Exercise 3.1. A sequence $(\varphi_n)_n$ in $C_c^\infty(\mathbb{R}^d)$ is called a *δ-sequence* if
(a) $\varphi_n \geq 0$ for $n \in \mathbb{N}$,
(b) $\operatorname{spt}\varphi_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]^d$ for $n \in \mathbb{N}$,
(c) $\int_{\mathbb{R}^d} \varphi_n = 1$ for $n \in \mathbb{N}$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{spt}\varphi \subseteq [-1,1]^d$, $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Prove that $(\varphi_n)_n$ given by $\varphi_n(x) := n^d \varphi(n^d x)$ for $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ defines a δ-sequence. Moreover, give an example for such a function φ.

Solution. There is a typo in the exercise: It should be $\varphi_n(x) = n^d\varphi(nx)$ instead of $\varphi_n(x) = n^d\varphi(n^d x)$!

- The properties $\varphi_n \geq 0$ and $\operatorname{spt}\varphi_n \subseteq [-1/n, 1/n]^d$ for all $n \in \mathbb{N}$ follow immediately from the assumptions $\varphi \geq 0$ and $\operatorname{spt}\varphi \subseteq [-1,1]^d$.
- Moreover, for all $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^d} \varphi_n(x)\mathrm{d}x = \int_{\mathbb{R}^d} n^d\varphi(nx)\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(y)\mathrm{d}y = 1$$

  by change of variables. Hence, the sequence $(\varphi_n)_n$ is a δ-sequence.
- An example for a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\operatorname{spt}\varphi_n \subseteq [-1,1]^d$, $\varphi \geq 0$, and $\|\varphi\|_{L_1(\mathbb{R}^d)} = 1$ is

$$\varphi(x) = \begin{cases} C\mathrm{e}^{-1/(1-|x|^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

  where C is chosen so that $\|\varphi\|_{L_1(\mathbb{R}^d)} = 1$.

Exercise 3.2. It is well-known that $\{\mathbf{1}_I\,;\, I \text{ } d\text{-dimensional bounded interval}\}$ is total in $L_2(\mathbb{R}^d)$ (note that the linear span of which is a smaller set than $S(\mathbb{R}^d)$).
(a) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $f \in L_2(\mathbb{R}^d)$. Prove that $f*\varphi \in C_c^\infty(\mathbb{R})$ with $\partial^\alpha (f*\varphi) = f*\partial^\alpha\varphi$ for all $\alpha \in \mathbb{N}_0^d$, where $\partial^\alpha\varphi = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}\varphi$. Moreover, prove that $\operatorname{spt} f*\varphi \subseteq \operatorname{spt} f + \operatorname{spt}\varphi$.
(b) Let $(\varphi_n)_n$ be a δ-sequence and $f \in L_2(\mathbb{R}^d)$. Show that $f*\varphi_n \to f$ in $L_2(\mathbb{R}^d)$ as $n \to \infty$.
Hint: Prove that $\mathbf{1}_I * \varphi_n \to \mathbf{1}_I$ in $L_2(\mathbb{R}^d)$ for all d-dimensional intervals and use that $\|f*\varphi_n\|_2 \leq \|f\|_2$ (see also ...).
(c) Prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$.

Solution. (a) ▶ Let $\varphi \in C_c^\infty(\mathrm{R}^d)$ and $f \in L_2(\mathrm{R})^d$. Prove that $f*\varphi \in C_c^\infty(\mathbb{R})$:

We rather give a class of counterexamples: For $\varphi \in C_c^\infty(\mathrm{R}^d)$, $\varphi \geq 0$, $\varphi \neq 0$ and $f > 0$ a.e., $f*\varphi \notin C_c^\infty(\mathbb{R}^d)$.

▶ ... with $\partial^\alpha (f*\varphi) = f*\partial^\alpha\varphi$ for all $\alpha \in \mathbb{N}_0^d$, where $\partial^\alpha\varphi = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}\varphi$.

