3 The Time Derivative

It is the aim of this lecture to define a derivative operator on a suitable L_2-space, which will be used as the derivative with respect to the temporal variable in our applications. As we want to deal with Hilbert space-valued functions, we start by introducing the concept of Bochner–Lebesgue spaces, which generalises the classical scalar-valued L_p-spaces to the Banach space-valued case.

3.1 Bochner–Lebesgue Spaces

Throughout, let (Ω, Σ, μ) be a σ-finite measure space and X a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We are aiming to define the spaces $L_p(\mu; X)$ for $1 \leqslant p \leqslant \infty$. This is the space of (equivalence classes of) measurable functions attaining values in X, which are p-integrable (if $p < \infty$), or essentially bounded (if $p = \infty$) with respect to the measure μ. We begin by defining the space of simple functions on Ω with values in X and the notion of Bochner-measurability.

Definition. For a function $f\colon \Omega \to X$ and $x \in X$ we set

$$A_{f,x} \coloneqq f^{-1}[\{x\}].$$

A function $f\colon \Omega \to X$ is called *simple* if $f[\Omega]$ is finite and for each $x \in X \setminus \{0\}$ the set $A_{f,x}$ belongs to Σ and has finite measure. We denote the set of simple functions by $S(\mu; X)$. A function $f\colon \Omega \to X$ is called *Bochner-measurable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that

$$f_n(\omega) \to f(\omega) \quad (n \to \infty)$$

for μ-a.e. $\omega \in \Omega$. If μ is absolutely continuous with respect to Lebesgue measure on $\Omega = \mathbb{R}^d$ we will just write $S(\mathbb{R}^d; X)$.

Remark 3.1.1. (a) It is easy to check that $S(\mu; X)$ is a vector space and an $S(\mu; \mathbb{K})$-module; that is, for $f \in S(\mu; X)$ and $g \in S(\mu; \mathbb{K})$ we have $g \cdot f \in S(\mu; X)$.

(b) If $f\colon \Omega \to X$ is Bochner-measurable, then $\|f(\cdot)\|_X\colon \Omega \to \mathbb{R}$ is μ-measurable. Indeed, since

$$\|f(\cdot)\|_X = \lim_{n\to\infty} \|f_n(\cdot)\|_X$$

for μ-a.e. $\omega \in \Omega$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$, it suffices to show that $\|f_n(\cdot)\|_X$ is measurable for all $n \in \mathbb{N}$. The latter follows since $A_{f,x} \cap A_{f,y} = \varnothing$ for $x \neq y$ and thus

$$\|f_n(\cdot)\|_X = \sum_{x\in f_n[\Omega]} \|x\|_X \cdot \mathbb{1}_{A_{f,x}}$$

is a real-valued simple function.

(c) If one deals with arbitrary measure spaces, the definition of simple functions has to be weakened by allowing the sets $A_{f,x}$ to have infinite measure. However, since in the applications to follow we only work with weighted Lebesgue measures, we restrict ourselves to σ-finite measure spaces.

Definition (Bochner–Lebesgue spaces)**.** For $p \in [1, \infty]$ we define

$$\mathcal{L}_p(\mu; X) \coloneqq \{f : \Omega \to X \,;\, f \text{ Bochner-measurable}, \, \|f(\cdot)\|_X \in \mathcal{L}_p(\mu)\},$$

as well as

$$L_p(\mu; X) \coloneqq \mathcal{L}_p(\mu; X)/\!\sim,$$

where $\sim$ denotes the usual equivalence relation of equality μ-almost everywhere.

Proposition 3.1.2. *Let $p \in [1, \infty]$. Then*

$$\|f\|_p \coloneqq \begin{cases} \left(\int_\Omega \|f(\omega)\|_X^p \, \mathrm{d}\mu(\omega)\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname{ess-sup}_{\omega \in \Omega} \|f(\omega)\|_X, & \text{if } p = \infty, \end{cases}$$

defines a seminorm on $\mathcal{L}_p(\mu; X)$ where $\|f\|_p = 0$ if and only if $f = 0$ μ-a.e. Consequently, $\|\cdot\|_p$ defines a norm on $L_p(\mu; X)$. Moreover, $(L_p(\mu; X), \|\cdot\|_p)$ is a Banach space and if $X = H$ is a Hilbert space, then so too is $L_2(\mu; H)$ with the scalar product given by

$$\langle f, g \rangle_2 \coloneqq \int_\Omega \langle f(\omega), g(\omega) \rangle_H \, \mathrm{d}\mu(\omega) \quad (f, g \in L_2(\mu; H)).$$

Proof. We just show the completeness of $L_p(\mu; X)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L_p(\mu; X)$ such that $\sum_{n=1}^\infty \|f_n\|_p < \infty$. We set

$$g_n(\omega) \coloneqq \|f_n(\omega)\| \quad (n \in \mathbb{N}, \omega \in \Omega).$$

Then $(g_n)_{n \in \mathbb{N}}$ is a sequence in $L_p(\mu)$ such that $\sum_{n=1}^\infty \|g_n\|_p < \infty$. By the completeness of $L_p(\mu)$ we infer that

$$g \coloneqq \sum_{n=1}^\infty g_n$$

exists and is an element in $L_p(\mu)$. In particular, $g(\omega) < \infty$ for μ-a.e. $\omega \in \Omega$ and thus,

$$\sum_{n=1}^\infty \|f_n(\omega)\|_X = \sum_{n=1}^\infty g_n(\omega) < \infty$$

for μ-a.e. $\omega \in \Omega$. By the completeness of X we can define

$$f(\omega) \coloneqq \sum_{n=1}^\infty f_n(\omega)$$

for μ-a.e. $\omega \in \Omega$. We need to prove that $f \in L_p(\mu; X)$ and that $\sum_{n=1}^k f_n \to f$ in $L_p(\mu; X)$ as $k \to \infty$. For this, it suffices to prove that

$$\sum_{n=k}^{\infty} f_n \in L_p(\mu; X) \text{ and } \sum_{n=k}^{\infty} f_n \to 0 \text{ in } L_p(\mu; X) \text{ as } k \to \infty. \tag{3.1}$$

