5 The Fourier–Laplace Transformation and Material Law Operators

In this lecture we introduce the Fourier–Laplace transformation and use it to define operator-valued functions of $\partial_{t,\nu}$; the so-called material law operators. These operators will play a crucial role when we deal with partial differential equations. In the equations of classical mathematical physics, like the heat equation, wave equation or Maxwell's equation, the involved material parameters, such as heat conductivity or permeability of the underlying medium, are incorporated within these operators and hence the name "material law". We start our lecture by defining the Fourier transformation and proving Plancharel's theorem in the Hilbert space-valued case, which states that the Fourier transformation defines a unitary operator on $L_2(\mathbb{R}; H)$.
Throughout, let H be a complex Hilbert space.

5.1 The Fourier Transformation

We start by defining the Fourier transformation on $L_1(\mathbb{R}; H)$.

Definition. For $f \in L_1(\mathbb{R}; H)$ we define the *Fourier transform* $\widehat{f}$ *of* f by

$$\widehat{f}(s) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t \quad (s \in \mathbb{R}).$$

Lemma 5.1.1 (Riemann–Lebesgue)**.** *Let* $f \in L_1(\mathbb{R}; H)$. *Then* $\widehat{f} \in C_{\mathrm{b}}(\mathbb{R}; H)$ *and* $\lim_{|t|\to\infty} \left\| \widehat{f}(t) \right\| \to 0$. *Moreover,*

$$\left\| \widehat{f} \right\|_\infty \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_1.$$

Proof. First, note that $\widehat{f}$ is continuous by dominated convergence and bounded with

$$\left\| \widehat{f} \right\|_\infty \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_1.$$

This shows that the mapping

$$\begin{aligned} L_1(\mathbb{R}; H) &\to C_{\mathrm{b}}(\mathbb{R}; H) \\ f &\mapsto \widehat{f} \end{aligned} \tag{5.1}$$

defines a bounded linear operator, where $C_{\mathrm{b}}(\mathbb{R};H)$ denotes the space of all bounded continuous functions on $\mathbb{R}$ with values in H. Moreover, for $\varphi \in C_{\mathrm{c}}^1(\mathbb{R};H)$ we compute

$$\widehat{\varphi}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} \varphi(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}s} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} \varphi'(t)\,\mathrm{d}t$$

for $s \neq 0$ and thus,

$$\limsup_{|s|\to\infty} \|\widehat{\varphi}(s)\| \leqslant \limsup_{|s|\to\infty} \frac{1}{|s|} \frac{1}{\sqrt{2\pi}} \left\|\varphi'\right\|_1 = 0,$$

which shows that $\lim_{|s|\to\infty} \|\widehat{\varphi}(s)\| = 0$. By the facts that $C_{\mathrm{c}}^1(\mathbb{R};H)$ is dense in $L_1(\mathbb{R};H)$ (see Lemma 3.1.7), $\left\{f \in C_{\mathrm{b}}(\mathbb{R};H)\,;\, \lim_{|t|\to\infty} \|f(t)\| = 0\right\}$ is a closed subspace of $C_{\mathrm{b}}(\mathbb{R};H)$ and the operator in (5.1) is bounded, the assertion follows. □

It is our main goal to extend the definition of the Fourier transformation to functions in $L_2(\mathbb{R};H)$. For doing so, we make use of the Schwartz space of rapidly decreasing functions.

Definition. We define

$$\mathcal{S}(\mathbb{R};H) \coloneqq \left\{ f \in C^\infty(\mathbb{R};H)\,;\, \forall n,k \in \mathbb{N} :\ \left(t \mapsto t^k f^{(n)}(t)\right) \in C_{\mathrm{b}}(\mathbb{R};H) \right\},$$

to be the *Schwartz space* of rapidly decreasing functions on $\mathbb{R}$ with values in H.

As usual we abbreviate $\mathcal{S}(\mathbb{R}) \coloneqq \mathcal{S}(\mathbb{R};\mathbb{K})$.

Remark 5.1.2. $\mathcal{S}(\mathbb{R};H)$ is a Fréchet space with respect to the seminorms

$$\mathcal{S}(\mathbb{R};H) \ni f \mapsto \sup_{t\in\mathbb{R}} \left\| t^k f^{(n)}(t) \right\| \quad (k,n \in \mathbb{N}).$$

Moreover, $\mathcal{S}(\mathbb{R};H) \subseteq \bigcap_{p\in[1,\infty]} L_p(\mathbb{R};H)$. Indeed, $\mathcal{S}(\mathbb{R};H) \subseteq L_\infty(\mathbb{R};H)$ by definition, and for $p < \infty$ we have that

$$\begin{aligned} \int_{\mathbb{R}} \|f(t)\|^p \,\mathrm{d}t &= \int_{\mathbb{R}} \frac{1}{(1+|t|)^{pk}} \left\| (1+|t|)^k f(t) \right\|^p \mathrm{d}t \\ &\leqslant \sup_{t\in\mathbb{R}} \left\| (1+|t|)^k f(t) \right\|^p \int_{\mathbb{R}} \frac{1}{(1+|t|)^{pk}} \,\mathrm{d}t < \infty \end{aligned}$$

if we choose $k > \frac{1}{p}$.

