6 Solution Theory for Evolutionary Equations

In this lecture, we shall discuss and present the first major result of this year's internet seminar: Picard's theorem on the solution theory for evolutionary equations which is the main result of [6]. In order to stress the applicability of this theorem, we shall deal with applications first and provide a proof of the actual result afterwards. With an initial interest in applications in mind, we start off with the introduction of some vector-analytic operators.

6.1 First Order Sobolev Spaces

Throughout this section let $\Omega \subseteq \mathbb{R}^d$ be an open set.

Definition. We define

$$\begin{aligned} \operatorname{grad}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^d \\ \phi &\mapsto (\partial_j \phi)_{j \in \{1,\dots,d\}}, \end{aligned}$$

$$\begin{aligned} \operatorname{div}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega) \\ (\phi_j)_{j \in \{1,\dots,d\}} &\mapsto \sum_{j \in \{1,\dots,d\}} \partial_j \phi_j, \end{aligned}$$

and if $d = 3$,

$$\begin{aligned} \operatorname{curl}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^3 \subseteq L_2(\Omega)^3 &\to L_2(\Omega)^3 \\ (\phi_j)_{j \in \{1,2,3\}} &\mapsto \begin{pmatrix} \partial_2 \phi_3 - \partial_3 \phi_2 \\ \partial_3 \phi_1 - \partial_1 \phi_3 \\ \partial_1 \phi_2 - \partial_2 \phi_1 \end{pmatrix}. \end{aligned}$$

Furthermore, we put

$$\operatorname{div} := -\operatorname{grad}_{\mathrm{c}}^*, \quad \operatorname{grad} := -\operatorname{div}_{\mathrm{c}}^*, \quad \operatorname{curl} := \operatorname{curl}_{\mathrm{c}}^*$$

and

$$\operatorname{div}_0 := -\operatorname{grad}^*, \quad \operatorname{grad}_0 := -\operatorname{div}^*, \quad \operatorname{curl}_0 := \operatorname{curl}^*.$$

Proposition 6.1.1. *The relations* $\operatorname{div}, \operatorname{div}_0, \operatorname{grad}, \operatorname{grad}_0, \operatorname{curl}$ *and* curl_0 *are all densely defined, closed linear operators.*

Proof. The operators $\operatorname{grad}_c, \operatorname{div}_c$ and curl_c are densely defined by Exercise 6.3. Thus, $\operatorname{div}, \operatorname{grad}$ and curl are closed linear operators. Moreover, it follows from integration by parts that $\operatorname{grad}_c \subseteq \operatorname{grad}$, $\operatorname{div}_c \subseteq \operatorname{div}$ and $\operatorname{curl}_c \subseteq \operatorname{curl}$. Thus, $\operatorname{div}, \operatorname{grad}$ and curl are also densely defined. This, in turn, implies that $\operatorname{grad}_c, \operatorname{div}_c$ and curl_c are closable with respective closures $\operatorname{grad}_0, \operatorname{div}_0$ and curl_0. □

We shall describe the domains of these operators in more detail in the next theorem.

Theorem 6.1.2. *If $f \in L_2(\Omega)$ and $g = (g_j)_{j\in\{1,\dots,d\}} \in L_2(\Omega)^d$ then the following statements hold:*

(a) *$f \in \operatorname{dom}(\operatorname{grad})$ and $g = \operatorname{grad} f$ if and only if*

$$\forall \phi \in C_c^\infty(\Omega), j \in \{1,\dots,d\}\colon\ -\int_\Omega f \partial_j \phi = \int_\Omega g_j \phi.$$

(b) *$f \in \operatorname{dom}(\operatorname{grad}_0)$ and $g = \operatorname{grad}_0 f$ if and only if $f \in \operatorname{dom}(\operatorname{grad})$ and there exists $(f_k)_k$ in $C_c^\infty(\Omega)$ such that $f_k \to f$ in $L_2(\Omega)$ and $g = \lim_{k\to\infty} \operatorname{grad} f_k$ in $L_2(\Omega)^d$.*

(c) *$g \in \operatorname{dom}(\operatorname{div})$ and $f = \operatorname{div} g$ if and only if*

$$\forall \phi \in C_c^\infty(\Omega)\colon\ -\int_\Omega g \cdot \operatorname{grad} \phi = \int_\Omega f \phi.$$

(d) *$g \in \operatorname{dom}(\operatorname{div}_0)$ and $f = \operatorname{div}_0 g$ if and only if $g \in \operatorname{dom}(\operatorname{div})$ and there exists $(g_k)_k$ in $C_c^\infty(\Omega)^d$ such that $g_k \to g$ in $L_2(\Omega)^d$ and $f = \lim_{k\to\infty} \operatorname{div} g_k$ in $L_2(\Omega)$.*

If $d = 3$ and $f, g \in L_2(\Omega)^3$ then the following statements hold:

(e) *$f \in \operatorname{dom}(\operatorname{curl})$ and $g = \operatorname{curl} f$ if and only if*

$$\forall \phi \in C_c^\infty(\Omega)^3\colon\ \int_\Omega f \cdot \operatorname{curl} \phi = \int_\Omega g \cdot \phi.$$

(f) *$f \in \operatorname{dom}(\operatorname{curl}_0)$ and $g = \operatorname{curl}_0 f$ if and only if $f \in \operatorname{dom}(\operatorname{curl})$ and there exists $(f_k)_k$ in $C_c^\infty(\Omega)^3$ such that $f_k \to f$ in $L_2(\Omega)^3$ and $g = \lim_{k\to\infty} \operatorname{curl} f_k$ in $L_2(\Omega)^3$.*

All the statements in Theorem 6.1.2 are elementary consequences of the integration by parts formula and the definitions of the adjoint. We ask the reader to prove these statements in Exercise 6.4.

