= \|L(z)y\|_H^2 \frac{1}{2\operatorname{Re} z}.\end{aligned}$$

Hence,

$$\|L(z)y\|_H = \sqrt{2\operatorname{Re} z}\left\|\mathbb{1}_{\mathbb{R}_{\geqslant 0}}(\mathrm{m}) T^* g_{z,y}\right\|_{L_2(\mathbb{R};H)} \leqslant \sqrt{2\operatorname{Re} z}\|T^*\| \|g_{z,y}\|_{L_2(\mathbb{R};H)} = \|T\| \|y\|_H,$$

which proves that $L(z) \in L(H)$ with $\|L(z)\| \leqslant \|T\|$ for each $z \in \mathbb{C}_{\mathrm{Re}>0}$. Let now $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$. Then $Tf \in L_2(\mathbb{R}_{\geqslant 0}; H)$ by causality and for $y \in H, z \in \mathbb{C}_{\mathrm{Re}>0}$ we compute

$$\begin{aligned}\langle y, (\mathcal{L}Tf)(z)\rangle_H &= \frac{1}{\sqrt{2\pi}} \left\langle y, \int_0^\infty \mathrm{e}^{-zt} (Tf)(t) \,\mathrm{d}t \right\rangle_H = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left\langle \mathrm{e}^{-z^* t} y, (Tf)(t) \right\rangle_H \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \langle g_{z^*,y}, Tf\rangle_{L_2(\mathbb{R};H)} = \frac{1}{\sqrt{2\pi}} \langle T^* g_{z^*,y}, f\rangle_{L_2(\mathbb{R};H)} \\ &= \frac{1}{\sqrt{2\pi}} \left\langle \mathrm{e}^{-z^*(\cdot)} L(z^*)y, f \right\rangle_{L_2(\mathbb{R};H)} = \langle L(z^*)y, (\mathcal{L}f)(z)\rangle_H \\ &= \langle y, L(z^*)^* (\mathcal{L}f)(z)\rangle_H .\end{aligned}$$

Hence, setting $M(z) := L(z^*)^* \in L(H)$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$, we derive

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>0}, f \in L_2(\mathbb{R}_{\geqslant 0}; H)).$$

Since $\|M(z)\| = \|L(z^*)\| \leqslant \|T\|$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$, we obtain the boundedness of M. We now prove the holomorphy of M. For doing so, we apply the latter formula to $f = \sqrt{2\pi} g_{1,y}$ for $y \in H$. Then $(\mathcal{L}f)(z) = \frac{1}{z+1} y$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$ and hence,

$$M(z)y = (z+1)(\mathcal{L}Tf)(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Since the right-hand side is holomorphic by Corollary 8.1.3, $M(\cdot)y$ is holomorphic for each $y \in H$ and thus, M is holomorphic by Exercise 5.3. □

We can now prove our main result of this section.

Proof of Theorem 8.2.1. We just prove the existence of a function M. The proof of its uniqueness is left as Exercise 8.3.

We first prove the assertion for $\nu_0 = 0$. So, let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. According to Theorem 8.2.4 we find $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Let now $\varphi \in C_c^\infty(\mathbb{R}; H)$ and set $a := \inf \operatorname{spt} \varphi$. Then $\tau_a \varphi \in L_2(\mathbb{R}_{\geqslant 0}; H)$, and for $\nu > 0$ we compute

$$\begin{aligned} \mathcal{L}_\nu T\varphi &= \mathcal{L}_\nu \tau_{-a} T \tau_a \varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a} \mathcal{L}_\nu T \tau_a \varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a} M(\mathrm{im}+\nu) \mathcal{L}_\nu \tau_a \varphi \\ &= M(\mathrm{im}+\nu)\mathcal{L}_\nu \varphi. \end{aligned} \tag{8.3}$$

The latter implies

$$\|T\varphi\|_{L_{2,\nu}(\mathbb{R};H)} = \|\mathcal{L}_\nu T\varphi\|_{L_2(\mathbb{R};H)} = \|M(\mathrm{im}+\nu)\mathcal{L}_\nu\varphi\|_{L_2(\mathbb{R};H)} \leqslant \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>0}} \|\varphi\|_{L_{2,\nu}(\mathbb{R};H)}$$

and hence, $T|_{C_c^\infty(\mathbb{R};H)}$ has a unique continuous extension $T_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$. Using (8.3) we obtain

$$T_\nu = \mathcal{L}_\nu^* M(\mathrm{im}+\nu)\mathcal{L}_\nu = M(\partial_{t,\nu})$$

by approximation.
Let now $\nu_0 \in \mathbb{R}$. Then the operator

$$\widetilde{T} := \mathrm{e}^{-\nu_0 \mathrm{m}} T \mathrm{e}^{\nu_0 \mathrm{m}} \in L(L_2(\mathbb{R}; H))$$

is causal and autonomous as well. Thus, $\widetilde{T}|_{C_c^\infty(\mathbb{R};H)}$ has continuous extensions $\widetilde{T}_\rho \in L(L_{2,\rho}(\mathbb{R}; H))$ for each $\rho > 0$ and there is $\widetilde{M}\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that $\widetilde{T}_\rho = \widetilde{M}(\partial_{t,\rho})$ for each $\rho > 0$. Using $T|_{C_c^\infty(\mathbb{R};H)} = \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}|_{C_c^\infty(\mathbb{R};H)} \mathrm{e}^{-\nu_0 \mathrm{m}}$, we derive that $T|_{C_c^\infty(\mathbb{R};H)}$ has the unique continuous extension $T_\nu = \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} \in L(L_{2,\nu}(\mathbb{R}; H))$ for each $\nu > \nu_0$ and

$$\begin{aligned} \mathcal{L}_\nu T_\nu &= \mathcal{L}_\nu \mathrm{e}^{\nu_0 \mathrm{m}} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} = \mathcal{L}_{\nu-\nu_0} \widetilde{T}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} = \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_{\nu-\nu_0} \mathrm{e}^{-\nu_0 \mathrm{m}} \\ &= \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_\nu. \end{aligned}$$