Proposition 5.1.3. *Let $f \in \mathcal{S}(\mathbb{R};H)$. Then $\widehat{f} \in \mathcal{S}(\mathbb{R};H)$ and the mapping*

$$\mathcal{S}(\mathbb{R};H) \to \mathcal{S}(\mathbb{R};H), \quad f \mapsto \widehat{f}$$

is bijective. Moreover, for $f, g \in L_1(\mathbb{R};H)$ we have that

$$\int_{\mathbb{R}} \left\langle \widehat{f}(t), g(t) \right\rangle \,\mathrm{d}t = \int_{\mathbb{R}} \langle f(t), \widehat{g}(-t) \rangle \,\mathrm{d}t. \tag{5.2}$$

Additionally, if $f, \widehat{f} \in L_1(\mathbb{R};H)$ then

$$f(t) = \widehat{\widehat{f}}(-t) \quad (t \in \mathbb{R}). \tag{5.3}$$

Proof. By Exercise 5.1 we have

$$\widehat{f}\,'(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t = -\mathrm{i}\big(\widehat{t \mapsto tf(t)}\big)(s) \quad (s \in \mathbb{R}) \tag{5.4}$$

and

$$s\widehat{f}(s) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{i}s)\,\mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t = -\mathrm{i}\widehat{f'}(s) \quad (s \in \mathbb{R}). \tag{5.5}$$

Using these formulas, one can show that $\widehat{f} \in \mathcal{S}(\mathbb{R}; H)$. Since the bijectivity of the Fourier transformation on $\mathcal{S}(\mathbb{R}; H)$ would follow from (5.3), it suffices to prove the formulas (5.2) and (5.3). Let $f, g \in L_1(\mathbb{R}; H)$. Then we compute using Proposition 3.1.5 and Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}} \left\langle \widehat{f}(t), g(t) \right\rangle \mathrm{d}t &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left\langle \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s, g(t) \right\rangle \mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}st} \left\langle f(s), g(t) \right\rangle \mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left\langle f(s), \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st} g(t)\,\mathrm{d}t \right\rangle \mathrm{d}s \\
&= \int_{\mathbb{R}} \left\langle f(s), \widehat{g}(-s) \right\rangle \mathrm{d}s,
\end{aligned}$$

which yields (5.2). For proving formula (5.3), we consider the function γ defined by $\gamma(t) := \mathrm{e}^{-\frac{t^2}{2}}$ for $t \in \mathbb{R}$. Clearly, $\gamma \in \mathcal{S}(\mathbb{R})$. We claim that $\widehat{\gamma} = \gamma$. Indeed, we observe that γ solves the initial value problem $y' + ty = 0$ subject to $y(0) = 1$; if we can show that $\widehat{\gamma}$ solves the same initial value problem, then their equality would follow from the uniqueness of the solution. First, we observe that $\widehat{\gamma}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t = 1$. Second, we compute using the formulas (5.4) and (5.5) that

$$\widehat{\gamma}'(s) = -\mathrm{i}\big(\widehat{t \mapsto t\gamma(t)}\big)(s) = \mathrm{i}\widehat{\gamma'}(s) = -s\widehat{\gamma}(s) \quad (s \in \mathbb{R}).$$

Altogether, we have shown that $\widehat{\gamma}$ solves the same initial value problem as γ and hence, $\widehat{\gamma} = \gamma$. Let now $f \in L_1(\mathbb{R}; H)$ with $\widehat{f} \in L_1(\mathbb{R}; H)$, $a > 0$ and $x \in H$. Then we compute

$$\begin{aligned}
\left\langle \int_{\mathbb{R}} \widehat{f}(t)\gamma(at)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t, x \right\rangle &= \int_{\mathbb{R}} \left\langle \widehat{f}(t), \gamma(at)x\mathrm{e}^{-\mathrm{i}st} \right\rangle \mathrm{d}t = \int_{\mathbb{R}} \left\langle f(t), \big(\widehat{\gamma(a\cdot)x\mathrm{e}^{-\mathrm{i}s(\cdot)}}\big)(-t) \right\rangle \mathrm{d}t \\
&= \int_{\mathbb{R}} \left\langle f(t), \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma(ar)x\mathrm{e}^{-\mathrm{i}sr}\mathrm{e}^{\mathrm{i}tr}\,\mathrm{d}r \right\rangle \mathrm{d}t \\
&= \frac{1}{a} \int_{\mathbb{R}} \left\langle f(t), \widehat{\gamma}\left(\frac{s-t}{a}\right) x \right\rangle \mathrm{d}t = \frac{1}{a} \int_{\mathbb{R}} \left\langle f(t), \gamma\left(\frac{s-t}{a}\right) x \right\rangle \mathrm{d}t \\
&= \int_{\mathbb{R}} \left\langle f(s-at), \gamma(t)\,x \right\rangle \mathrm{d}t = \left\langle \int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t, x \right\rangle
\end{aligned}$$

for each $s \in \mathbb{R}$, where we have used (5.2) in the first line. Since this holds for all $x \in H$ we get

$$\int_{\mathbb{R}} \widehat{f}(t)\gamma(at)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = \int_{\mathbb{R}} f(s-at)\gamma(t)\;\mathrm{d}t \quad (s \in \mathbb{R}).$$