Remark 6.1.3. We remark here that, classically, the following notation has been introduced:

$$\begin{aligned} H^1(\Omega) &:= \operatorname{dom}(\operatorname{grad}),\\ H_0^1(\Omega) &:= \operatorname{dom}(\operatorname{grad}_0),\\ H(\operatorname{div},\Omega) &:= \operatorname{dom}(\operatorname{div}),\\ H(\operatorname{curl},\Omega) &:= \operatorname{dom}(\operatorname{curl}). \end{aligned}$$

Following the rationale of appending zero as an index for $H_0^1(\Omega)$, we shall also use

$$H_0(\operatorname{div},\Omega) := \operatorname{dom}(\operatorname{div}_0),$$
$$H_0(\operatorname{curl},\Omega) := \operatorname{dom}(\operatorname{curl}_0).$$

We do, however, caution the reader that other authors also use $H_0(\operatorname{div},\Omega)$ and $H_0(\operatorname{curl},\Omega)$ to denote the kernel of div and curl. All the spaces just defined are so-called Sobolev spaces. We note that for $d = 3$ we clearly have $H^1(\Omega) \subseteq H(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)$. On the other hand, note that $H(\operatorname{div},\Omega)$ is neither a sub- nor a superset of $H(\operatorname{curl},\Omega)$.

Remark 6.1.4. We emphasise that $H_0^1(\Omega) \subseteq H^1(\Omega)$ is a proper inclusion for many open Ω. The '0' in the index is a reminder of '0'-boundary conditions. In fact, the only difference between these two spaces lies in the behaviour of their elements at the boundary of Ω. The space H_0^1 signifies all H^1-functions vanishing at $\partial\Omega$ in a generalised sense. The corresponding statements are true for the inclusions $H_0(\operatorname{div},\Omega) \subseteq H(\operatorname{div},\Omega)$ and $H_0(\operatorname{curl},\Omega) \subseteq H(\operatorname{curl},\Omega)$. The space $H_0(\operatorname{div},\Omega)$ describes $H(\operatorname{div},\Omega)$-vector fields with vanishing normal component and to lie in $H_0(\operatorname{curl},\Omega)$ provides a handy generalisation of vanishing tangential component. We will anticipate these abstractions, when we apply the solution theory of evolutionary equations for particular cases. In a later lecture we will come back to this issue when we discuss inhomogeneous boundary value problems.

For later use, we record the following relationships between the vector-analytical operators introduced above.

Proposition 6.1.5. *Let $d = 3$. We have the following inclusions:*

$$\overline{\operatorname{ran}}(\operatorname{curl}_0) \subseteq \ker(\operatorname{div}_0),$$
$$\overline{\operatorname{ran}}(\operatorname{grad}_0) \subseteq \ker(\operatorname{curl}_0),$$
$$\overline{\operatorname{ran}}(\operatorname{curl}) \subseteq \ker(\operatorname{div}),$$
$$\overline{\operatorname{ran}}(\operatorname{grad}) \subseteq \ker(\operatorname{curl}).$$

Proof. It is elementary to show that for given $\psi \in C_c^\infty(\Omega)^3$ and $\phi \in C_c^\infty(\Omega)$ we have $\operatorname{div}_0 \operatorname{curl}_0 \psi = 0$ as well as $\operatorname{curl}_0 \operatorname{grad}_0 \phi = 0$. Thus, we obtain $\operatorname{ran}(\operatorname{curl}_c) \subseteq \ker(\operatorname{div}_0)$ and $\operatorname{ran}(\operatorname{grad}_c) \subseteq \ker(\operatorname{curl}_0)$. Since $\ker(\operatorname{div}_0)$ and $\ker(\operatorname{curl}_0)$ are closed, and $C_c^\infty(\Omega)^3$ and $C_c^\infty(\Omega)$ are cores for curl_0 and grad_0 respectively, we obtain the first two inclusions. The last two inclusions follow from the first two by taking into account the orthogonal decompositions

$$L_2(\Omega)^3 = \overline{\operatorname{ran}}(\operatorname{grad}) \oplus \ker(\operatorname{div}_0) = \ker(\operatorname{curl}) \oplus \overline{\operatorname{ran}}(\operatorname{curl}_0)$$

and

$$L_2(\Omega)^3 = \overline{\operatorname{ran}}(\operatorname{grad}_0) \oplus \ker(\operatorname{div}) = \ker(\operatorname{curl}_0) \oplus \overline{\operatorname{ran}}(\operatorname{curl}). \qquad \square$$

6.2 Well-Posedness of Evolutionary Equations and Applications

The solution theory of evolutionary equations is contained in the next result, Picard's theorem. This result is central for all the derivations to come. In fact, with the notation

of Theorem 6.2.1, we shall prove that for all (well-behaved) F there is a unique solution of

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)\, U = F.$$

The solution U depends continuously and causally on the choice of F.
In order to formulate the result, for a Hilbert space H, $\nu \in \mathbb{R}$ and a given operator $A\colon \operatorname{dom}(A) \subseteq H \to H$ we define its extended operator in $L_{2,\nu}(\mathbb{R}; H)$, again denoted by A, by

$$\begin{aligned} L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A)) \subseteq L_{2,\nu}(\mathbb{R}; H) &\to L_{2,\nu}(\mathbb{R}; H) \\ f &\mapsto \big(t \mapsto Af(t)\big). \end{aligned}$$

We have collected some properties of extended operators in Exercise 6.1 and Exercise 6.2.