Indeed, this would imply both that $f - \sum_{n=1}^k f_n \in L_p(\mu; X)$ and the desired convergence result. We prove (3.1) for $p < \infty$ and $p = \infty$ separately.
First, let $p = \infty$. For each $n \in \mathbb{N}$ we have $f_n \in L_\infty(\mu; X)$ and thus $\|f_n(\omega)\|_X \leqslant \|f_n\|_\infty$ for all $\omega \in \Omega \setminus N_n$ and some null set $N_n \subseteq \Omega$. We set $N \coloneqq \bigcup_{n=1}^\infty N_n$, which is again a null set. For $k \in \mathbb{N}$ and $\omega \in \Omega \setminus N$ we then estimate

$$\left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \leqslant \sum_{n=k}^{\infty} \|f_n(\omega)\|_X \leqslant \sum_{n=k}^{\infty} \|f_n\|_\infty,$$

which yields (3.1).
Now, let $p < \infty$. For $k \in \mathbb{N}$ we estimate

$$\begin{aligned}
\left(\int_\Omega \left(\left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} &\leqslant \left(\int_\Omega \left(\sum_{n=k}^{\infty} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega \lim_{m\to\infty} \left(\sum_{n=k}^{m} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&= \lim_{m\to\infty} \left(\int_\Omega \left(\sum_{n=k}^{m} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&\leqslant \lim_{m\to\infty} \sum_{n=k}^{m} \|f_n\|_p = \sum_{n=k}^{\infty} \|f_n\|_p,
\end{aligned}$$

where we have used monotone convergence in the third line. This estimate yields (3.1). □

We now want to define an X-valued integral for functions in $L_1(\mu; X)$; the so-called Bochner-integral. To do this, we need the following density result.

Lemma 3.1.3. *The space $S(\mu; X)$ is dense in $L_p(\mu; X)$ for $p \in [1, \infty)$.*

Proof. Let $f \in L_p(\mu; X)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$ for some null set $N \subseteq \Omega$. For $n \in \mathbb{N}$ we define the set

$$I_n \coloneqq \{\omega \in \Omega\,;\, \|f_n(\omega)\|_X \leqslant 2\|f(\omega)\|_X\} \in \Sigma,$$

and set $\widetilde{f_n} \coloneqq f_n \mathbb{1}_{I_n}$. Then $\widetilde{f_n} \in S(\mu; X)$ and we claim that $\widetilde{f_n}(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$. Indeed, if $f(\omega) = 0$ then $\widetilde{f_n}(\omega) = 0$ also and the claim follows. If $f(\omega) \neq 0$,

then there is some $n_0 \in \mathbb{N}$ such that $\|f_n(\omega)\|_X \leqslant 2\|f(\omega)\|_X$ for $n \geq n_0$, and hence $\omega \in \bigcap_{n \geqslant n_0} I_n$. Consequently $\widetilde{f}_n(\omega) = f_n(\omega) \to f(\omega)$. By dominated convergence, it now follows that

$$\int_\Omega \left\| \widetilde{f}_n(\omega) - f(\omega) \right\|_X^p \mathrm{d}\mu(\omega) \to 0 \quad (n \to \infty),$$

which proves the claim. □

Proposition 3.1.4. *The mapping*[1]

$$\int_\Omega \mathrm{d}\mu \colon S(\mu; X) \subseteq L_1(\mu; X) \to X$$
$$f \mapsto \sum_{x \in X} x \cdot \mu(A_{f,x})$$

is linear and continuous, and thus has a unique continuous linear extension to $L_1(\mu; X)$, called the Bochner-integral. *Moreover,*

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\|_X \leqslant \|f\|_1 \quad (f \in L_1(\mu; X)),$$

and for $A \in \Sigma, f \in L_1(\mu; X)$ we set

$$\int_A f \,\mathrm{d}\mu \coloneqq \int_\Omega f \cdot \mathbb{1}_A \,\mathrm{d}\mu.$$

Proof. We first show linearity. Let $f, g \in S(\mu; X)$ and $\lambda \in \mathbb{K}$. We then compute

$$\begin{aligned}
\int_\Omega (\lambda f + g) \,\mathrm{d}\mu &= \sum_{x \in X} x \cdot \mu(A_{\lambda f + g, x}) = \sum_{x \in X} x \cdot \mu\left(\bigcup_{y \in X} (A_{f,y} \cap A_{g, x - \lambda y}) \right) \\
&= \sum_{x \in X} \sum_{y \in X} x \cdot \mu(A_{f,y} \cap A_{g, x - \lambda y}) \\
&= \sum_{y \in X} \sum_{x \in X} \lambda y \cdot \mu(A_{f,y} \cap A_{g, x - \lambda y}) + \sum_{x \in X} \sum_{y \in X} (x - \lambda y)\, \mu(A_{f,y} \cap A_{g, x - \lambda y}) \\
&= \lambda \sum_{y \in X} y \cdot \mu\left(A_{f,y} \cap \bigcup_{x \in X} A_{g, x - \lambda y} \right) + \sum_{z \in X} z \cdot \mu\left(\bigcup_{y \in X} A_{f,y} \cap A_{g,z} \right) \\
&= \lambda \int_\Omega f \,\mathrm{d}\mu + \int_\Omega g \,\mathrm{d}\mu,
\end{aligned}$$

where we have used the fact that all occuring unions and sums are finite. In order to prove continuity, let $f \in S(\mu; X)$. We estimate