Letting $a \to 0$ in the latter equality, we obtain

$$\int_{\mathbb{R}} \widehat{f}(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = \lim_{a\to 0}\int_{\mathbb{R}} f(s-at)\gamma(t)\;\mathrm{d}t \quad (s \in \mathbb{R}), \tag{5.6}$$

where we have used dominated convergence for the term on the left-hand side. In order to compute the limit on the right-hand side, we first observe that

$$\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} f(s-at)\gamma(t)\;\mathrm{d}t \right\| \mathrm{d}s \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \|f(s-at)\|\,\mathrm{d}s\,\gamma(t)\,\mathrm{d}t = \|f\|_1\|\gamma\|_1,$$

and hence, for each $a > 0$ the operator

$$\begin{aligned} S_a \colon L_1(\mathbb{R};H) &\to L_1(\mathbb{R};H),\\ f &\mapsto \left(s \mapsto \int_{\mathbb{R}} f(s-at)\gamma(t)\;\mathrm{d}t\right) \end{aligned}$$

is bounded by $\|\gamma\|_1$. Moreover, since $S_a\psi \to \psi(\cdot)\|\gamma\|_1$ as $a \to 0$ for $\psi \in C_{\mathrm{c}}(\mathbb{R};H)$, we infer that

$$S_a f \to f(\cdot)\|\gamma\|_1 \quad (a \to 0)$$

for each $f \in L_1(\mathbb{R};H)$. Hence, passing to a suitable sequence $(a_n)_n$ in $\mathbb{R}_{>0}$ tending to 0, we get

$$\lim_{n\to\infty} \left(S_{a_n} f\right)(s) \to f(s)\|\gamma\|_1 \quad (\text{a.e. } s \in \mathbb{R}).$$

Using this identity for the right-hand side of (5.6), we get

$$\int_{\mathbb{R}} \widehat{f}(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = f(s)\|\gamma\|_1 \quad (\text{a.e. } s \in \mathbb{R}),$$

and since $\|\gamma\|_1 = \sqrt{2\pi}$, we derive (5.3). □

With these preparations at hand, we are now able to prove the main theorem of this section.

Theorem 5.1.4 (Plancharel)**.** *The mapping*

$$\mathcal{F} \colon \mathcal{S}(\mathbb{R};H) \subseteq L_2(\mathbb{R};H) \to L_2(\mathbb{R};H),\ f \mapsto \widehat{f}$$

extends to a unitary operator on $L_2(\mathbb{R};H)$, again denoted by $\mathcal{F}$, the Fourier transformation*. Moreover, $\mathcal{F}^* = \mathcal{F}^{-1}$ is given by $f \mapsto \widehat{f}(-\cdot)$.*

Proof. Using (5.2) and (5.3) we obtain that

$$\left\langle \widehat{f}, \widehat{g} \right\rangle_2 = \int_{\mathbb{R}} \left\langle \widehat{f}(t), \widehat{g}(t) \right\rangle \, \mathrm{d}t = \int_{\mathbb{R}} \left\langle f(t), \widehat{\widehat{g}}(-t) \right\rangle \, \mathrm{d}t = \int_{\mathbb{R}} \langle f(t), g(t) \rangle \, \mathrm{d}t = \langle f, g \rangle_2$$

for all $f, g \in \mathcal{S}(\mathbb{R}; H)$ and thus, in particular,

$$\|f\|_2 = \|\mathcal{F} f\|_2. \tag{5.7}$$

Moreover, $\operatorname{dom}(\mathcal{F}) = \operatorname{ran}(\mathcal{F}) = \mathcal{S}(\mathbb{R}; H)$ is dense in $L_2(\mathbb{R}; H)$ and hence, the assertion follows by Exercise 5.2. □

Remark 5.1.5. We emphasise that for $f \in L_2(\mathbb{R}; H)$ the Fourier transform $\mathcal{F} f$ is not given by the integral expression for L_1-functions, simply because the integral does not need to exist. However, by dominated convergence

$$\mathcal{F} f = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathrm{e}^{-\mathrm{i}t(\cdot)} f(t) \, \mathrm{d}t,$$

where the limit is taken in $L_2(\mathbb{R}; H)$.

5.2 The Fourier–Laplace Transformation and its Relation to the Time Derivative

We now use the Fourier transformation to define an analogue transformation on our exponentially weighted L_2-type spaces; the so-called Fourier–Laplace transformation. We recall from Corollary 3.2.7 that for $\nu \in \mathbb{R}$ the mapping

$$\exp(-\nu \mathrm{m}) \colon L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H), \ f \mapsto \big(t \mapsto \mathrm{e}^{-\nu t} f(t)\big)$$

is unitary. In a similar fashion, we obtain that

$$\exp(-\nu \mathrm{m}) \colon L_{1,\nu}(\mathbb{R}; H) \to L_1(\mathbb{R}; H), \ f \mapsto \big(t \mapsto \mathrm{e}^{-\nu t} f(t)\big)$$

defines an isometry.