Theorem 6.2.1 (Picard). *Let $\nu_0 \in \mathbb{R}$ and H be a Hilbert space. Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law with $\mathrm{s_b}(M) < \nu_0$ and let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. Assume that*

$$\operatorname{Re} \langle \phi, zM(z)\phi \rangle_H \geqslant c\|\phi\|_H^2 \quad (\phi \in H, z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0})$$

for some $c > 0$. Then for all $\nu \geqslant \nu_0$ the operator $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable and

$$S_\nu := \overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}^{-1} \in L(L_{2,\nu}(\mathbb{R}; H)).$$

Furthermore, S_ν is causal and satisfies $\|S_\nu\|_{L(L_{2,\nu})} \leqslant 1/c$, and for all $F \in \operatorname{dom}(\partial_{t,\nu})$ we have

$$S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A).$$

Furthermore, for $\eta, \nu \geqslant \nu_0$ and $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ we have that $S_\nu F = S_\eta F$.

The property that $S_\nu F = S_\eta F$ for all $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ where $\eta, \nu \geqslant \nu_0$, for some $\nu_0 \in \mathbb{R}$, will be referred to as S_ν being *eventually independent of ν* in what follows.

Definition. Let H be a Hilbert space and $T \in L(H)$. If T is selfadjoint, we write $T \geqslant c$ for some $c \in \mathbb{R}$ if

$$\forall x \in H : \langle x, Tx \rangle_H \geqslant c\|x\|_H^2.$$

Moreover, we define the *real part of T* by $\operatorname{Re} T := \frac{1}{2}(T + T^*)$.

Note that if H is a Hilbert space and $T \in L(H)$ then $\operatorname{Re} T$ is selfadjoint. Moreover,

$$\langle x, \operatorname{Re} Tx \rangle_H = \operatorname{Re} \langle x, Tx \rangle \quad (x \in H).$$

Hence, in Theorem 6.2.1 the assumption on the material law can be rephrased as

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0}).$$

The following operators will be prototypical examples needed for the applications of the previous theorem.

Proposition 6.2.2. *Let H_0, H_1 be Hilbert spaces.*

(a) *Let $B\colon \operatorname{dom}(B) \subseteq H_0 \to H_1$, $C\colon \operatorname{dom}(C) \subseteq H_1 \to H_0$ be densely defined linear operators. Then*

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} : \operatorname{dom}(B) \times \operatorname{dom}(C) \subseteq H_0 \times H_1 \to H_0 \times H_1$$
$$(\phi, \psi) \mapsto (C\psi, B\phi)$$

is densely defined, and we have

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix}.$$

(b) *Let $a \in L(H_0)$, and $c > 0$. Assume $\operatorname{Re} a \geqslant c$. Then $a^{-1} \in L(H_0)$ and $\operatorname{Re} a^{-1} \geqslant c\|a\|^{-2}$.*

Proof. The proof of the first statement can be done in two steps. First, notice that the inclusion $\begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*$ follows immediately. If, on the other hand, $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*\right)$ with $\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ we get for all $x \in \operatorname{dom}(B)$ that

$$\begin{aligned} \langle Bx, \psi \rangle_{H_1} &= \left\langle \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} \\ &= \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_{H_0 \times H_1} = \langle x, \xi \rangle_{H_0}. \end{aligned}$$

Hence, $\psi \in \operatorname{dom}(B^*)$ and $B^*\psi = \xi$. Similarly, we obtain $\phi \in \operatorname{dom}(C^*)$ and $C^*\phi = \zeta$. For the second statement, we compute for all $\phi \in H_0$ using the Cauchy–Schwarz inequality

$$\|\phi\| \|a\phi\|_{H_0} \geqslant |\langle \phi, a\phi \rangle|_{H_0} \geqslant \operatorname{Re} \langle \phi, a\phi \rangle_{H_0} \geqslant c \langle \phi, \phi \rangle_{H_0} = c\|\phi\|_{H_0}^2.$$

Thus, a is one-to-one. Since $\operatorname{Re} a = \operatorname{Re} a^*$ it follows that a^* is one-to-one, as well. Thus, we get that a has dense range. The inequality

$$\|a\phi\|_{H_0} \geqslant c\|\phi\|_{H_0}$$

implies that a^{-1} is bounded. Hence, as a^{-1} is closed, $\operatorname{ran}(a) = \operatorname{dom}(a^{-1})$ is closed and $a^{-1} \in L(H_0)$. To conclude, let $\psi \in H_0$ and put $\phi := a^{-1}\psi$. Then $\|\psi\|_{H_0} = \left\|aa^{-1}\psi\right\|_{H_0} \leqslant \|a\| \left\|a^{-1}\psi\right\|_{H_0}$ and so

$$\begin{aligned} \operatorname{Re} \left\langle \psi, a^{-1}\psi \right\rangle_{H_0} &= \operatorname{Re} \langle a\phi, \phi \rangle_{H_0} = \operatorname{Re} \langle \phi, a\phi \rangle_{H_0} \geqslant c \langle \phi, \phi \rangle_{H_0} = c \left\langle a^{-1}\psi, a^{-1}\psi \right\rangle_{H_0} \\ &\geqslant c \frac{1}{\|a\|^2} \|\psi\|_{H_0}^2. \end{aligned}$$