Hence,

$$T_\nu = M(\partial_{t,\nu})$$

for the holomorphic and bounded function M given by $M(z) := \widetilde{M}(z - \nu_0)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. □

8.3 Comments

The stated Theorem of Paley and Wiener is of course not the only theorem characterising properties of the support of L_2-functions in terms of their Fourier or Laplace transform. For instance, a similar result holds for functions having compact support, see e.g. [5, 19.3 Theorem] and Exercise 8.7. These theorems provide a nice connection between L_2-functions and spaces of holomorphic functions in the form of the so-called Hardy spaces.

In this lecture we just introduced the Hardy space $\mathcal{H}_2$ and it is not surprising that there are also the Hardy spaces $\mathcal{H}_p$ for $1 \leqslant p \leqslant \infty$. We refer to [2] for this topic.
The representation result presented in the second part of this lecture was originally proved by Fourès and Segal in 1955, [3]. In this article the authors prove an analogous representation result for causal operators on $L_2(\mathbb{R}^d; H)$, where causality is defined with respect to a closed and convex cone on $\mathbb{R}^d$. The quite elementary proof for $d = 1$ presented here is due to Weiss in 1991, [6].

Exercises

Exercise 8.1. Let $\Lambda \subseteq \mathbb{R}_{>0}$ be a set with an accumulation point in $\mathbb{R}_{>0}$. Prove that $\{(x \mapsto \mathrm{e}^{-\lambda x}) \,;\, \lambda \in \Lambda\}$ is a total set in $L_1(\mathbb{R}_{\geqslant 0})$.
Hint: Use that the set is total if and only if

$$\forall f \in L_\infty(\mathbb{R}_{\geqslant 0}) : \left(\forall \lambda \in \Lambda : \int_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x = 0 \Rightarrow f = 0 \right).$$

Exercise 8.2. Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Moreover, let $\nu > \mathrm{s_b}(M)$. Show that $\lim_{\rho \to \nu+} M(\mathrm{i}m + \rho) = M(\mathrm{i}m + \nu)$ where the limit is meant in the strong operator topology on $L_2(\mathbb{R}; H)$.

Exercise 8.3. Prove the uniqueness statement in Theorem 8.2.1.

Exercise 8.4. Give an example of a continuous and bounded function $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ such that the corresponding operator $M(\partial_{t,\nu})$ is not causal for any $\nu > 0$.

Exercise 8.5. Prove the following distributional variant of the Paley–Wiener theorem: Let $\nu_0 > 0$, $k \in \mathbb{N}$, $f\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathbb{C}$, and set $h(z) := \frac{1}{z^k} f(z)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. We assume that $h \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; \mathbb{C})$. For $\nu > \nu_0$ we define the distribution $u\colon C_\mathrm{c}^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$u(\psi) := \left\langle \mathcal{L}_\nu^* h(\mathrm{i}\cdot + \nu), \partial_{t,\nu}^* \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} \quad (\psi \in C_\mathrm{c}^\infty(\mathbb{R}; \mathbb{C})).$$

Prove that $\operatorname{spt} u \subseteq \mathbb{R}_{\geqslant 0}$, where

$$\operatorname{spt} u := \mathbb{R} \setminus \bigcup \{U \subseteq \mathbb{R} \text{ open} \,;\, \forall\, \psi \in C_\mathrm{c}^\infty(U; \mathbb{C}) : u(\psi) = 0\}.$$

What is u if $f = \mathbb{1}_{\mathbb{C}_{\mathrm{Re}>\nu_0}}$?

Exercise 8.6. Let $g \in L_2(\mathbb{R}), a > 0$ such that $\operatorname{spt} g \subseteq [-a, a]$. Show that $f := \mathcal{F}g$ extends to a holomorphic function $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ with $\widetilde{f}(\mathrm{i}t) = f(t)$ for each $t \in \mathbb{R}$ such that

$$\exists C \geqslant 0 \, \forall z \in \mathbb{C} : \, |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}.$$

Exercise 8.7. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic such that

(a) $\exists C \geqslant 0,\, a > 0 \, \forall z \in \mathbb{C} : \, |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}$,

(b) $f(\mathrm{i}\cdot) \in L_2(\mathbb{R})$.

Prove that $g := \mathcal{F}^* f(\mathrm{i}\cdot)$ satisfies $\operatorname{spt} g \subseteq [-a, a]$.
Hint: Apply Theorem 8.1.2 to the function $h : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}$ given by

$$h(z) := \mathrm{e}^{-za} \frac{f(z)}{z+1} \quad (z \in \mathbb{C}_{\mathrm{Re}>0})$$

to derive that $\operatorname{spt} g \subseteq \mathbb{R}_{\geqslant -a}$.
Note: The assertion even holds true if one replaces condition (a) by

$$\exists C \geqslant 0,\, a > 0\, \forall z \in \mathbb{C} :\, |f(z)| \leqslant C\mathrm{e}^{a|z|}.$$

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