Definition. Let $\nu \in \mathbb{R}$. We define the *Fourier–Laplace transformation* as

$$\mathcal{L}_\nu \colon L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H), \ f \mapsto \mathcal{F} \exp(-\nu \mathrm{m}) f.$$

We can also consider the Fourier–Laplace transformation as a mapping from $L_{1,\nu}(\mathbb{R}; H)$ to $C_{\mathrm{b}}(\mathbb{R}; H)$; that is,

$$\mathcal{L}_\nu \colon L_{1,\nu}(\mathbb{R}; H) \to C_{\mathrm{b}}(\mathbb{R}; H), \ f \mapsto \mathcal{F} \exp(-\nu \mathrm{m}) f.$$

Remark 5.2.1. Note that $\mathcal{L}_\nu$ is unitary as an operator from $L_{2,\nu}(\mathbb{R};H)$ to $L_2(\mathbb{R};H)$ since it is the composition of two unitary operators. If not explicitly written otherwise, we shall use this unitary realisation of $\mathcal{L}_\nu$. For $\varphi \in L_{1,\nu}(\mathbb{R};H)$, we have the expression

$$(\mathcal{L}_\nu \varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s} \varphi(s)\,\mathrm{d}s \quad (t \in \mathbb{R}),$$

which shows that $\mathcal{L}_\nu$ can be interpreted as a shifted variant of the Fourier transformation, where the real part in the exponent equals ν instead of zero.

Our next goal is to show that the Fourier–Laplace transformation provides a spectral representation of our time derivative, $\partial_{t,\nu}$.

Definition. Let $V\colon \mathbb{R} \to \mathbb{K}$ be measurable. We define the *multiplication-by-V operator* as

$$V(\mathrm{m})\colon \operatorname{dom}(V(\mathrm{m})) \subseteq L_2(\mathbb{R};H) \to L_2(\mathbb{R};H),\ f \mapsto \big(t \mapsto V(t)f(t)\big)$$

with

$$\operatorname{dom}(V(\mathrm{m})) \coloneqq \left\{f \in L_2(\mathbb{R};H)\,;\ \big(t \mapsto V(t)f(t)\big) \in L_2(\mathbb{R};H)\right\}.$$

In particular, if V is the identity on $\mathbb{R}$ we will just write m instead of id(m) and call it the *multiplication-by-the-argument operator*.

Remark 5.2.2. Note that the multiplication-by-V operator is a vector-valued analogue of the multiplication operator seen in Example 2.4.2 and Example 2.4.3. As a consequence of Example 2.4.2, m is selfadjoint. Moreover, when $\dim(H) > 0$, by Example 2.4.3, we have that

$$\sigma(\mathrm{m}) = \mathbb{R}.$$

(Note that both statements generalise in a straight-forward way to the vector-valued situation.)

Theorem 5.2.3. *Let $\nu \in \mathbb{R}$. Then*

$$\partial_{t,\nu} = \mathcal{L}_\nu^* (\mathrm{im} + \nu) \mathcal{L}_\nu.$$

In particular,

$$\sigma(\partial_{t,\nu}) = \{\mathrm{i}t + \nu\,;\ t \in \mathbb{R}\}.$$

Proof. We first prove the assertion for $\nu \neq 0$ and show that

$$I_\nu = \mathcal{L}_\nu^* \left(\frac{1}{\mathrm{im} + \nu}\right) \mathcal{L}_\nu.$$

The assertion will then follow by Example 2.4.2. Note that $\frac{1}{\mathrm{im}+\nu} \in L(L_2(\mathbb{R};H))$ by Example 2.4.2, and hence, both operators are bounded and defined on the whole of $L_{2,\nu}(\mathbb{R};H)$. Thus, it suffices to prove the equality on a dense subset of $L_{2,\nu}(\mathbb{R};H)$, like $C_\mathrm{c}(\mathbb{R};H)$. We will just do the computation for the case when $\nu > 0$. So, let $\varphi \in C_\mathrm{c}(\mathbb{R};H)$ and compute

$$(\mathcal{L}_\nu I_\nu \varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s} \int_{-\infty}^{s} \varphi(r)\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{r}^{\infty} \mathrm{e}^{-(\mathrm{i}t+\nu)s}\,\mathrm{d}s\,\varphi(r)\,\mathrm{d}r$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}t+\nu} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)r} \varphi(r)\,\mathrm{d}r = \frac{1}{\mathrm{i}t+\nu} \left(\mathcal{L}_\nu \varphi\right)(t)$$

for $t \in \mathbb{R}$. For $\nu < 0$ the computation is analogous. In the case when $\nu = 0$ we observe that

$$\begin{aligned} \partial_{t,0} &= \exp(-\nu\mathrm{m})(\partial_{t,\nu} - \nu)\exp(-\nu\mathrm{m})^{-1} = \exp(-\nu\mathrm{m})\mathcal{L}_\nu^*(\mathrm{im}+\nu-\nu)\mathcal{L}_\nu \exp(-\nu\mathrm{m})^{-1} \\ &= \mathcal{L}_0^*(\mathrm{im})\mathcal{L}_0. \end{aligned} \qquad \square$$

5.3 Material Law Operators

Using the multiplication operator representation of $\partial_{t,\nu}$ via the Fourier–Laplace transformation, we can assign a functional calculus to this operator. We will do this in the following and define operator-valued functions of $\partial_{t,\nu}$. The class of functions used for this calculus are the so-called material laws. We begin by defining this function class.

Definition. A mapping $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ is called a *material law* if

(a) $\operatorname{dom}(M)$ is open and M is holomorphic (i.e. complex differentiable; see also Exercise 5.3),

(b) there exists some $\nu \in \mathbb{R}$ such that $\mathbb{C}_{\mathrm{Re}>\nu} \subseteq \operatorname{dom}(M)$ and

$$\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} := \sup_{z\in\mathbb{C}_{\mathrm{Re}>\nu}} \|M(z)\| < \infty.$$

Moreover, we set

$$\mathrm{s_b}\,(M) := \inf\{\nu \in \mathbb{R}\,;\ \text{(b) holds}\}$$

to be the *abscissa of boundedness* of M.