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\|_X = \left\| \sum_{x \in f[\Omega]} x \cdot \mu(A_{f,x}) \right\| \leqslant \sum_{x \in f[\Omega]} \|x\|_X \mu(A_{f,x}) = \int_\Omega \sum_{x \in f[\Omega]} \|x\|_X \mathbb{1}_{A_{f,x}} \,\mathrm{d}\mu$$

[1] Note that the sum is indeed finite and all summands are well-defined if we set $0_X \cdot \infty = 0_X$.

$$= \int_\Omega \|f(\cdot)\|_X \,\mathrm{d}\mu = \|f\|_1.$$

The remaining assertions now follow from Lemma 3.1.3 by continuous extension (see Corollary 2.1.5). □

The next proposition tells us how the Bochner-integral of a function behaves if we compose the function with a bounded or closed linear operator first. For what follows, let $X' := L(X, \mathbb{K})$ denote the dual space of X.

Proposition 3.1.5. *Let $f \in L_1(\mu; X)$, Y Banach space.*

(a) *Let $B \in L(X, Y)$. Then $B \circ f \in L_1(\mu; Y)$ and*

$$\int_\Omega B \circ f \,\mathrm{d}\mu = B \int_\Omega f \,\mathrm{d}\mu.$$

(b) *If $X_0 \subseteq X$ is a closed subspace and $f(\omega) \in X_0$ for μ-a.e. $\omega \in \Omega$, then $\int_\Omega f \,\mathrm{d}\mu \in X_0$.*

(c) *(Theorem of Hille) Let $A\colon \mathrm{dom}(A) \subseteq X \to Y$ be a closed linear operator and assume that $f(\omega) \in \mathrm{dom}(A)$ for μ-a.e. $\omega \in \Omega$ and that $A \circ f \in L_1(\mu; Y)$. Then $\int_\Omega f \,\mathrm{d}\mu \in \mathrm{dom}(A)$ and*

$$A \int_\Omega f \,\mathrm{d}\mu = \int_\Omega A \circ f \,\mathrm{d}\mu.$$

Proof. (a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $S(\mu; X)$ such that $f_n \to f$ μ-a.e. Then $B \circ f_n \in S(\mu; Y)$ since

$$(B \circ f_n)\,[\Omega] = B\big[f_n[\Omega]\big]$$

is finite and for $y \in (B \circ f_n)[\Omega] \setminus \{0\}$ we have that

$$A_{B\circ f_n, y} = \bigcup_{x \in B^{-1}[\{y\}] \cap f_n[\Omega]} A_{f_n, x} \in \Sigma,$$

and thus, $\mu(A_{B\circ f_n, y}) < \infty$. Due to the continuity of B we have that $B \circ f_n \to B \circ f$ μ-a.e., hence $B \circ f$ is Bochner-measurable. Moreover, $\|B \circ f(\cdot)\|_Y \leqslant \|B\| \|f(\cdot)\|_X$, which yields that $B \circ f \in L_1(\mu; Y)$. By continuity of both B and $\int_\Omega \mathrm{d}\mu$, it suffices to check the interchanging property for any $f \in S(\mu; X)$ alone. However, this is clear, since

$$\int_\Omega B \circ f \,\mathrm{d}\mu = \sum_{x \in X} Bx \cdot \mu(A_{f,x}) = B \sum_{x \in X} x \cdot \mu(A_{f,x}) = B \int_\Omega f \,\mathrm{d}\mu.$$

(b) Let $x' \in X'$ with $x'|_{X_0} = 0$. It follows from (a) that

$$x'\left(\int_\Omega f \,\mathrm{d}\mu\right) = \int_\Omega x' \circ f \,\mathrm{d}\mu = 0,$$

and since x' was arbitrary, it follows that $\int_\Omega f \,\mathrm{d}\mu \in X_0$ from the Theorem of Hahn–Banach.

(c) Consider the space $L_1(\mu; X \times Y)$. By assumption, it follows that

$$(f, A \circ f) \in L_1(\mu; X \times Y).$$

However, $(f, A \circ f)(\omega) = (f(\omega), (A \circ f)(\omega)) \in A \subseteq X \times Y$ for μ-a.e. $\omega \in \Omega$, and since A is closed we can use (b) to derive that

$$\int_\Omega (f, A \circ f) \, \mathrm{d}\mu \in A. \tag{3.2}$$

Let π_1, π_2 be the projection from $X \times Y$ to X and Y, respectively. It then follows from part (a) that

$$\pi_1 \left(\int_\Omega (f, A \circ f) \, \mathrm{d}\mu \right) = \int_\Omega \pi_1 (f, A \circ f) \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu,$$

and analogously for π_2. Using these equalities we derive from (3.2) that $\int_\Omega f \, \mathrm{d}\mu \in \mathrm{dom}(A)$ and that $A \int_\Omega f \, \mathrm{d}\mu = \int_\Omega A \circ f \, \mathrm{d}\mu$. □

As a consequence of the latter proposition, we derive the fundamental theorem of calculus for Banach space-valued functions.