□

The Heat Equation

The first example we will consider is the heat equation in an open subset $\Omega \subseteq \mathbb{R}^d$. Under a heat source, $Q\colon \mathbb{R}\times\Omega \to \mathbb{R}$, the heat distribution, $\theta\colon \mathbb{R}\times\Omega\to\mathbb{R}$, satisfies the so-called heat-flux-balance

$$\partial_t \theta + \operatorname{div} q = Q.$$

Here, $q\colon \mathbb{R}\times\Omega\to\mathbb{R}^d$ is the heat flux which is connected to θ via Fourier's law

$$q = -a \operatorname{grad} \theta,$$

where $a\colon \Omega\to\mathbb{R}^{d\times d}$ is the heat conductivity, which is measurable, bounded and uniformly strictly positive in the sense that

$$\operatorname{Re} a(x) \geqslant c$$

for all $x\in\Omega$ and some $c>0$ in the sense of positive definiteness. Moreover, we assume that Ω is thermally isolated, which is modelled by requiring that the normal component of q vanishes at $\partial\Omega$; that is, $q\in\operatorname{dom}(\operatorname{div}_0)$. Written as a block matrix and incorporating the boundary condition, we obtain

$$\left(\partial_t \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix} Q\\ 0\end{pmatrix}.$$

Theorem 6.2.3. *For all $\nu>0$, the operator*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega)\times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof. The assertion follows from Theorem 6.2.1 applied to

$$M(z) = \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + z^{-1}\begin{pmatrix} 0 & 0\\ 0 & a^{-1}\end{pmatrix} \quad\text{and}\quad A = \begin{pmatrix} 0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}.$$

Note that M is a material law with $\mathrm{s_b}(M) = 0$ by Example 5.3.1. The required real part condition in Theorem 6.2.1 follows from Proposition 6.2.2(b). Moreover, A is skew-selfadjoint by Proposition 6.2.2(a). □

Remark 6.2.4. Assume that $Q\in\operatorname{dom}(\partial_{t,\nu})$. It then follows from Theorem 6.2.1 that

$$\begin{aligned}\begin{pmatrix}\theta\\ q\end{pmatrix} &\coloneqq \overline{\left(\partial_{t,\nu}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & a^{-1}\end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right)}^{-1}\begin{pmatrix} Q\\ 0\end{pmatrix}\\ &\in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right).\end{aligned} \tag{6.1}$$

Then, it follows that θ and q satisfy the heat-flux-balance and Fourier's law in the sense that $\theta \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{grad})$ and $q \in \operatorname{dom}(\operatorname{div}_0)$ and

$$\begin{aligned} \partial_t \theta + \operatorname{div}_0 q &= Q, \\ q &= -a \operatorname{grad} \theta. \end{aligned}$$

In fact, in order to deduce the latter equality it is sufficient to have

$$\begin{pmatrix} \theta \\ q \end{pmatrix} \in \operatorname{dom}\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix} \right).$$

This regularity result is true even for $Q \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$; see [7].

The Scalar Wave Equation

The classical scalar wave equation in a medium $\Omega \subseteq \mathbb{R}^d$ (think, for instance, of a vibrating string ($d = 1$) or membrane ($d = 2$)) consists of the equation of the balance of momentum where the acceleration of the (vertical) displacement, $u\colon \mathbb{R} \times \Omega \to \mathbb{R}$, is balanced by external forces, $f\colon \mathbb{R} \times \Omega \to \mathbb{R}$, and the divergence of the stress, $\sigma\colon \mathbb{R} \times \Omega \to \mathbb{R}^d$, in such a way that

$$\partial_t^2 u - \operatorname{div} \sigma = f.$$

The stress is related to u via the following so-called stress-strain relation (here Hooke's law)

$$\sigma = T \operatorname{grad} u,$$

where the so-called elasticity tensor, $T\colon \Omega \to \mathbb{R}^{d \times d}$, is bounded, measurable, and satisfies

$$T(x) = T(x)^* \geqslant c$$

for some $c > 0$ uniformly in $x \in \Omega$. The quantity $\operatorname{grad} u$ is referred to as the strain. We think of u as being fixed at $\partial\Omega$ ("clamped boundary condition"). This is modelled by $u \in \operatorname{dom}(\operatorname{grad}_0)$.
Using $v := \partial_t u$ as an unknown, we can rewrite the balance of momentum and Hooke's law as 2×2-block-operator matrix equation

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The solution theory of evolutionary equations for the wave equation now reads as follows:

Theorem 6.2.5. *Let $\Omega \subseteq \mathbb{R}^d$ be open, and T as indicated above. Then, for all $\nu > 0$,*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof. We apply Theorem 6.2.1 to $A = -\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$, which is skew-selfadjoint by Proposition 6.2.2(a), and $M(z) = \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix}$, which defines a material law with $\mathrm{s_b}(M) = -\infty$. The positive definiteness constraint needed in Theorem 6.2.1 is satisfied by Proposition 6.2.2(b) on account of the selfadjointness of T, which implies the same for T^{-1}. Indeed, for $\nu_0 > 0$ and $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}$ we estimate

$$\begin{aligned} \mathrm{Re}\,\langle (x,y), zM(z)(x,y)\rangle_{L_2(\Omega)\times L_2(\Omega)^d} &= \mathrm{Re}\,\langle x, zx\rangle_{L_2(\Omega)} + \mathrm{Re}\,\left\langle y, zT^{-1}y\right\rangle_{L_2(\Omega)^d} \\ &\geqslant \nu_0 \|x\|^2_{L_2(\Omega)} + \nu_0 \frac{c}{\|T\|^2}\|y\|^2_{L_2(\Omega)^d} \\ &\geqslant \nu_0 \min\{1, c/\|T\|^2\}\|(x,y)\|^2_{L_2(\Omega)\times L_2(\Omega)^d} \end{aligned}$$

for each $(x,y) \in L_2(\Omega) \times L_2(\Omega)^d$, where we used the selfadjointness of T^{-1} in the second line. □

