2 Unbounded operators

We will gather some information on operators in Banach and Hilbert spaces. Throughout this chapter let X_0, X_1, and X_2 be Banach spaces and H_0, H_1, and H_2 be Hilbert spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

2.1 Operators in Banach spaces

The main difference of continuous (or bounded) linear operators, that is,

$$L(X_0, X_1) \coloneqq \left\{ B\colon X_0 \to X_1\,;\, B \text{ linear}, \ \|B\| \coloneqq \sup_{x \in X_0 \setminus \{0\}} \frac{\|Bx\|}{\|x\|} < \infty \right\}$$

(with the usual abbreviation $L(X_0) \coloneqq L(X_0, X_0)$) and the set of unbounded linear operators is that the latter are defined only on a subset of X_0. In order to define unbounded linear operators, we will first take a more general point of view and introduce (linear) relations. This perspective will turn out to be the natural setting later on.

Definition. A subset $A \subseteq X_0 \times X_1$ is called a *relation in X_0 and X_1*. We define the *domain*, *range* and *kernel of A* as follows

$$\begin{aligned} \operatorname{dom}(A) &\coloneqq \{x \in X_0\,;\, \exists y \in X_1\colon (x, y) \in A\}\,, \\ \operatorname{ran}(A) &\coloneqq \{y \in X_1\,;\, \exists x \in X_0\colon (x, y) \in A\}\,, \\ \ker(A) &\coloneqq \{x \in X_0\,;\, (x, 0) \in A\}\,. \end{aligned}$$

The *image, $A[M]$, of a set $M \subseteq X_0$ under A* is given by

$$A[M] \coloneqq \{y \in X_1\,;\, \exists x \in M\colon (x, y) \in A\}\,.$$

A relation A is called *bounded*, if for all bounded $M \subseteq X_0$ the set $A[M] \subseteq X_1$ is bounded. For a given relation A we define the *inverse relation*

$$A^{-1} \coloneqq \{(y, x) \in X_1 \times X_0\,;\, (x, y) \in A\}\,.$$

A relation A is called *linear*, if for all $(x, y), (u, v) \in A$, $\lambda \in \mathbb{K}$ we have $(x+\lambda u, y+\lambda v) \in A$. A linear relation A is called *linear operator* or just *operator from X_0 to X_1*, if

$$A[\{0\}] = \{y \in X_1\,;\, (0, y) \in A\} = \{0\}.$$

In this case, we also write

$$A\colon \operatorname{dom}(A) \subseteq X_0 \to X_1$$

to denote a linear operator from X_0 to X_1. Moreover, we shall write $Ax = y$ instead of $(x, y) \in A$ in this case. A linear operator A, which is not bounded, is called *unbounded.*

For completeness, we also define the sum, composition, and scalar multiples of relations.

Definition. Let $A \subseteq X_0 \times X_1$, $B \subseteq X_0 \times X_1$ and $C \subseteq X_1 \times X_2$ be relations, $\lambda \in \mathbb{K}$. Then we define

$$\begin{aligned} A + B &\coloneqq \{(x, y + w) \in X_0 \times X_1\,;\, (x, y) \in A, (x, w) \in B\}\,, \\ \lambda A &\coloneqq \{(x, \lambda y) \in X_0 \times X_1\,;\, (x, y) \in A\}\,, \\ CA &\coloneqq \{(x, z) \in X_0 \times X_2\,;\, \exists y \in X_1\colon (x, y) \in A, (y, z) \in C\}\,. \end{aligned}$$

For a relation $A \subseteq X_0 \times X_1$ we will use the abbreviation $-A := -1A$ (so that the minus sign only acts on the second component). We now proceed with topological notions for relations.

Definition. Let $A \subseteq X_0 \times X_1$ be a relation. A is called *densely defined*, if $\operatorname{dom}(A)$ is dense in X_0. We call A *closed*, if A is a closed subset of the direct sum of the Banach spaces X_0 and X_1. If A is a linear operator then we will call A *closable*, whenever $\overline{A} \subseteq X_0 \times X_1$ is a linear operator.

Proposition 2.1.1. *Let $A \subseteq X_0 \times X_1$ be a relation, $C \in L(X_2, X_0)$ and $B \in L(X_0, X_1)$. Then the following statements hold.*

(a) *A is closed if and only if A^{-1} is closed;*

(b) *A is closed if and only if $A + B$ is closed;*

(c) *if A is closed, then AC is closed.*

Proof. The first statement follows upon realising that $X_0 \times X_1 \ni (x, y) \mapsto (y, x) \in X_1 \times X_0$ is an isomorphism. For the second statement, it suffices to show that the closedness of A implies the same for $A+B$. Let $((x_n, y_n))_n$ be a sequence in $A+B$ convergent in $X_0 \times X_1$ to some (x, y). Since $B \in L(X_0, X_1)$, it follows that $((x_n, y_n - Bx_n))_n$ in A is convergent to $(x, y - Bx)$ in $X_0 \times X_1$. Since A is closed, $(x, y - Bx) \in A$. Thus, $(x, y) \in A + B$.
For the third statement of the present proposition, let $((w_n, y_n))_n$ be a sequence in AC convergent in $X_2 \times X_1$ to some (w, y). Since C is continuous, $(Cw_n)_n$ converges to Cw. Hence, $(Cw_n, y_n) \to (Cw, y)$ in $X_0 \times X_1$ and since $(Cw_n, y_n) \in A$ and A is closed, it follows that $(Cw, y) \in A$. Equivalently, $(w, y) \in AC$, which yields closedness of AC. □

We shall gather some other elementary facts about closed operators in the following. We will make use of the following notion.