Corollary 3.1.6 (fundamental theorem of calculus)**.** *Let $a, b \in \mathbb{R}, a < b$ and consider the measure space $([a,b], \mathcal{B}([a,b]), \lambda)$, where $\mathcal{B}([a,b])$ denotes the Borel-σ-algebra of $[a,b]$ and λ is the Lebesgue measure. Let $f\colon [a,b] \to X$ be continuously differentiable.*[2] *Then*

$$f(b) - f(a) = \int_{[a,b]} f' \, \mathrm{d}\lambda.$$

Proof. Note first of all that continuous functions are Bochner-measurable. Thus, the integral on the right-hand side is well-defined. Let $\varphi \in X'$. Then $\varphi \circ f\colon [a,b] \to \mathbb{K}$ is continuously differentiable, and $(\varphi \circ f)'(t) = (\varphi \circ f')(t)$. Using Proposition 3.1.5 (a) together with the fundamental theorem of calculus for the scalar-valued case we get

$$\varphi \left(\int_{[a,b]} f' \, \mathrm{d}\lambda \right) = \int_{[a,b]} (\varphi \circ f') \, \mathrm{d}\lambda = \varphi(f(b)) - \varphi(f(a)) = \varphi(f(b) - f(a)).$$

Since this holds for all $\varphi \in X'$, the assertion follows from the Theorem of Hahn–Banach. □

We conclude this section with a density result, which will be useful throughout the course.

Lemma 3.1.7. *Let $\mathcal{D} \subseteq L_2(\mu)$ be total in $L_2(\mu)$ and H a Hilbert space. Then the set $\{\varphi(\cdot)x\,;\, x \in H,\, \varphi \in \mathcal{D}\}$ is total in $L_2(\mu; H)$.*

[2] By this we mean that f is continuous on $[a,b]$, continuously differentiable on (a,b) and f' has a continuous extension to $[a,b]$.

Proof. Let $f \in L_2(\mu; H)$ and assume that

$$\langle f, \varphi(\cdot)x \rangle_2 = 0$$

for each $x \in H, \varphi \in \mathcal{D}$. We need to prove that $f = 0$ for that $\operatorname{lin}\{\varphi(\cdot)x\,;\, x \in H,\, \varphi \in \mathcal{D}\}$ is dense in $L_2(\mu; H)$. First we show that there is a null set N such that $f[\Omega \setminus N]$ is contained in a separable subspace H_0 of H. Indeed, let $(f_n)_{n\in\mathbb{N}}$ be in $S(\mu; X)$ where $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$ for some null set $N \subseteq \Omega$. As $f_n[\Omega]$ is finite for every $n \in \mathbb{N}$, we infer that

$$f[\Omega \setminus N] \subseteq \overline{\operatorname{lin}} \bigcup_{n\in\mathbb{N}} f_n[\Omega] =: H_0,$$

which is clearly separable. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in H_0. For $n \in \mathbb{N}$ consider the function $g_n := \langle f(\cdot), x_n \rangle_H \in L_2(\mu)$. We then have that

$$\begin{aligned}\langle g_n, \varphi \rangle_{L_2(\mu)} &= \int_\Omega \langle f(\omega), x_n \rangle_H \, \varphi(\omega) \, \mathrm{d}\mu(\omega) = \int_\Omega \langle f(\omega), \varphi(\omega) x_n \rangle_H \, \mathrm{d}\mu(\omega) \\ &= \langle f, \varphi(\cdot) x_n \rangle_2 = 0,\end{aligned}$$

for each $\varphi \in \mathcal{D}$. Since $\mathcal{D}$ is total in $L_2(\mu)$ it follows that $g_n(\omega) = 0$ for $\omega \in \Omega \setminus N_n$ for some null set $N_n \subseteq \Omega$. Setting $\widetilde{N} := \bigcup_{n\in\mathbb{N}} N_n \cup N$ and using the fact that $(x_n)_{n\in\mathbb{N}}$ is dense in H_0, it follows that $f(\omega) = 0$ for $\omega \in \Omega \setminus \widetilde{N}$, which shows the claim. □

Lemma 3.1.8. *Let $\mathcal{D} \subseteq L_2(\mu)$ be total in $L_2(\mu)$, H a Hilbert space, $D_0 \subseteq H$ total in H. Then $\{\varphi(\cdot)x\,;\, x \in D_0, \varphi \in \mathcal{D}\}$ is total in $L_2(\mu; H)$.*

Proof. The proof follows upon realising that the set $\{\varphi(\cdot)x\,;\, x \in D_0,\, \varphi \in \mathcal{D}\}$ is total in the set $\{\varphi(\cdot)x\,;\, x \in H,\, \varphi \in \mathcal{D}\}$. From here we just apply Lemma 3.1.7. □

3.2 The Time Derivative as a Normal Operator

Now let H be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $\nu \in \mathbb{R}$ and $p \in [1, \infty)$ we define the measure

$$\mu_{p,\nu}(A) := \int_A \mathrm{e}^{-p\nu t} \, \mathrm{d}\lambda(t)$$

for A in the Borel-σ-algebra, $\mathcal{B}(\mathbb{R})$, of $\mathbb{R}$. As our underlying Hilbert space for the time derivative we set

$$L_{2,\nu}(\mathbb{R}; H) := L_2(\mu_{2,\nu}; H).$$

In the same way we define

$$L_{p,\nu}(\mathbb{R}; H) := L_p(\mu_{\nu,p}; H)$$

for $p \in [1, \infty)$. If $H = \mathbb{K}$ we abbreviate $L_{p,\nu}(\mathbb{R}) := L_{p,\nu}(\mathbb{R}; \mathbb{K})$. We start our considerations with the definition of convolution operators in $L_{2,\nu}(\mathbb{R}; H)$.