Remark 6.2.6. Let $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$, $\nu > 0$, and define

$$\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} = \overline{\left(\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right)}^{-1} \begin{pmatrix} \partial_{t,\nu}^{-1} f \\ 0 \end{pmatrix}.$$

By Theorem 6.2.1, we obtain $\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right)$. Hence, we have

$$\begin{aligned} \partial_{t,\nu} u - \operatorname{div} \widetilde{\sigma} &= \partial_{t,\nu}^{-1} f \\ \partial_{t,\nu} T^{-1} \widetilde{\sigma} &= \operatorname{grad}_0 u \end{aligned}$$

or

$$\begin{aligned} \partial_{t,\nu} u - \operatorname{div} \widetilde{\sigma} &= \partial_{t,\nu}^{-1} f \\ \widetilde{\sigma} &= T\partial_{t,\nu}^{-1} \operatorname{grad}_0 u. \end{aligned}$$

Thus, formally, after another time-differentiation and the setting of $\sigma = \partial_{t,\nu}\widetilde{\sigma}$ we obtain a solution of the wave equation, (u, σ). Notice, however, that differentiating $\operatorname{div} \widetilde{\sigma}$ cannot be done without any additional knowledge of the regularity of $\widetilde{\sigma}$. In fact, in order to arrive at the balance of momentum equation, one would need to have $\operatorname{div} \widetilde{\sigma} \in \operatorname{dom}(\partial_{t,\nu})$. However, one only has $\widetilde{\sigma} \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{div})$. It is an elementary argument, see [10, Lemma 4.6], that we in fact have $\operatorname{div} \partial_{t,\nu}^{-1} = \overline{\partial_{t,\nu}^{-1} \operatorname{div}}$, which suggests that, in general, $\operatorname{div} \widetilde{\sigma} \notin \operatorname{dom}(\partial_{t,\nu})$, see Exercise 6.6.

Maxwell's Equations

The final example in this lecture forms the archetypical evolutionary equation – Maxwell's equations in a medium $\Omega \subseteq \mathbb{R}^3$. In order to see this (and to finally conclude the 2×2-block matrix formulation historically due to the work of [4, 9, 3]), we start out with

Faraday's law of induction, which relates the unknown electric field, $E\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, to the magnetic induction, $B\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, via

$$\partial_t B + \operatorname{curl} E = 0.$$

We assume that the medium is contained in a perfect conductor, which is reflected in the so-called electric boundary condition which asks for the vanishing of the tangential component of E at the boundary. This is modelled by $E\in\operatorname{dom}(\operatorname{curl}_0)$. The next constituent of Maxwell's equations is Ampere's law

$$\partial_t D + J_c - \operatorname{curl} H = J_0,$$

which relates the unknown electric displacement, $D\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, charge, $J_c\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, and magnetic field, $H\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$, to the (given) external currents, $J_0\colon \mathbb{R}\times\Omega\to\mathbb{R}^3$. Maxwell's equations are completed by constitutive relations specific to each material at hand. Indeed, the (bounded, measurable) dielectricity, $\varepsilon\colon\Omega\to\mathbb{R}^{3\times3}$, and the (bounded, measurable) magnetic permeability, $\mu\colon\Omega\to\mathbb{R}^{3\times3}$, are symmetric matrix-valued functions which couple the electric displacement to the electric field and the magnetic field to the magnetic induction via

$$D=\varepsilon E, \text{ and } B=\mu H.$$

Finally, Ohm's law relates the charge to the electric field via the (bounded, measurable) electric conductivity, $\sigma\colon\Omega\to\mathbb{R}^{3\times3}$, as

$$J_c=\sigma E.$$

All in all, in terms of (E,H), Maxwell's equations read

$$\left(\partial_t\begin{pmatrix}\varepsilon&0\\0&\mu\end{pmatrix}+\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}+\begin{pmatrix}0&-\operatorname{curl}\\\operatorname{curl}_0&0\end{pmatrix}\right)\begin{pmatrix}E\\H\end{pmatrix}=\begin{pmatrix}J_0\\0\end{pmatrix}.$$

For the time being, we shall assume that there exists $c>0$ and $\nu_0>0$ such that for all $\nu\geqslant\nu_0$ we have

$$\nu\varepsilon(x)+\operatorname{Re}\sigma(x)\geqslant c,\quad \mu(x)\geqslant c\quad (x\in\Omega)$$

in the sense of positive definiteness. Note that the latter condition allows particularly for $\varepsilon=0$ on certain regions, if $\operatorname{Re}\sigma$ compensates for this. This situation is referred to as the eddy current approximation in these regions. With the above preparations at hand, we may now formulate the well-posedness result concerning Maxwell's equations.