Example 5.3.1. (a) Polynomials in z^{-1}: Let $n \in \mathbb{N}_0$, $M_0, \ldots, M_n \in L(H)$. Then

$$M(z) := \sum_{k=0}^{n} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\})$$

defines a material law with

$$\mathrm{s_b}\,(M) = \begin{cases} -\infty, & \text{if } M_1 = \ldots = M_n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Series in z^{-1}: Let $(M_k)_{k\in\mathbb{N}}$ in $L(H)$ such that $\sum_{k=0}^{\infty} \|M_k\| r^{-k} < \infty$ for some $r > 0$. Then

$$M(z) := \sum_{k=0}^{\infty} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\})$$

defines a material law with $\mathrm{s_b}\,(M) \leqslant r$.

(c) Exponentials: Let $h \in \mathbb{R}, M_0 \in L(H)$ where $M_0 \neq 0$ and set

$$M(z) \coloneqq M_0 \mathrm{e}^{zh} \quad (z \in \mathbb{C}).$$

Then M is a material law if and only if $h \leqslant 0$. In this case, $\mathrm{s_b}\,(M) = -\infty$.

(d) Fourier transforms: Let $\nu \in \mathbb{R}$ and $k \in L_{1,\nu}(\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geq 0}$. Then

$$M(z) \coloneqq \widehat{k}(z) \coloneqq \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-zt} k(t)\,\mathrm{d}t \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu})$$

defines a material law with $\mathrm{s_b}\,(M) \leqslant \nu$.

(e) Fractional powers: Let $M_0 \in L(H)$, $M_0 \neq 0$, $\alpha \in \mathbb{R}$ and set

$$M(z) \coloneqq M_0 z^{-\alpha} \quad (z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}),$$

where we set

$$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^{-\alpha} \coloneqq r^{-\alpha}\mathrm{e}^{-\mathrm{i}\alpha\theta} \quad (r > 0, \theta \in (-\pi, \pi)).$$

Then M is a material law if and only if $\alpha \geqslant 0$ and

$$\mathrm{s_b}\,(M) = \begin{cases} -\infty & \text{if } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For material laws M we now define the corresponding material law operators in terms of the functional calculus induced by the spectral representation of $\partial_{t,\nu}$.

Proposition 5.3.2. *Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then, for $\nu > \mathrm{s_b}\,(M)$, the operator*

$$M(\mathrm{im} + \nu)\colon L_2(\mathbb{R}; H) \to L_2(\mathbb{R}; H),\ f \mapsto \big(t \mapsto M(\mathrm{i}t + \nu) f(t)\big)$$

is bounded. Moreover, we set

$$M(\partial_{t,\nu}) \coloneqq \mathcal{L}_\nu^* M(\mathrm{im} + \nu)\mathcal{L}_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$$

and obtain

$$\|M(\partial_{t,\nu})\| \leqslant \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}.$$

Proof. The proof is clear. □

Example 5.3.3. We now revisit the material laws presented in Example 5.3.1 and compute their corresponding operators, $M(\partial_{t,\nu})$.

(a) Let $n \in \mathbb{N}_0$, $M_0, \ldots, M_n \in L(H)$ and

$$M(z) := \sum_{k=0}^{n} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\}).$$

Then, for $\nu > 0$, one obviously has

$$M(\partial_{t,\nu}) = \sum_{k=0}^{n} \partial_{t,\nu}^{-k} M_k,$$

due to Theorem 5.2.3.

(b) Let $(M_k)_{k\in\mathbb{N}}$ in $L(H)$ such that $\sum_{k=0}^{\infty} \|M_k\| r^{-k} < \infty$ for some $r > 0$ and

$$M(z) := \sum_{k=0}^{\infty} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\}).$$

Then, for $\nu > r$, one has

$$M(\partial_{t,\nu}) = \sum_{k=0}^{\infty} \partial_{t,\nu}^{-k} M_k$$

again on account of Theorem 5.2.3.

(c) Let $h \leqslant 0, M_0 \in L(H)$ and

$$M(z) := M_0 \mathrm{e}^{zh} \quad (z \in \mathbb{C}).$$

Then, for $\nu \in \mathbb{R}$, we have

$$M(\partial_{t,\nu}) = M_0 \tau_h,$$

where

$$\tau_h \colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H),\ f \mapsto \big(t \mapsto f(t+h)\big).$$

Indeed, for $\varphi \in C_{\mathrm{c}}(\mathbb{R}; H)$ we compute

$$\begin{aligned}(\mathcal{L}_\nu M_0 \tau_h \varphi)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s} M_0 \varphi(s+h)\,\mathrm{d}s \\ &= M_0 \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)(s-h)} \varphi(s)\,\mathrm{d}s = M(\mathrm{i}t+\nu)(\mathcal{L}_\nu \varphi)(t)\end{aligned}$$

for all $t \in \mathbb{R}$, where we have used Proposition 3.1.5 in the second line. Hence,

$$M_0 \tau_h \varphi = \mathcal{L}_\nu^* M(\mathrm{im}+\nu) \mathcal{L}_\nu \varphi = M(\partial_{t,\nu})\varphi$$

and since $C_{\mathrm{c}}(\mathbb{R}; H)$ is dense in $L_{2,\nu}(\mathbb{R}; H)$ the assertion follows.