This is, of course, textbook stuff ;) therefore we only highlight two ideas:

Distributional derivatives:

For any testfunction $\psi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}\langle f * \varphi, \partial^\alpha \psi\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\varphi(x-y)dy\partial^\alpha\psi(x)dx \\ &= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d}\varphi(x-y)\partial^\alpha\psi(x)dxdy \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d}\partial^\alpha\varphi(x-y)\psi(x)dxdy \\ &= (-1)^{|\alpha|}\langle f * \partial^\alpha\varphi, \psi\rangle,\end{aligned}$$

where we used Fubini (note that the integrability condition on f together with the compact supports of the test functions guarantee the required integrability) in the second line and integration by parts in the third line.

The identity thus obtained gives the asserted equality

$$\partial^\alpha (f * \varphi) = f * \partial^\alpha\varphi$$

in the distributional sense. By Sobolev embedding theorems, we also get the assertion for classical derivatives.

Classical derivatives:

It is easy to see that it suffices to consider the case $d = 1$ and one differentiation, which we denote by the usual $'$. By assumption, the function

$$g : \mathbb{R} \to \mathbb{R}, g(t) := \int_{\mathbb{R}} f(s)\varphi'(t-s)\mathrm{d}s$$

is continuous. In fact, this is in a certain sense the real heart of the matter, it follows by a standard application of Lebesgue's theorem. We use the familiar convention that $\int_0^t ...\mathrm{d}s = -\int_{[t,0]} ...\mathrm{d}s$ provided $t < 0$. The fundamental theorem of calculus gives that $G : x \mapsto \int_0^x g(t)\mathrm{d}t$ is differentiable with derivative $g = f * \varphi'$ and a short calculation gives that $G = f * \varphi$ up to an additive constant. In fact, using Fubini and the fundamental theorem of calculus once more:

$$\begin{aligned}\int_0^x g(t)\mathrm{d}t &= \int_0^x\int_{\mathbb{R}} f(s)\varphi'(t-s)\mathrm{d}s\ \mathrm{d}t \\ &= \int_{\mathbb{R}} f(s)\int_0^x \varphi'(t-s)\mathrm{d}t\ \mathrm{d}s \\ &= \int_{\mathbb{R}} f(s)(\varphi(x-s)-\varphi(-s))\mathrm{d}s \\ &= f*\varphi(x) - f*\varphi(0),\end{aligned}$$

giving the claim.

► ... prove that $\operatorname{spt} f * \varphi \subseteq \operatorname{spt} f + \operatorname{spt} \varphi$:

The claim is obvious for continuous functions by looking at the integrand that defines $f * \varphi$. For the case in question, pick a measurable representative $\hat{f}$ of the equivalence class f with the additional property that $\hat{f} = 0$ outside $\operatorname{spt}(f)$. Then

$$\hat{f}(\cdot)\varphi(x - \cdot) = 0 \text{ unless } x \in \operatorname{spt}(f) + \operatorname{spt}(\varphi) \quad (\clubsuit);$$

indeed, $\hat{f}(y)\varphi(x-y) \neq 0$ implies that $y \in \operatorname{spt}(f)$ and $x - y \in \operatorname{spt}(\varphi)$. Property ($\clubsuit$) gives that $f * \varphi$ vanishes on the complement of $\operatorname{spt} f + \operatorname{spt} \varphi$. Since the latter is a closed set, we are done.

(b) ► Fix I, a bounded interval as in the assertion. Note that for $x \in I^\circ$, it follows that $\operatorname{spt}(\varphi_n(x - \cdot)) \subset I$ for large enough n so that $\mathbf{1}_I * \varphi_n(x) \to 1$ as $n \to \infty$. By the same reason, $\mathbf{1}_I * \varphi_n(x) \to 0$ as $n \to \infty$ for any $x \in (\bar{I})^c$. As the boundary of I has vanishing Lebesgue measure, $\mathbf{1}_I * \varphi_n \to \mathbf{1}_I$ pointwise a.e. and an appeal to Lebesgue's theorem shows that $\mathbf{1}_I * \varphi_n \to \mathbf{1}_I$ in L_2. (If you haven't discovered a majorizing function by now, try $\mathbf{1}_J$ for $J = I + [-1, 1]^d$;)

This already settles (c) since it follows that the closure of $C_c^\infty(\mathbb{R}^d)$ contains $\{\mathbf{1}_I\,;\, I \text{ } d\text{-dimensional bounded interval}\}$.