Theorem 6.2.7. *Let $\Omega\subseteq\mathbb{R}^3$ be open and $\nu\geqslant\nu_0$. Then*

$$\partial_{t,\nu}\begin{pmatrix}\varepsilon&0\\0&\mu\end{pmatrix}+\begin{pmatrix}\sigma&0\\0&0\end{pmatrix}+\begin{pmatrix}0&-\operatorname{curl}\\\operatorname{curl}_0&0\end{pmatrix}$$

is densely defined and closable in $L_{2,\nu}\left(\mathbb{R};L_2(\Omega)^3\times L_2(\Omega)^3\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof. The assertion follows from Theorem 6.2.1 applied to the material law

$$M(z) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + z^{-1} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and the skew-selfadjoint operator

$$A = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}. \qquad \square$$

Remark 6.2.8. In the physics literature (see e.g. [1, Chapter 18]), Maxwell's equations are usually complemented by Gauss' law,

$$\operatorname{div}_0 B = 0,$$

as well as the introduction of the charge density, $\rho = \operatorname{div} \varepsilon E$, and the current, $J = J_0 - J_c$, by the continuity equation

$$\partial_t \rho = \operatorname{div} J.$$

We shall argue in the following that these equations are *automatically* satisfied if (E, H) is a solution to Maxwell's equation. Indeed, assuming $J_0 \in \operatorname{dom}(\partial_{t,\nu})$, then, as a consequence of Theorem 6.2.1, we have that

$$\begin{aligned} \begin{pmatrix} E \\ H \end{pmatrix} &= \overline{\left(\partial_{t,\nu} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \right)}^{-1} \begin{pmatrix} J_0 \\ 0 \end{pmatrix} \\ &\in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \right). \end{aligned}$$

Reformulating the latter equation yields

$$\begin{aligned} B = \mu H &= -\partial_{t,\nu}^{-1} \operatorname{curl}_0 E, \\ \varepsilon E &= \partial_{t,\nu}^{-1} \left(-\sigma E + J_0 + \operatorname{curl} H \right) = \partial_{t,\nu}^{-1} J + \partial_{t,\nu}^{-1} \operatorname{curl} H. \end{aligned}$$

Since $\operatorname{curl}_0 E \in \operatorname{ran}(\operatorname{curl}_0)$, we have by Proposition 3.1.5 that $\partial_{t,\nu}^{-1} \operatorname{curl}_0 E \in \overline{\operatorname{ran}}(\operatorname{curl}_0)$. Thus, by Proposition 6.1.5, we obtain

$$\operatorname{div}_0 B = \operatorname{div}_0 \left(-\partial_{t,\nu}^{-1} \operatorname{curl}_0 E \right) = 0.$$

Similarly, we deduce that

$$\rho = \operatorname{div} \varepsilon E = \operatorname{div} \partial_{t,\nu}^{-1} J.$$

If, in addition, we have that $J \in \operatorname{dom}(\operatorname{div})$, we recover the continuity equation. In general, the continuity equation is satisfied in the integrated sense just derived.

We shall keep the list of examples to that for now. In the course of this internet seminar, we will see more (involved) examples. Furthermore, we will study the boundary conditions more deeply and shall relate the conditions introduced abstractly here to more classical formulations involving trace spaces.

6.3 Proof of Picard's Theorem

In this section we shall prove the well-posedness theorem. For this, we recall an elementary result from functional analysis. It is remindful of the Lax–Milgram lemma.

Proposition 6.3.1. *Let H be a Hilbert space and $B\colon \operatorname{dom}(B) \subseteq H \to H$ densely defined and closed with $\operatorname{dom}(B) \supseteq \operatorname{dom}(B^*)$. Assume there exists $c > 0$ such that*

$$\operatorname{Re}\langle \phi, B\phi\rangle_H \geqslant c\|\phi\|_H^2 \quad (\phi \in \operatorname{dom}(B)).$$

Then $B^{-1} \in L(H)$ and $\|B^{-1}\| \leqslant 1/c$.

Proof. The proof is a refinement of the argument in Proposition 6.2.2. In fact, the assumed inequality implies closedness of the range of B as well as continuous invertibility with $B^{-1}\colon \operatorname{ran}(B) \to H$. The fact that $\operatorname{ran}(B)$ is dense in H follows from the fact that $\operatorname{Re}\langle \phi, B^*\phi\rangle_H \geqslant c\|\phi\|_H^2$ for all $\phi \in \operatorname{dom}(B^*) \subseteq \operatorname{dom}(B)$ which, in turn, also follows from the assumed inequality. □

Proof of Theorem 6.2.1. Let $\nu \geqslant \nu_0$. We first prove that $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable for each $\nu \geqslant \nu_0$. By unitary equivalence, it suffices to show that $(\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A$ is closable. For doing so, let $(U_n)_n$ be a sequence in $\operatorname{dom}((\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A)$ with $U_n \to 0$ and $((\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A)U_n \to F$ in $L_2(\mathbb{R}; H)$ for some $F \in L_2(\mathbb{R}; H)$. For $R > 0$ we observe that $\mathbb{1}_{[-R,R]}U_n \in \operatorname{dom}((\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A)$ with

$$\begin{aligned}\big((\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A\big)\mathbb{1}_{[-R,R]}U_n &= \mathbb{1}_{[-R,R]}\big((\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A\big)U_n \\ &\to \mathbb{1}_{[-R,R]}F\end{aligned}$$

as $n \to \infty$. However, on $L_2([-R, R]; H)$ the operator $(\operatorname{im}+\nu)M(\operatorname{im}+\nu)$ is bounded and hence, $(\operatorname{im} + \nu)M(\operatorname{im} + \nu) + A$ is closed. Since $\mathbb{1}_{[-R,R]}U_n \to 0$ we infer that