(d) Let $\nu \in \mathbb{R}$ and $k \in L_{1,\nu}(\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geq 0}$ and

$$M(z) := \widehat{k}(z) := \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-zt} k(t) \,\mathrm{d}t \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu}).$$

Then, by Exercise 5.4,

$$M(\partial_{t,\mu}) = k*$$

for each $\mu > \nu$.

(e) Let $M_0 \in L(H)$, $\alpha > 0$ and

$$M(z) := M_0 z^{-\alpha} \quad (z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

Then for $\nu > 0$ we have

$$\left(M(\partial_{t,\nu})f\right)(t) = M_0 \int_{-\infty}^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} f(s) \,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}) \tag{5.8}$$

for each $f \in L_{2,\nu}(\mathbb{R}; H)$; see Exercise 5.5. This formula gives rise to the definition

$$\left(\partial_{t,\nu}^{-\alpha} f\right)(t) := \int_{-\infty}^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} f(s) \,\mathrm{d}s \quad (t \in \mathbb{R}),$$

which is known as the *(Riemann–Liouville) fractional integral of order α*.

Throughout the previous examples, the operator $M(\partial_{t,\nu})$ did not depend on the actual value of ν. Indeed, this is true for all material laws. In order to see this, we need the following lemma.

Lemma 5.3.4. *Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$, and set $U := \{z \in \mathbb{C}\,;\, \operatorname{Re} z \in (\mu, \nu)\}$. Moreover, let $g\colon \overline{U} \to H$ be continuous and holomorphic on U such that $g(\mathrm{i}\cdot+\nu), g(\mathrm{i}\cdot+\mu) \in L_2(\mathbb{R}; H)$ and there exists a sequence $(R_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{\geqslant 0}$ such that $R_n \to \infty$ and*

$$\int_\mu^\nu \|g(\pm iR_n + \rho)\| \,\mathrm{d}\rho \to 0 \quad (n \to \infty). \tag{5.9}$$

Then

$$\mathcal{L}_\mu^* g(\mathrm{i}\cdot+\mu) = \mathcal{L}_\nu^* g(\mathrm{i}\cdot+\nu).$$

Proof. Let $t \in \mathbb{R}$. By Cauchy's integral theorem, we have that

$$\int_{\gamma_{R_n}} g(z) \mathrm{e}^{zt} \,\mathrm{d}z = 0,$$

where γ_{R_n} is the rectangular closed path with corners $\pm \mathrm{i}R_n + \mu, \pm \mathrm{i}R_n + \nu$. Thus, we have that

$$\begin{aligned} &\mathrm{i}\int_{-R_n}^{R_n} g(\mathrm{i}s+\nu)\mathrm{e}^{(\mathrm{i}s+\nu)t} \,\mathrm{d}s - \mathrm{i}\int_{-R_n}^{R_n} g(\mathrm{i}s+\mu)\mathrm{e}^{(\mathrm{i}s+\mu)t} \,\mathrm{d}s \\ &= -\int_\mu^\nu g(-\mathrm{i}R_n+\rho)\mathrm{e}^{(-\mathrm{i}R_n+\rho)t} \,\mathrm{d}\rho + \int_\mu^\nu g(\mathrm{i}R_n+\rho)\mathrm{e}^{(\mathrm{i}R_n+\rho)t} \,\mathrm{d}\rho. \end{aligned} \tag{5.10}$$

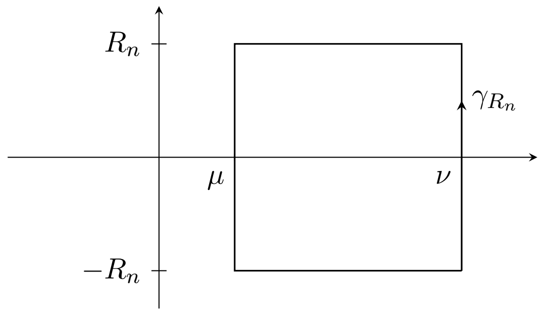

Figure 5.1: Curve γ_{R_n}.

Note that the left-hand side of (5.10) is nothing but

$$\sqrt{2\pi}\mathrm{i}\left(\left(\mathcal{L}_\nu^* \mathbb{1}_{[-R_n,R_n]} g(\mathrm{i}\cdot+\nu)\right)(t) - \left(\mathcal{L}_\mu^* \mathbb{1}_{[-R_n,R_n]} g(\mathrm{i}\cdot+\mu)\right)(t)\right),$$

and hence, there is a subsequence of $(R_n)_n$ (which we do not relabel) such that the left-hand side of (5.10) tends to

$$\sqrt{2\pi}\mathrm{i}\left(\left(\mathcal{L}_\nu^* g(\mathrm{i}\cdot+\nu)\right)(t) - \left(\mathcal{L}_\mu^* g(\mathrm{i}\cdot+\mu)\right)(t)\right)$$

for almost every $t \in \mathbb{R}$ as $n \to \infty$. As such, all we need to show is that the right-hand side of (5.10) tends to 0 as $n \to \infty$, which obviously follows by (5.9). □

Theorem 5.3.5. *Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then, for $\mu, \nu > \mathrm{s_b}(M)$ and $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$, we have*

$$M(\partial_{t,\nu})f = M(\partial_{t,\mu})f.$$

Moreover, $M(\partial_{t,\nu})$ is causal for all $\nu > \mathrm{s_b}(M)$.