Lemma 3.2.1. *Let $k \in L_{1,\nu}(\mathbb{R})$. We define the convolution operator*

$$k*\colon L_{2,\nu}(\mathbb{R};H) \to L_{2,\nu}(\mathbb{R};H)$$

by

$$(k*f)(t) \coloneqq \int_{\mathbb{R}} k(s)f(t-s)\,\mathrm{d}s,$$

which exists for a.e. $t \in \mathbb{R}$. Then, $k*$ is linear and bounded such that $\|k*\| \leqslant \|k\|_{L_{1,\nu}(\mathbb{R})}$.

Proof. We first prove that $s \mapsto k(s)f(t-s) \in L_1(\mathbb{R};H)$ for a.e. $t \in \mathbb{R}$. The measurability is clear since k and f are both measurable. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|k(s)f(t-s)\|_H \,\mathrm{d}s \right)^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} \|f(t-s)\|_H \mathrm{e}^{-\nu(t-s)}\,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |k(s)| \mathrm{e}^{-\nu s}\,\mathrm{d}s \right) \left(\int_{\mathbb{R}} |k(s)| \mathrm{e}^{-\nu s} \|f(t-s)\|_H^2 \mathrm{e}^{-2\nu(t-s)}\,\mathrm{d}s \right) \mathrm{d}t \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})} \int_{\mathbb{R}} |k(s)| \int_{\mathbb{R}} \|f(t-s)\|^2 \mathrm{e}^{-2\nu(t-s)}\,\mathrm{d}t\, \mathrm{e}^{-\nu s}\,\mathrm{d}s \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})}^2 \|f\|_{L_{2,\nu}(\mathbb{R};H)}^2,
\end{aligned}$$

which on the one hand proves that

$$\int_{\mathbb{R}} \|k(s)f(t-s)\|_H \,\mathrm{d}s < \infty$$

for a.e. $t \in \mathbb{R}$ and on the other hand shows the norm estimate. Since the linearity of $k*$ is clear the proof is done. □

Definition. For $\nu \neq 0$ we define the operator

$$I_\nu\colon L_{2,\nu}(\mathbb{R};H) \to L_{2,\nu}(\mathbb{R};H)$$

by

$$I_\nu \coloneqq \begin{cases} \mathbb{1}_{[0,\infty)}*, & \text{if } \nu > 0, \\ -\mathbb{1}_{(-\infty,0]}*, & \text{if } \nu < 0. \end{cases}$$

Note that, by Lemma 3.2.1, I_ν is bounded with $\|I_\nu\| \leqslant \frac{1}{|\nu|}$.

Remark 3.2.2. For $\nu > 0$, $f \in L_{2,\nu}(\mathbb{R},H)$ and $t \in \mathbb{R}$ we have

$$I_\nu f(t) = \mathbb{1}_{[0,\infty)} * f(t) = \int_0^\infty f(t-s)\,\mathrm{d}s = \int_{-\infty}^t f(s)\,\mathrm{d}s.$$

Analogously, for $\nu < 0$, $f \in L_{2,\nu}(\mathbb{R},H)$ and $t \in \mathbb{R}$ we have

$$I_\nu f(t) = -\int_t^\infty f(s)\,\mathrm{d}s.$$

Proposition 3.2.3. *Let $\nu \neq 0$. Then I_ν is one-to-one and $C_c^1(\mathbb{R};H)$, the space of continuously differentiable, compactly supported functions on $\mathbb{R}$ with values in H, is in the range of I_ν.*

Proof. We just prove the assertion for the case when $\nu > 0$. Let $f \in L_{2,\nu}(\mathbb{R};H)$ satisfy $I_\nu f = 0$. In particular, we obtain for all $t \in \mathbb{R} \setminus N$ that $0 = I_\nu f(t) = \int_{-\infty}^t f(s)\,\mathrm{d}s$ for some Lebesgue null set, $N \subseteq \mathbb{R}$. Then for $a, b \in \mathbb{R} \setminus N$ with $a < b$ and $x \in H$ we have that

$$\begin{aligned}\left\langle f, \mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]} \cdot x \right\rangle_{L_{2,\nu}(\mathbb{R};H)} &= \int_{\mathbb{R}} \left\langle f(t), \mathrm{e}^{2\nu t}\mathbb{1}_{[a,b]}(t) \cdot x \right\rangle_H \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &= \left\langle \int_a^b f(t)\,\mathrm{d}t, x \right\rangle_H \\ &= \left\langle (I_\nu f)(b) - (I_\nu f)(a), x \right\rangle_H = 0.\end{aligned}$$

Thus $f = 0$. Indeed, since $\mathbb{R} \setminus N$ is dense in $\mathbb{R}$, $\left\{\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]}\,;\, a, b \in \mathbb{R} \setminus N\right\}$ is total in $L_{2,\nu}(\mathbb{R})$. Hence, $\left\{\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{[a,b]} \cdot x\,;\, a, b \in \mathbb{R} \setminus N,\, x \in H\right\}$ is total in $L_{2,\nu}(\mathbb{R};H)$ by Lemma 3.1.7. This proves the injectivity of I_ν. Moreover, if $\varphi \in C_c^1(\mathbb{R};H)$ then by Corollary 3.1.6 we have

$$\varphi(t) = \int_{-\infty}^t \varphi'(s)\,\mathrm{d}s = \left(I_\nu \varphi'\right)(t) \quad (t \in \mathbb{R}). \qquad \square$$