$$\mathbb{1}_{[-R,R]}F = 0$$

and since $R > 0$ was chosen arbitrarily, we obtain $F = 0$. Hence, $(\operatorname{im}+\nu)M(\operatorname{im}+\nu)+A$ is closable.
Let now $z \in \mathbb{C}_{\operatorname{Re}\geqslant\nu}$. Define $B(z) := zM(z) + A$. Since $M(z) \in L(H)$ it follows that $B(z)^* = (zM(z))^* - A$ and $\operatorname{dom}(B(z)) = \operatorname{dom}(B(z)^*) = \operatorname{dom}(A)$. Moreover, for all $\phi \in \operatorname{dom}(A)$ we have

$$\operatorname{Re}\langle \phi, B(z)\phi\rangle_H = \operatorname{Re}\langle \phi, (zM(z) + A)\,\phi\rangle_H = \operatorname{Re}\langle \phi, zM(z)\phi\rangle_H \geqslant c\|\phi\|_H^2,$$

due to the skew-selfadjointness of A. Thus, by Proposition 6.3.1 applied to $B(z)$ instead of B, we deduce that

$$N\colon \mathbb{C}_{\operatorname{Re}\geqslant\nu} \ni z \mapsto B(z)^{-1}$$

is bounded assuming values in $L(H)$ with norm bounded by $1/c$. By Exercise 6.5, we have that N is holomorphic. Thus, N is a material law and $\|N(\partial_{t,\nu})\| \leqslant 1/c$ by Proposition 5.3.2. Moreover, Theorem 5.3.5 implies that $N(\partial_{t,\nu})$ is independent of ν and causal.

Next, if $f \in \operatorname{dom}(\partial_{t,\nu})$, it follows that $(\mathrm{i}\mathrm{m} + \nu)\,\mathcal{L}_\nu f \in L_2(\mathbb{R}; H)$. Hence, for all $t \in \mathbb{R}$ we obtain

$$\begin{aligned} AN(\mathrm{i}t + \nu)\mathcal{L}_\nu f(t) &= A\big((\mathrm{i}t + \nu)\,M(\mathrm{i}t + \nu) + A\big)^{-1}\mathcal{L}_\nu f(t) \\ &= \mathcal{L}_\nu f(t) - (\mathrm{i}t + \nu)\,M(\mathrm{i}t + \nu)N(\mathrm{i}t + \nu)\mathcal{L}_\nu f(t). \end{aligned}$$

Thus, by the boundedness of M and N, we deduce $N(\mathrm{i}\cdot + \nu)\mathcal{L}_\nu f \in L_2(\mathbb{R}; \operatorname{dom}(A))$. This implies $N(\partial_{t,\nu})f \in L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ by Exercise 6.2. Similarly, but more easily, it follows that $(\mathrm{i}\cdot + \nu)\,N(\mathrm{i}\cdot + \nu)\mathcal{L}_\nu f \in L_2(\mathbb{R}; H)$ also.
Thus, to complete the proof, it remains to show that

$$N(\partial_{t,\nu}) = \overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A}^{-1}.$$

Applying $\mathcal{L}_\nu$ to both sides and using Exercise 6.2, the latter is equivalent to

$$N(\mathrm{i}\mathrm{m} + \nu) = \overline{(\mathrm{i}\mathrm{m} + \nu)\,M(\mathrm{i}\mathrm{m} + \nu) + A}^{-1}.$$

We remark that the left-hand side is the "multiplication operator" associated with the function $t \mapsto ((\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)+A)^{-1}$, while the right-hand side is the closure of the relation $((\mathrm{i}\mathrm{m}+\nu)M(\mathrm{i}\mathrm{m}+\nu)+A)^{-1}$. We will prove the equality by showing both inclusions.

First, let $(F, U) \in \overline{(\mathrm{i}\mathrm{m} + \nu)\,M(\mathrm{i}\mathrm{m} + \nu) + A}^{-1}$. Then we find $\big((U_n, F_n)\big)_n$ converging to (U, F) in $L_2(\mathbb{R}; H \times H)$, such that for all $n \in \mathbb{N}$ we have

$$(\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu)\,U_n + AU_n = F_n \in L_2(\mathbb{R}; H).$$

Thus, $U_n \in \operatorname{dom}\left(B\,(\mathrm{i}\mathrm{m} + \nu)\right)$ and, hence, $(F_n, U_n) \in N\,(\mathrm{i}\mathrm{m} + \nu)$. The closedness of $N\,(\mathrm{i}\mathrm{m} + \nu)$, thus implies $(F, U) \in N\,(\mathrm{i}\mathrm{m} + \nu)$.
On the other hand, let $F \in L_2(\mathbb{R}; H)$. For $n \in \mathbb{N}$ we define $G_n := N(\mathrm{i}\mathrm{m} + \nu)\mathbb{1}_{[-n,n]}F$. By the continuity of $N\,(\mathrm{i}\mathrm{m} + \nu)$, we obtain $G_n \to N(\mathrm{i}\mathrm{m} + \nu)F$ as $n \to \infty$. Note that $G_n \in \operatorname{dom}\big((\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu)\big)$ as M is bounded and G_n is supported in $[-n, n]$, only. Next, we confirm that $G_n \in L_2(\mathbb{R}; \operatorname{dom}(A))$. For this, we note that for almost every $t \in \mathbb{R}$ we have that $G_n(t) \in \operatorname{dom}(A)$ by the definition of $N(\mathrm{i}\mathrm{m} + \nu)$. Moreover, we estimate

$$\int_{\mathbb{R}} \big\|\big((\mathrm{i}t + \nu)\,M\,(\mathrm{i}t + \nu) + A\big)G_n(t)\big\|_H^2\,\mathrm{d}t = \int_{\mathbb{R}} \big\|\mathbb{1}_{[-n,n]}(t)F(t)\big\|_H^2\,\mathrm{d}t \leqslant \|F\|_{L_2(\mathbb{R};H)}^2.$$