Proof. Let $\mu < \nu$. We prove the assertion for $f = \mathbb{1}_{[a,b]} \cdot x$ with $a < b$ and $x \in H$ first. For $\rho \in \mathbb{R}$ we compute

$$(\mathcal{L}_\rho f)(t) = \frac{1}{\sqrt{2\pi}} \int_a^b x \mathrm{e}^{-(\mathrm{i}t+\rho)s}\,\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}t+\rho}\left(\mathrm{e}^{-(\mathrm{i}t+\rho)a} - \mathrm{e}^{-(\mathrm{i}t+\rho)b}\right)x \quad (t \in \mathbb{R}\setminus\{0\}).$$

Moreover, we define

$$g(z) \coloneqq \frac{1}{\sqrt{2\pi}} M(z) x \frac{1}{z}\left(\mathrm{e}^{-za} - \mathrm{e}^{-zb}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}\geqslant\mu}\setminus\{0\})$$

and prove that g satisfies the assumptions of Lemma 5.3.4. First, we note that g is bounded on $\mathbb{C}_{\mathrm{Re}\geqslant\mu}\setminus\{0\}$. Indeed, we only need to prove that it is also bounded near 0 provided that $\mu \leqslant 0$. To that end, we observe

$$\frac{1}{z}(\mathrm{e}^{-za} - \mathrm{e}^{-zb}) = \mathrm{e}^{-za}\frac{1-\mathrm{e}^{-z(b-a)}}{z} \to b-a \quad (z \to 0).$$

Thus, g is also bounded near 0. In particular, $z=0$ is a removable singularity and, hence, g can be extended holomorphically to $\mathbb{C}_{\mathrm{Re}\geqslant\mu}$. Moreover, for $\rho\geqslant\mu$ we have that

$$\int_{\mathbb{R}}\|g(\mathrm{i}t+\rho)\|^2\,\mathrm{d}t=\int_{-1}^{1}\|g(\mathrm{i}t+\rho)\|^2\,\mathrm{d}t+\int_{|t|>1}\|g(\mathrm{i}t+\rho)\|^2\,\mathrm{d}t.$$

The first term on the right-hand side is finite since g is bounded, while the second term can be estimated by

$$\int_{|t|>1}\|g(\mathrm{i}t+\rho)\|^2\,\mathrm{d}t\leqslant\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\mu}}\|x\|\frac{\mathrm{e}^{-\rho a}+\mathrm{e}^{-\rho b}}{\sqrt{2\pi}}\int_{|t|>1}\frac{1}{t^2+\rho^2}\,\mathrm{d}t<\infty.$$

This proves that $g(\mathrm{i}\cdot+\rho)\in L_2(\mathbb{R};H)$ for each $\rho\geqslant\mu$ and hence, in particular, for $\rho=\mu$ and $\rho=\nu$. Finally, for $\rho\geqslant\mu$ we have that

$$\|g(\mathrm{i}t+\rho)\|\leqslant\frac{1}{\sqrt{2\pi}}\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\mu}}\|x\|\frac{1}{\sqrt{t^2+\rho^2}}\left(\mathrm{e}^{-\rho a}+\mathrm{e}^{-\rho b}\right)\to 0\quad(|t|\to\infty),$$

which together with the boundedness of g yields (5.9) by dominated convergence. This shows that g satisfies the assumptions of Lemma 5.3.4 and thus

$$M(\partial_{t,\nu})f=\mathcal{L}_\nu^*g(\mathrm{i}\cdot+\nu)=\mathcal{L}_\mu^*g(\mathrm{i}\cdot+\mu)=M(\partial_{t,\mu})f.$$

By linearity, this equality extends to $S(\mathbb{R};H)$ and so,

$$F\colon S(\mathbb{R};H)\to\bigcap_{\nu\geqslant\mu}L_{2,\nu}(\mathbb{R};H),\ f\mapsto M(\partial_{t,\nu})f$$

is well-defined. Moreover, F is uniformly Lipschitz continuous (with $\sup_{\nu\geqslant\mu}\|F^\nu\|=\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\mu}}$) and hence, the assertions follow from Lemma 4.2.3. □

5.4 Comments

The Fourier and the Fourier–Laplace transformation introduced in this chapter are used to define an operator-valued functional calculus for the time derivative, $\partial_{t,\nu}$. This functional calculus can be defined since the Fourier–Laplace transformation provides the unitary transformation yielding the spectral representation of the time derivative as multiplication operator. This fact was already noticed in [5], which eventually led to evolutionary equations in [4].

We emphasise that we have used the fundamental property that both $\mathcal{F}$ and $\mathcal{L}_\nu$ are unitary. It is noteworthy that the Fourier transformation is unitary on $L_2(\mathbb{R};X)$ if and only if X is a Hilbert space, see [3]. In the Banach space-valued case one has to further restrict the class of functions used to define a functional calculus. For the topic of functional calculus we refer to the 21st ISem [1] by Markus Haase and to his monograph, [2].