► Consider now the linear contractions (use Lemma 3.2.1)

$$J_n : L_2 \to L_2, f \mapsto \varphi_n * f.$$

As $J_n \to Id$ on the dense linear hull of $\{\mathbf{1}_I\,;\, I \text{ } d\text{-dimensional bounded interval}\}$, denoted by D, by what we just proved, the unifrom boundedness implies that $J_n \to Id$ on L_2, the assertion. (In fact, fix $f \in L_2$ and $\varepsilon > 0$; take $g \in D$ such that $\|f - g\| \leq \frac{1}{3}\varepsilon$. By definition of D and the convergence established above, there is $n_0 \in \mathbb{N}$ such that $\|J_n g - g\| \leq \frac{1}{3}\varepsilon$ for all $n \geq n_0$. For such n it follows that

$$\|J_n f - f\| \leq \|J_n f - J_n g\| + \|J_n g - g\| + \|g - f\| \leq \varepsilon,$$

since $\|J_n f - J_n g\| \leq \|J_n\| \, \|g - f\|$ and we are done.

(c) See above.

Exercise 3.3. Let $a < b$, X_0, X_1, X_2 be Banach spaces, $f\colon (a, b) \to X_0$ and $g\colon (a, b) \to X_1$ both continuously differentiable, $\ell\colon X_0 \times X_1 \to X_2$ bilinear and continuous. Prove that $h\colon (a, b) \to X_2$ given by

$$h(t) := \ell(f(t), g(t)) \quad (t \in (a, b))$$

is continuously differentiable with

$$h'(t) = \ell(f'(t), g(t)) + \ell(f(t), g'(t)) \quad (t \in (a, b)).$$

If f, f', g, g' have continuous extensions to $[a, b]$, prove the integration by parts formula:

$$\int_a^b \ell(f'(t), g(t))\mathrm{d}t = \ell(f(b), g(b)) - \ell(f(a), g(a)) - \int_a^b \ell(f(t), g'(t))\mathrm{d}t.$$

Solution. ▶ We first prove differentiability of h and the formula for h'. Our proof below actually gives a local version and we do not need continuity of the derivatives!

Fix $t_0 \in (a,b)$ and let $t \in (a,b)$. Then bilinearity of ℓ gives that

$$\begin{aligned} h(t) - h(t_0) &= \ell(f(t), g(t)) - \ell(f(t_0), g(t_0)) \\ &= \ell(f(t), g(t)) - \ell(f(t_0), g(t)) + \ell(f(t_0), g(t)) - \ell(f(t_0), g(t_0)) \\ &= \ell(f(t) - f(t_0), g(t)) + \ell(f(t_0), g(t) - g(t_0)). \end{aligned}$$

If f is differentiable at t_0, it is continuous at t_0 as well, so that:

$$f(t) \to f(t_0), \frac{f(t) - f(t_0)}{t - t_0} \to f'(t_0) \text{ as } t \to t_0.$$

With the respective convergence for g, bilinearity and continuity of ℓ we get that

$$\begin{aligned} \frac{h(t) - h(t_0)}{t - t_0} &= \frac{1}{t - t_0}\ell(f(t) - f(t_0), g(t)) + \frac{1}{t - t_0}\ell(f(t_0), g(t) - g(t_0)) \\ &= \ell\left(\frac{f(t) - f(t_0)}{t - t_0}, g(t)\right) + \ell\left(f(t_0), \frac{g(t) - g(t_0)}{t - t_0}\right) \\ &\to \ell\left(f'(t_0), g(t_0)\right) + \ell\left(f(t_0), g'(t_0)\right), \end{aligned}$$

for $t \to t_0$, the asserted product rule.