Definition. For $\nu \neq 0$ we define the *time derivative*, $\partial_{t,\nu}$, on $L_{2,\nu}(\mathbb{R};H)$ by

$$\partial_{t,\nu} := I_\nu^{-1}.$$

Note that by Lemma 3.2.1 and Proposition 3.2.3, $\partial_{t,\nu}$ is a closed linear operator for which $C_c^1(\mathbb{R};H) \subseteq \mathrm{dom}(\partial_{t,\nu})$. Since

$$C_c^1(\mathbb{R};H) \supseteq \mathrm{lin}\left\{\varphi \cdot x\,;\, \varphi \in C_c^1(\mathbb{R}),\, x \in H\right\}$$

we infer that $\partial_{t,\nu}$ is densely defined by Lemma 3.1.7 and Exercise 3.2. Moreover, since $I_\nu \varphi' = \varphi$ for $\varphi \in C_c^1(\mathbb{R};H)$ we get that

$$\partial_{t,\nu}\varphi = \varphi';$$

that is, $\partial_{t,\nu}$ extends the classical derivative of continuously differentiable functions. We shall discuss the actual domain of $\partial_{t,\nu}$ in the next lecture.

Proposition 3.2.4. *Let $\nu \neq 0$. Then $\mathcal{D}_H := \mathrm{lin}\left\{\varphi \cdot x\,;\, \varphi \in C_c^\infty(\mathbb{R}),\, x \in H\right\}$ is a core for $\partial_{t,\nu}$. Here, $C_c^\infty(\mathbb{R})$ denotes the space of arbitrarily differentiable functions on $\mathbb{R}$ with compact support.*

Proof. We first prove that

$$\left\{\varphi'\,;\, \varphi \in C_c^\infty(\mathbb{R})\right\} \tag{3.3}$$

is dense in $L_{2,\nu}(\mathbb{R})$. As $C_c^\infty(\mathbb{R})$ is dense in $L_{2,\nu}(\mathbb{R})$ (see Exercise 3.2), it suffices to approximate functions in $C_c^\infty(\mathbb{R})$. For this, let $f \in C_c^\infty(\mathbb{R})$. Moreover, we choose $\psi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi = 1$ and

$$\operatorname{spt}\psi \subseteq \begin{cases} \mathbb{R}_{\geq 0}, & \text{if } \nu > 0, \\ \mathbb{R}_{\leq 0}, & \text{if } \nu < 0, \end{cases}$$

where $\operatorname{spt}\psi$ denotes the support of ψ. For $n \in \mathbb{N}$ we define $\psi_n(t) \coloneqq \frac{1}{n}\psi\left(\frac{t}{n}\right)$ for $t \in \mathbb{R}$. Notice we still have that $\psi_n \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \psi_n = 1$. We now define

$$\varphi_n(t) \coloneqq \int_{-\infty}^t f(s)\,\mathrm{d}s - \left(\int_{\mathbb{R}} f\right)\int_{-\infty}^t \psi_n(s)\,\mathrm{d}s \quad (t \in \mathbb{R}).$$

Then $\varphi_n \in C_c^\infty(\mathbb{R})$ for each $n \in \mathbb{N}$ and

$$\varphi_n'(t) = f(t) - \left(\int_{\mathbb{R}} f\right)\psi_n(t) \quad (t \in \mathbb{R}).$$

Consequently,

$$\left\|\varphi_n' - f\right\|_{L_{2,\nu}(\mathbb{R})} = \left|\int_{\mathbb{R}} f\right| \|\psi_n\|_{L_{2,\nu}(\mathbb{R})},$$

and since

$$\begin{aligned} \|\psi_n\|_{L_{2,\nu}(\mathbb{R})}^2 &= \frac{1}{n^2}\int_{\mathbb{R}} \left|\psi\left(\frac{t}{n}\right)\right|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &= \frac{1}{n}\int_{\mathbb{R}} |\psi(t)|^2 \mathrm{e}^{-2\nu n t}\,\mathrm{d}t \to 0 \quad (n \to \infty) \end{aligned}$$

by the support condition imposed on ψ, the density of (3.3) in $L_{2,\nu}(\mathbb{R})$ follows. By Lemma 3.1.7 we have that

$$\left\{\varphi' \cdot x\,;\, \varphi \in C_c^\infty(\mathbb{R}),\, x \in H\right\}$$

is total in $L_{2,\nu}(\mathbb{R}; H)$ and so $\partial_{t,\nu}[\mathcal{D}_H]$ is dense in $L_{2,\nu}(\mathbb{R}; H)$. Now let $f \in \operatorname{dom}(\partial_{t,\nu})$ and $\varepsilon > 0$. By what we have shown above there exists some $\varphi \in \mathcal{D}_H$ such that

$$\|\partial_{t,\nu}\varphi - \partial_{t,\nu} f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \varepsilon.$$

Since $\partial_{t,\nu}^{-1} = I_\nu$ is bounded with $\left\|\partial_{t,\nu}^{-1}\right\| \leqslant \frac{1}{|\nu|}$, the latter implies that

$$\|\varphi - f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \frac{\varepsilon}{|\nu|},$$

and hence, $\mathcal{D}_H$ is indeed a core for $\partial_{t,\nu}$. □

Finally, we want to compute the adjoint of $\partial_{t,\nu}$. For this, we first consider the adjoint of its inverse, I_ν.