Exercises

Exercise 5.1. Let (Ω, Σ, μ) be a measure space, X a Banach space and $I \subseteq \mathbb{R}$ an open interval. Let $g\colon I \times \Omega \to X$ such that $g(t, \cdot) \in L_1(\mu; X)$ for each $t \in I$, and define

$$h\colon I \to X,\ t \mapsto \int_\Omega g(t, \omega)\, \mathrm{d}\mu(\omega).$$

(a) Assume that $g(\cdot, \omega)$ is continuous for μ-almost every $\omega \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|g(t, \omega)\| \leqslant f(\omega) \quad (t \in I, \omega \in \Omega).$$

Prove that h is continuous.

(b) Assume that $g(\cdot, \omega)$ is differentiable for μ-almost every $\omega \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|\partial_t g(t, \omega)\| \leqslant f(\omega) \quad (t \in I, \omega \in \Omega).$$

Prove that h is differentiable with

$$h'(t) = \int_\Omega \partial_t g(t, \omega)\, \mathrm{d}\mu(\omega).$$

Exercise 5.2. Let H_0, H_1 be two Hilbert spaces and $U\colon \operatorname{dom}(U) \subseteq H_0 \to H_1$ linear such that

- $\operatorname{dom}(U)$ is dense in H_0 and $\operatorname{ran}(U)$ is dense in H_1.
- $\forall x \in \operatorname{dom}(U) : \ \|Ux\|_{H_1} = \|x\|_{H_0}$.

Show that U can be uniquely extended to a unitary operator between H_0 and H_1.

Exercise 5.3. Let $\Omega \subseteq \mathbb{C}$ be open, X a complex Banach space and $f\colon \Omega \to X$. Prove that the following statements are equivalent:

(i) f is holomorphic.

(ii) For all $x' \in X'$ the mapping $x' \circ f\colon \Omega \to \mathbb{C}$ is holomorphic.

(iii) f is locally bounded and $x' \circ f\colon \Omega \to \mathbb{C}$ is holomorphic for all $x' \in D$, where $D \subseteq X'$ is a norming set[1] for X.

(iv) f is analytic, i.e. for each $z_0 \in \Omega$ there is $r > 0$ and $(a_n)_n$ in X with $B(z_0, r) \subseteq \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (z \in B(z_0, r)).$$

[1] $D \subseteq X'$ is called a norming set for X, if $\|x\| = \sup_{x' \in D \setminus \{0\}} \frac{1}{\|x'\|} |x'(x)|$ for each $x \in X$. Note that X' is norming for X by the Hahn–Banach theorem.

Assume now that $X = L(X_1, X_2)$ for two complex Banach spaces X_1, X_2, let $D_1 \subseteq X_1$ be dense and $D_2 \subseteq X_2'$ norming for X_2. Prove that the statements (a) to (d) are equivalent to

(v) f is locally bounded and $\Omega \ni z \mapsto x_2'(f(z)(x_1)) \in \mathbb{C}$ is holomorphic.

Exercise 5.4. Let $\nu \in \mathbb{R}$ and $k \in L_{1,\nu}(\mathbb{R})$. Prove that

$$\mathcal{L}_\nu (k * f) = \sqrt{2\pi}\, (\mathcal{L}_\nu k) \cdot (\mathcal{L}_\nu f)$$

for $f \in L_{2,\nu}(\mathbb{R}; H)$.

Exercise 5.5. Let $\alpha > 0$ and define $g_\alpha(t) \coloneqq \mathbb{1}_{[0,\infty)}(t) t^{\alpha-1}$ for $t \in \mathbb{R}$. Show that $g_\alpha \in L_{1,\nu}(\mathbb{R})$ for each $\nu > 0$ and that

$$(\mathcal{L}_\nu g_\alpha)(t) = \frac{1}{\sqrt{2\pi}} \Gamma(\alpha)(\mathrm{i}t + \nu)^{-\alpha}.$$

Use this formula and Exercise 5.4 to derive (5.8).
Hint: To compute the Fourier–Laplace transform of g_α, derive that $\mathcal{L}_\nu g_\alpha$ solves a first order ordinary differential equation and use separation of variables to solve this equation.

Exercise 5.6. Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ and $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$. Moreover, set $U \coloneqq \{z \in \mathbb{C}\,;\, \mu < \operatorname{Re} z < \nu\}$. Show that $f \in \bigcap_{\mu<\rho<\nu} L_{2,\rho}(\mathbb{R}; H) \cap L_{1,\rho}(\mathbb{R}; H)$ and that

$$U \ni z \mapsto (\mathcal{L}_{\operatorname{Re} z} f)(\operatorname{Im} z)$$

is holomorphic.

Exercise 5.7. Let H_0, H_1 be Hilbert spaces and $T\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ linear and bounded. We call T *autonomous* if $T\tau_h = \tau_h T$ for each $h \in \mathbb{R}$ (τ_h denotes the translation operator defined in Example 5.3.3). Prove that for autonomous T, the following statements are equivalent:

(i) T is causal.

(ii) For all $f \in L_{2,\nu}(\mathbb{R}; H_0)$ with $\operatorname{spt} f \subseteq [0, \infty)$ one has $\operatorname{spt} Tf \subseteq [0, \infty)$.

Moreover, prove that for a material law M, the operator $M(\partial_{t,\nu})$ is autonomous for each $\nu > \mathrm{s_b}(M)$.

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