▶ For the integration by parts formula, we now assume the mentioned continuity properties. The fundamental theorem of calculus gives that, for $a < a' \leq b' < b$:

$$\ell(f(b'), g(b')) - \ell(f(a'), g(a')) = \int_{a'}^{b'} \ell(f'(t), g(t))\mathrm{d}t + \int_{a'}^{b'} \ell(f(t), g'(t))\mathrm{d}t.$$

Rearranging the terms and the limits $a' \to a$ and $b' \to b$ proves the claim.

Exercise 3.4. For $\nu \neq 0$, show that $\|I_\nu\| = 1/|\nu|$.

Solution. From Lemma 3.2.1 we already know that $\|I_\nu\| \leq 1/|\nu|$. In order to prove the reverse inequality we assume that $\nu > 0$. (The case $\nu < 0$ is similar.) Note that in the vector valued case, one has to assume that $H \neq \{0\}$, since otherwise the claim is incorrect! Moreover, we could restrict ourselves to the scalar case, as can be seen from the following. Let $\psi \in H$ with $\|\psi\| = 1$ and define for $\epsilon > 0$ the function $f_\epsilon \in L_{2,\nu}(\mathbb{R}, H)$ by

$$f_\epsilon(t) = \mathrm{e}^{\nu t}\epsilon^{1/2}\mathbf{1}_{[-1/2,1/2]}(\epsilon t)\psi.$$

We will show that this family can be used to get the lower bound by an elementary calculation. First, for $\epsilon > 0$ we find that

$$\|f_\epsilon\|^2_{L_{2,\nu}(\mathbb{R},H)} = \int_{\mathbb{R}} \epsilon\mathbf{1}_{[-1/2,1/2]}(\epsilon t)\mathrm{d}t = 1,$$

and for $\epsilon > 0$ and $t \in \mathbb{R}$

$$\left\| \int_{-\infty}^{t} f_\epsilon(s) \mathrm{d}s \right\| = \left(\int_{-\infty}^{t} \mathrm{e}^{\nu s} \epsilon^{1/2} \mathbf{1}_{[-1/2,1/2]}(\epsilon s) \mathrm{d}s \right)^2$$

$$= \begin{cases} 0 & \text{if } t \leq 1/(2\epsilon), \\ \frac{\epsilon}{\nu^2} \left(\mathrm{e}^{\nu t} - \mathrm{e}^{-\nu/(2\epsilon)} \right)^2 & \text{if } -1/(2\epsilon) < t < 1/(2\epsilon), \\ \frac{\epsilon}{\nu^2} \left(\mathrm{e}^{\nu/(2\epsilon)} - \mathrm{e}^{-\nu/(2\epsilon)} \right)^2 & \text{if } t \geq 1/(2\epsilon). \end{cases}$$

For $\epsilon > 0$ we calculate

$$\|I_\nu f_\epsilon\|^2_{L_{2,\nu}(\mathbb{R},H)} = \int_{\mathbb{R}} \mathrm{e}^{-2\nu t} \left\| \int_{-\infty}^{t} f_\epsilon(s) \mathrm{d}s \right\|^2 \mathrm{d}t$$

$$= \frac{\epsilon}{\nu^2} \int_{-1/(2\epsilon)}^{1/(2\epsilon)} \mathrm{e}^{-2\nu t} \left(\mathrm{e}^{\nu t} - \mathrm{e}^{-\nu/(2\epsilon)} \right)^2 \mathrm{d}t + \frac{\epsilon}{\nu^2} \left(\mathrm{e}^{\nu/(2\epsilon)} - \mathrm{e}^{-\nu/(2\epsilon)} \right)^2 \int_{1/(2\epsilon)}^{\infty} \mathrm{e}^{-2\nu t} \mathrm{d}t$$

$$= \frac{\epsilon}{\nu^2} \left(\frac{1}{\epsilon} - \frac{1}{2\nu} \left(\mathrm{e}^{-2\nu/\epsilon} - 1 \right) + \frac{2}{\nu} \left(\mathrm{e}^{-\nu/\epsilon} - 1 \right) \right) + \frac{\epsilon}{2\nu^3} \left(1 - \mathrm{e}^{-\nu/\epsilon} \right)^2.$$

Thus, we get that $\|I_\nu f_\epsilon\|^2_{L_{2,\nu}(\mathbb{R},H)} \to 1/\nu^2$ as $\epsilon \to 0$. This implies $\|I_\nu\| \geq 1/\nu$.