Lemma 3.2.5. *Let $\nu \neq 0$. Then $\operatorname{ran}(I_\nu^*) \subseteq \operatorname{ran}(I_\nu)$. More precisely, we have*

$$I_\nu^* f = -I_\nu f + 2\nu I_\nu I_\nu^* f \quad (f \in L_{2,\nu}(\mathbb{R}; H)).$$

Proof. We just prove this equality for the case $\nu > 0$. For $f, g \in L_{2,\nu}(\mathbb{R}; H)$ we compute, using Proposition 3.1.5, that

$$\begin{aligned}
\langle I_\nu f, g\rangle_{L_{2,\nu}(\mathbb{R};H)} &= \int_{\mathbb{R}} \left\langle \int_{-\infty}^t f(s)\,\mathrm{d}s, g(t)\right\rangle_H \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}} \int_{-\infty}^t \langle f(s), g(t)\rangle_H \,\mathrm{e}^{-2\nu t}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_s^\infty \langle f(s), g(t)\rangle_H \,\mathrm{e}^{-2\nu(t-s)}\mathrm{e}^{-2\nu s}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}} \left\langle f(s), \int_s^\infty g(t)\mathrm{e}^{2\nu(s-t)}\,\mathrm{d}t\right\rangle_H \mathrm{e}^{-2\nu s}\,\mathrm{d}s \\
&= \left\langle f, \left(\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{(-\infty,0]}\right) * g\right\rangle_{L_{2,\nu}(\mathbb{R};H)}
\end{aligned}$$

so that

$$I_\nu^* = \left(\mathrm{e}^{2\nu(\cdot)}\mathbb{1}_{(-\infty,0]}\right) * .$$

With the notation $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$, we thus obtain

$$\begin{aligned}
2\nu\left(I_\nu I_\nu^* f\right)(t) &= 2\nu \int_{-\infty}^t I_\nu^* f(s)\,\mathrm{d}s = 2\nu \int_{-\infty}^t \int_s^\infty f(r)\mathrm{e}^{2\nu(s-r)}\,\mathrm{d}r\,\mathrm{d}s \\
&= 2\nu \int_{\mathbb{R}} f(r)\mathrm{e}^{-2\nu r} \int_{-\infty}^{t\wedge r} \mathrm{e}^{2\nu s}\,\mathrm{d}s\,\mathrm{d}r = \int_{\mathbb{R}} f(r)\mathrm{e}^{-2\nu r}\mathrm{e}^{2\nu(t\wedge r)}\,\mathrm{d}r \\
&= \int_{-\infty}^t f(r)\,\mathrm{d}r + \int_t^\infty f(r)\mathrm{e}^{2\nu(t-r)}\,\mathrm{d}r = \left(I_\nu f\right)(t) + \left(I_\nu^* f\right)(t)
\end{aligned}$$

for $t \in \mathbb{R}$, which shows the claim. □

Corollary 3.2.6. *For $\nu \neq 0$ the adjoint of $\partial_{t,\nu}$ is given by*

$$\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

In particular, $\partial_{t,\nu}$ is a normal operator with $\operatorname{Re}\partial_{t,\nu} := \frac{1}{2}\left(\overline{\partial_{t,\nu} + \partial_{t,\nu}^*}\right) = \nu$.

Proof. Integrating by parts, one obtains

$$\begin{aligned}
\int_{\mathbb{R}} \langle \partial_{t,\nu}\varphi(t), \psi(t)\rangle\,\mathrm{e}^{-2\nu t}\,\mathrm{d}t &= \int_{\mathbb{R}} \left\langle \varphi'(t), \psi(t)\right\rangle \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left\langle \varphi(t), -\psi'(t) + 2\nu\psi(t)\right\rangle \mathrm{e}^{-2\nu t}\,\mathrm{d}t
\end{aligned}$$

for $\varphi, \psi \in C_{\mathrm{c}}^\infty(\mathbb{R}; H)$. Since $C_{\mathrm{c}}^\infty(\mathbb{R}; H)$ is a core for $\partial_{t,\nu}$ by Proposition 3.2.4, the latter shows

$$\partial_{t,\nu} \subseteq -\partial_{t,\nu}^* + 2\nu.$$

To show equality it suffices to prove that $\operatorname{dom}(\partial_{t,\nu}^*) \subseteq \operatorname{dom}(\partial_{t,\nu})$. The latter inclusion would follow if $\operatorname{ran}(I_\nu^*) \subseteq \operatorname{ran}(I_\nu)$, but this was already shown in Lemma 3.2.5, so the claim follows. □

Corollary 3.2.7. *For $\nu \in \mathbb{R}$ the mapping*

$$\begin{aligned} \exp(-\nu \mathrm{m}) : L_{2,\nu}(\mathbb{R}; H) &\to L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto \mathrm{e}^{-\nu t} f(t)) \end{aligned}$$

is unitary, and for $\nu, \mu \neq 0$ one has

$$\exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu) \exp(-\nu \mathrm{m})^{-1} = \exp(-\mu \mathrm{m})(\partial_{t,\mu} - \mu) \exp(-\mu \mathrm{m})^{-1}.$$

Moreover, the operator

$$\partial_{t,0} \coloneqq \exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu) \exp(-\nu \mathrm{m})^{-1}$$

is skew-selfadjoint for all $\nu \neq 0$.