Since $G_n \in \operatorname{dom}((\mathrm{i}\mathrm{m} + \nu)M(\mathrm{i}\mathrm{m} + \nu))$, the latter gives $G_n \in L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ and altogether we obtain $G_n \in \operatorname{dom}\big((\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu) + A\big)$. Moreover,

$$\big((\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu) + A\big)G_n = \mathbb{1}_{[-n,n]}F,$$

since this equality holds almost everywhere by the definition of $N(\mathrm{i}\mathrm{m} + \nu)$. Therefore, by passing to the limit as $n \to \infty$, we confirm that, indeed, $N(\mathrm{i}\mathrm{m} + \nu)F \in \operatorname{dom}\left(\overline{\big((\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu) + A\big)}\right)$ and $\overline{\big((\mathrm{i}\mathrm{m} + \nu)\,M\,(\mathrm{i}\mathrm{m} + \nu) + A\big)}N\,(\mathrm{i}\mathrm{m} + \nu)\,F = F$ or, to put it differently, $(F, N(\mathrm{i}\mathrm{m} + \nu)F) \in \overline{(\mathrm{i}\mathrm{m} + \nu)\,M(\mathrm{i}\mathrm{m} + \nu) + A}^{-1}$, which yields the assertion. □

Comments

The proof of Theorem 6.2.1 here is rather close to the strategy originally employed in [6], at least where existence and uniqueness are concerned. The causality part is a consequence of some observations detailed in [2, 11]. The original process of proving causality used the Theorem of Paley and Wiener, which we shall discuss later on.
The eddy current approximation has enjoyed great interest in the mathematical and physical community, in particular for the case when $\varepsilon = 0$ everywhere. The reason being that then Maxwell's equations are merely of parabolic type. We shall refer to [5] and the references therein for an extensive discussion.
Both Proposition 6.3.1 and the Lax–Milgram lemma have been put into a general perspective in [8].

Exercises

Exercise 6.1. Let (Ω, Σ, μ) be a σ-finite measure space and let H_0, H_1 be Hilbert spaces. Let $A\colon \operatorname{dom}(A) \subseteq H_0 \to H_1$ be densely defined and closed. Show that the operator

$$\begin{aligned} A_\mu\colon L_2(\mu; \operatorname{dom}(A)) \subseteq L_2(\mu; H_0) &\to L_2(\mu; H_1) \\ f &\mapsto \big(\omega \mapsto Af(\omega)\big) \end{aligned}$$

is densely defined and closed. Moreover, show that $(A_\mu)^* = (A^*)_\mu$.

Exercise 6.2. In the situation of Exercise 6.1, if $(\Omega_1, \Sigma_1, \mu_1)$ is another σ-finite measure space and $\mathcal{F}\colon L_2(\mu) \to L_2(\mu_1)$ is unitary, show that for $j \in \{0, 1\}$ there exists a unique unitary operator $\mathcal{F}_{H_j}\colon L_2(\mu; H_i) \to L_2(\mu_1; H_j)$ such that

$$\mathcal{F}_{H_j}(\phi x) = (\mathcal{F}\phi)x \quad (\phi \in L_2(\mu), x \in H_j).$$

Furthermore, prove that

$$\mathcal{F}_{H_1} A_\mu \mathcal{F}_{H_0}^* = A_{\mu_1}.$$

Exercise 6.3. Show that for $\Omega \subseteq \mathbb{R}^d$ open, the set $C_c^\infty(\Omega) \subseteq L_2(\Omega)$ is dense.

Exercise 6.4. Prove Theorem 6.1.2.

Exercise 6.5. Let H be a Hilbert space, $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint, and $c > 0$. Moreover, let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be holomorphic with

$$\operatorname{Re} M(z) \geqslant c \quad (z \in \operatorname{dom}(M)).$$

Show that $\operatorname{dom}(M) \ni z \mapsto (M(z) + A)^{-1}$ is holomorphic.

Exercise 6.6. Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be a densely defined and closed linear operator acting in Hilbert spaces H_0 and H_1. For $\nu > 0$ show that

$$\overline{\partial_{t,\nu}^{-1} C} = C \partial_{t,\nu}^{-1}.$$

Hint: Apply Exercise 6.2 and show $\overline{(\mathrm{im} + \nu)^{-1} C} = C(\mathrm{im} + \nu)^{-1}$ with a suitable approximation argument.

Exercise 6.7. In this exercise, we shall assume that

$$D \coloneqq \left\{\phi \in H^1(\Omega)\,;\, \operatorname{grad}\phi \in \operatorname{dom}(\operatorname{div}), \phi = \operatorname{div}\operatorname{grad}\phi\right\} \subseteq C^\infty(\Omega).$$

(a) Compute $H_0^1(\Omega)^\perp$ where the orthogonal complement is computed in $H^1(\Omega)$.

(b) Show that $C^\infty(\Omega) \cap H^1(\Omega) \subseteq H^1(\Omega)$ is dense.

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