Exercise 3.5. Prove Corollary 3.2.7, i.e., for $\nu \in \mathbb{R}$ the mapping

$$\exp(-\nu m) : L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H), f \mapsto (t \mapsto e^{-\nu t} f(t))$$

is unitary, and for $\nu, \mu \neq 0$:

$$\exp(-\nu m)(\partial_{t,\nu} - \nu) \exp(-\nu m)^{-1} = \exp(-\mu m)(\partial_{t,\mu} - \mu) \exp(-\mu m)^{-1} \qquad (*).$$

Moreover, the operator

$$\partial_{t,0} := \exp(-\nu m)(\partial_{t,\nu} - \nu) \exp(-\nu m)^{-1}$$

is skew-selfadjoint for all $\nu \neq 0$.

Solution. ▶ The first assertion follows by inspecting the definitions: clearly, $\exp(-\nu m)$ is linear and isometric and the inverse $\exp(-\nu m)^{-1}$ is just multiplication with the inverse function.

▶ For further use notice that $\exp(-\nu m)$ also maps test functions onto test functions, i.e.

$$\exp(-\nu m)(\mathcal{D}_H) = \mathcal{D}_H.$$

- By the previous observation and Proposition 3.2.4 is suffices to check equality $(*)$ for test functions, i.e. on the core $\mathcal{D}_H$. Using linearity, we see that it is enough to consider functions of the form φx, with $\varphi \in C_c^\infty(\mathbb{R})$ and $x \in H$, so finally, we can forget about H and consider the scalar case, $\varphi \in C_c^\infty(\mathbb{R})$. By slight abuse of notation we denote the mapping

$$\exp(\nu m) : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}), \psi \mapsto (t \mapsto \mathrm{e}^{\nu t}\psi(t)),$$

by the same symbol that was designated to its cousin between L_2–spaces. Obviously, $\exp(\nu m)\psi$ is a representative of the equivalence class $\exp(-\nu m)^{-1}\psi \in L_{2,\nu}$ for any $\psi \in C_c^\infty(\mathbb{R})$. The product rule gives:

$$\begin{aligned} \partial_{t,\nu}(\exp(\nu m)\psi) - \nu(\exp(\nu m)\psi) &= (\exp(\nu m)\psi)' - \nu(\exp(\nu m)\psi) \\ &= \exp(\nu m)\psi' \end{aligned}$$

where we used $\partial_{t,\nu}\varphi = \varphi'$, see the discussion preceeding Proposition 3.2.4. for the first equality sign and the product rule for the second equality. Therefore,

$$\exp(-\nu m)(\partial_{t,\nu} - \nu)\exp(-\nu m)^{-1}\psi = \psi'$$

holds for all $\nu \neq 0$, $\psi \in C_c^\infty(\mathbb{R})$ giving $(*)$ by our discussion above.

- From the previous observation we get that

$$A_\nu^0 := (\partial_{t,\nu} - \nu)|C_c^\infty(\mathbb{R})$$

is skew-symmetric. In fact, it is unitarily equivalent to $A^0 := C_c^\infty(\mathbb{R}) \ni \varphi \mapsto \varphi'$ in L_2 and this operator is skew-symmetric as can be seen from integration by parts. Since $C_c^\infty(\mathbb{R})$ is a core, it follows that $(\partial_{t,\nu} - \nu) =: A_\nu$ satisfies

$$A_\nu^* = (\overline{A_\nu^0})^* = (A_\nu^0)^* \supset -A_\nu^0.$$

Closedness of A_ν gives that

$$A_\nu^* \supset -A_\nu.$$

- An appeal to Corollary 3.2.6 implies that the domains of A_ν^* and $-A_\nu$ are equal, so that the claim of the exercise is established.