Proof. The proof is left as Exercise 3.5. □

3.3 Comments

Standard references for Bochner integration and related results are [1, 2].
Considering the derivative operator in an exponentially weighted space goes back (at least) to Morgenstern [3], where ordinary differential equations were considered in a classical setting. In fact, we shall return to this observation in the next lecture when we devote our study to some implications of the already developed concepts on ordinary and delay differential equations.
A first occurence of the derivative operator in exponentially weighted L^2-spaces can be found in [5], where a corresponding spectral theorem has been focussed on. We will prove in a later lecture that the spectral representation of the time-derivative as a multiplication operator can be realised by a shifted variant of the Fourier transformation – the so-called Fourier–Laplace transformation.
In an applied context, the time derivative operator discussed here has been introduced in [4].

Exercises

Exercise 3.1. A sequence $(\varphi_n)_n$ in $C_\mathrm{c}^\infty(\mathbb{R}^d)$ is called a *δ-sequence,* if

(a) $\varphi_n \geqslant 0$ for $n \in \mathbb{N}$,

(b) $\operatorname{spt} \varphi_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]^d$ for $n \in \mathbb{N}$,

(c) $\int_{\mathbb{R}^d} \varphi_n = 1$ for $n \in \mathbb{N}$.

Let $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$ with $\operatorname{spt} \varphi \subseteq [-1, 1]^d$, $\varphi \geqslant 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Prove that $(\varphi_n)_n$ given by $\varphi_n(x) \coloneqq n^d \varphi(n^d x)$ for $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ defines a δ-sequence. Moreover, give an example for such a function φ.

Exercise 3.2. It is well-known that $\{\mathbb{1}_I\,;\, I \text{ } d\text{-dimensional bounded interval}\}$ is total in $L_2(\mathbb{R}^d)$ (note that the linear span of which is a smaller set than $S(\mathbb{R}^d)$).

(a) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $f \in L_2(\mathbb{R}^d)$. Prove that $f*\varphi \in C_c^\infty(\mathbb{R})$ with $\partial^\alpha (f*\varphi) = f*\partial^\alpha\varphi$ for all $\alpha \in \mathbb{N}_0^d$, where $\partial^\alpha\varphi = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}\varphi$. Moreover, prove that $\operatorname{spt} f*\varphi \subseteq \operatorname{spt} f + \operatorname{spt}\varphi$.

(b) Let $(\varphi_n)_n$ be a δ-sequence and $f \in L_2(\mathbb{R}^d)$. Show that $f*\varphi_n \to f$ in $L_2(\mathbb{R}^d)$ as $n\to\infty$.
Hint: Prove that $\mathbb{1}_I * \varphi_n \to \mathbb{1}_I$ in $L_2(\mathbb{R}^d)$ for all d-dimensional intervals and use that $\|f*\varphi_n\|_2 \leqslant \|f\|_2$ (see also Lemma 3.2.1).

(c) Prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$.

Exercise 3.3. Let $a<b$, X_0, X_1, X_2 be Banach spaces, $f\colon (a,b)\to X_0$ and $g\colon (a,b)\to X_1$ both continuously differentiable, $\ell\colon X_0\times X_1 \to X_2$ bilinear and continuous. Prove that $h\colon (a,b)\to X_2$ given by

$$h(t) \coloneqq \ell(f(t), g(t)) \quad (t\in(a,b))$$

is continuously differentiable with

$$h'(t) = \ell(f'(t), g(t)) + \ell(f(t), g'(t)) \quad (t\in(a,b)).$$

If f, f', g, g' have continuous extensions to $[a,b]$, prove the integration by parts formula:

$$\int_a^b \ell(f'(t), g(t))\,\mathrm{d}t = \ell(f(b),g(b)) - \ell(f(a),g(a)) - \int_a^b \ell(f(t), g'(t))\,\mathrm{d}t.$$

Exercise 3.4. For $\nu\neq 0$, show that $\|I_\nu\| = \frac{1}{|\nu|}$.

Exercise 3.5. Prove Corollary 3.2.7.

Exercise 3.6. Let $\nu\in\mathbb{R}$ and H be a complex Hilbert space. Prove that $\sigma(\partial_{t,\nu}) \subseteq \{\mathrm{i}t+\nu\,;\, t\in\mathbb{R}\}$, where $\partial_{t,0}$ is defined in Corollary 3.2.7.
Hint: For $f\in\operatorname{dom}(\partial_{t,\nu}), z\in\mathbb{C}$ compute $\operatorname{Re}\langle (z-\partial_{t,\nu})f, f\rangle_{L_{2,\nu}(\mathbb{R};H)}$ by using Corollary 3.2.6. For proving the surjectivity of $z-\partial_{t,\nu}$ for a suitable z, use the formula

$$\overline{\operatorname{ran}}(z-\partial_{t,\nu}) = \ker(z^*-\partial_{t,\nu}^*)^\perp.$$

Remark: Later we will see that, actually, $\sigma(\partial_{t,\nu}) = \{\mathrm{i}t+\nu\,;\, t\in\mathbb{R}\}$.

Exercise 3.7. Consider the differential equation

$$\left(\partial_{t,\nu}^2 - 1\right) u = \mathbb{1}_{[-1,1]}.$$

Since $\partial_{t,\nu}^2 - 1 = (\partial_{t,\nu}-1)(\partial_{t,\nu}+1)$, it follows by Exercise 3.6 that there is a unique $u\in L_{2,\nu}(\mathbb{R})$ solving this equation if $\nu\notin\{-1,1\}$. Compute these solutions.
Hint: For $u\in\operatorname{dom}(\partial_{t,\nu})$ use the fact that u is necessarily continuous (which we shall establish in the next lecture).

References

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