Exercise 3.6. Let $\nu \in \mathbb{R}$ and H be a complex Hilbert space. Prove that $\sigma(\partial_{t,\nu}) \subseteq \{\mathrm{i}t + \nu\,;\, t \in \mathbb{R}\}$, where $\partial_{t,0}$ is defined in Corollary 3.2.7.
Hint: For $f \in \mathrm{dom}(\partial_{t,\nu}), z \in \mathbb{C}$ compute $\langle \mathrm{Re}\,(z - \partial_{t,\nu})f, f\rangle_{L_{2,\nu}(\mathbb{R};H)}$ by using Corollary 3.2.6. For proving the surjectivity of $z - \partial_{t,\nu}$ for a suitable z, use the formula

$$\overline{\mathrm{ran}}(z - \partial_{t,\nu}) = \ker(z^* - \partial_{t,\nu}^*)^\perp.$$

Remark: Later we will see that, actually, $\sigma(\partial_{t,\nu}) = \{\mathrm{i}t + \nu\,;\, t \in \mathbb{R}\}$.

Solution. From Corollary 3.2.7, see the previous exercise and solution, we know that for all $\nu \in \mathbb{R}$ the operator $\partial_{t,\nu} - \nu$ is skew-selfadjoint. (This was the claim for $\nu = 0$ and together with the unitary equivalence it extends to arbitrary ν.)

Hence, its spectrum is purely imaginary. This implies $\sigma(\partial_{t,\nu}) = \nu + \sigma(\partial_{t,\nu} - \nu) \subseteq \{\nu + \mathrm{i}t\,;\, t \in \mathbb{R}\}$.

Exercise 3.7. Consider the differential equation

$$\left(\partial_{t,\nu}^2 - 1\right) u = \mathbf{1}_{[-1,1]}.$$

Since $\partial_{t,\nu}^2 - 1 = (\partial_{t,\nu} - 1)\,(\partial_{t,\nu} + 1)$, it follows by that there is a unique $u \in L_{2,\nu}(\mathbb{R})$ solving this equation if $\nu \notin \{-1, 1\}$. Compute these solutions.

Hint: For $u \in \mathrm{dom}(\partial_{t,\nu})$ use the fact that u is necessarily continuous (which we shall establish in the next lecture).

Solution. We begin by remarking that it suffices to consider the case $\nu \in [0, \infty) \setminus \{1\}$. Indeed, let u be the unique solution in $\mathrm{dom}(\partial_{t,\nu})$ to the given ODE. Then clearly we have that $u(-\cdot)$ is the unique solution of the given ODE lying in the space $\mathrm{dom}(\partial_{t,-\nu})$.

According to Proposition 4.1.1, $\partial_{t,\nu} u$ agrees with the distributional derivative $\dot{u}$ of u if we view u as a distribution. By the characterization of $\mathrm{dom}(\partial_{t,\nu})$ given in Proposition 4.1.1, we need to find a distributional solution u of

$$\ddot{u} - u = \mathbf{1}_{[-1,1]},$$

such that $u, \dot{u} \in L_{2,\nu}(\mathbb{R})$. Since $u, \dot{u} \in \mathrm{dom}(\partial_{t,\nu})$, it also follows from Theorem 4.1.2 that u and $\dot{u}$ have to be continuous.

If we define the open sets $J_1 = [-\infty, -1)$, $J_2 = (-1, 1)$, $J_3 = (1, \infty)$, then the restrictions $u_1 = u|_{J_1}, u_2 = u|_{J_2}, u_3 = u|_{J_3}$ solve the ODEs

$$\ddot{u}_1 - u_1 = 0\,, \quad \ddot{u}_2 - u_2 = 1\,, \quad \ddot{u}_3 - u_3 = 0\,.$$

It is well known that the functions $\{e^t, e^{-t}\}$ form a fundamental system for the homogeneous ODE $\ddot{u} - u = 0$. It is therefore easy to see that there exist $a_j, b_j \in \mathbb{R}$ where $j = 1, 2, 3$ such that

$$\begin{aligned} u_1(t) &= a_1 e^t + b_1 e^{-t} \\ u_2(t) &= a_2 e^t + b_2 e^{-t} - 1 \\ u_3(t) &= a_3 e^t + b_3 e^{-t}\,. \end{aligned}$$

We therefore aim to choose the a_j, b_j such that

$$u(t) = \begin{cases} u_1(t) = a_1 e^t + b_1 e^{-t} & : \quad t \leq -1 \\ u_2(t) = a_2 e^t + b_2 e^{-t} - 1 & : \quad -1 < t \leq 1 \\ u_3(t) = a_3 e^t + b_3 e^{-t} & : \quad t > 1 \end{cases}$$

lies in $C^1(\mathbb{R})$. Moreover, we want to choose the a_j, b_j such that in addition $u, \dot{u} \in L_{2,\nu}(\mathbb{R})$ holds. By the above discussion, such a choice of the a_j, b_j gives us the unique solution of

$$\partial_{t,\nu}^2 u - u = \mathbf{1}_{[-1,1]}\,.$$

We start by assuming that $\nu > 1$. Since $e^t \mathbf{1}_{J_1}, e^{-t}\mathbf{1}_{J_1} \notin L_{2,\nu}$ in this case, it follows that we must choose $a_1 = b_1 = 0$ and thus $u_1 = 0$. In order to ensure continuity, we require that

$$\begin{aligned} 0 = u(-1) &= a_2 e^{-1} + b_2 e^{-1} - 1 \\ 0 = \dot{u}(-1) &= a_2 e^{-1} - b_2 e\,. \end{aligned}$$

This linear system has the unique solution $a_2 = -\frac{1}{2}e$, $b_2 = -\frac{1}{2}e^{-1}$. It follows that $u_2(t) = -1 - \cosh(t+1)$. To determine a_3, b_3, we need to solve the system

$$\begin{aligned} u(1) &= -1 - \cosh(2) = a_3 e - b_3 e^{-1} \\ \dot{u}(1) &= -1 - \sinh(2) = a_3 e - b_3 e^{-1}\,. \end{aligned}$$

The unique solution is given by $a_3 = -e^{-1} - \frac{1}{2}e$, $\quad b_3 = \frac{1}{2}e^{-1}$. One obtains that $u_3(t) = -e^{t-1} - \sinh(t+1)$. Therefore, for $\nu > 1$ the solution is given by

$$u(t) = \begin{cases} 0 & : \quad t \leq -1 \\ -1 - \cosh(t+1) & : \quad -1 < t \leq 1 \\ -e^{t-1} - \sinh(t+1) & : \quad t > 1 \end{cases}.$$

We now turn to the case $0 \leq \nu < 1$: Since we need to have $u \in L_{2,\nu}(\mathbb{R})$, it follows that $a_1 = b_3 = 0$. To ensure continuity,we therefore need to solve the linear system

$$\begin{array}{ccccc} u(-1) & = & a_1 e^{-1} & = & a_2 e^{-1} + b_2 e - 1 \\ \dot{u}(-1) & = & a_1 e^{-1} & = & a_2 e^{-1} - b_2 e \\ u(1) & = & b_3 e^{-1} & = & b_2 e^{-1} - 1 \\ \dot{u}(1) & = & -b_3 e^{-1} & = & a_2 e - b_2 e^{-1}\,. \end{array}$$

In matrix-vector notation, this is the system

$$\begin{pmatrix} e^{-1} & -e^{-1} & -e & 0 \\ e^{-1} & -e^{-1} & e & 0 \\ 0 & -e & -e^{-1} & e^{-1} \\ 0 & -e & e^{-1} & -e^{-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

The solution is given by

$$a_1 = -\frac{1}{2}e + \frac{1}{2}e^{-1}\,, \quad a_2 = \frac{1}{2}e^{-1}\,, \quad b_2 = \frac{1}{2}e^{-1}\,, \quad b_3 = -\frac{1}{2}e + \frac{1}{2}e^{-1}\,.$$

One obtains that

$$u(t) = \begin{cases} -\frac{1}{2}e^{t+1} + \frac{1}{2}e^{t-1} & : \quad t \leq -1 \\ \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t-1} - 1 & : \quad -1 < t \leq 1 \\ -\frac{1}{2}e^{-t+1} + \frac{1}{2}e^{-t-1} & : \quad t > 1\,. \end{cases}$$

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