Definition. Let $A\colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then the *graph norm* of A is defined by $\operatorname{dom}(A) \ni x \mapsto \|x\|_A \coloneqq \sqrt{\|x\|^2 + \|Ax\|^2}$.

Lemma 2.1.2. *Let $A\colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then A is closed if and only if $\operatorname{dom}(A)$ equipped with the graph norm is a Banach space if and only if for all $(x_n)_n$ in $\operatorname{dom}(A)$ convergent in X_0 such that $(Ax_n)_n$ is convergent in X_1 we have $\lim_{n\to\infty} x_n \in \operatorname{dom}(A)$ and $A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n$.*

Proof. For the first equivalence, it suffices to observe that $\mathrm{dom}(A) \ni x \mapsto (x, Ax) \in A$, where $\mathrm{dom}(A)$ is endowed with the graph norm, is an isomorphism. The last statement is an easy reformulation of the definition of closedness of $A \subseteq X_0 \times X_1$. ☐

Unless explicitly stated otherwise (e.g. in the form $\mathrm{dom}(A) \subseteq X_0$, where we regard $\mathrm{dom}(A)$ as a subspace of X_0), for closed operators A we always consider $\mathrm{dom}(A)$ as a Banach space on its own right; that is, we shall regard it as being endowed with the graph norm.

Lemma 2.1.3. *Let $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ be a closed linear operator. Then A is bounded if and only if $\mathrm{dom}(A) \subseteq X_0$ is closed.*

Proof. First of all note that boundedness of A is equivalent to the fact that the graph norm and the X_0-norm on $\mathrm{dom}(A)$ are equivalent. Hence, the closedness and boundedness of A implies that $\mathrm{dom}(A) \subseteq X_0$ is closed. On the other hand, the embedding

$$\iota\colon (\mathrm{dom}(A), \|\cdot\|_A) \hookrightarrow (\mathrm{dom}(A), \|\cdot\|_{X_0})$$

is continuous and bijective. Since the range is closed, the open mapping theorem implies that ι^{-1} is continuous. This yields the equivalence of the graph norm and the X_0-norm and, thus, the boundedness of A. ☐

For unbounded operators, obtaining a precise description of the domain may be difficult. However, there may be a subset of the domain which essentially (or approximately) describes the operator. This gives rise to the following notion of a core.

Definition. Let $A \subseteq X_0 \times X_1$. A set $D \subseteq \mathrm{dom}(A)$ is called a *core for A* provided $\overline{A \cap (D \times X_1)} = \overline{A}$.

Proposition 2.1.4. *Let $A \in L(X_0, X_1)$, and $D \subseteq X_0$ a dense linear subspace. Then D is a core for A.*

Corollary 2.1.5. *Let $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ be a densely defined, bounded linear operator. Then there exists a unique $B \in L(X_0, X_1)$ with $B \supseteq A$. In particular, we have $B = \overline{A}$ and*

$$\|B\| = \sup_{x \in \mathrm{dom}(A), x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The proofs of Proposition 2.1.4 and Corollary 2.1.5 are asked for in Exercise 2.2.

2.2 Operators in Hilbert spaces

Let us now focus on operators on Hilbert spaces. In this setting, we can additionally make use of scalar products $\langle \cdot, \cdot \rangle$, which in this course are considered to be linear in the second argument (and anti-linear in the first, in the case when $\mathbb{K} = \mathbb{C}$).
For a linear operator $A\colon \mathrm{dom}(A) \subseteq H_0 \to H_1$ the graph norm of A is induced by the scalar product

$$(x, y) \mapsto \langle x, y \rangle + \langle Ax, Ay \rangle,$$

known as the *graph scalar product of* A. If A is closed then $\mathrm{dom}(A)$ (equipped with the graph norm) is a Hilbert space.
Of course, no presentation of operators in Hilbert spaces would be complete without the central notion of the adjoint operator. We wish to pose the adjoint within the relational framework just established. The definition is as follows.

Definition. For a relation $A \subseteq H_0 \times H_1$ we define the *adjoint relation* A^* by

$$A^* := -\left(\left(A^{-1}\right)^{\perp}\right) \subseteq H_1 \times H_0,$$

where the orthogonal complement is computed in the direct sum of the Hilbert spaces H_1 and H_0; that is, the set $H_1 \times H_0$ endowed with the scalar product $\big((x,y),(u,v)\big) \mapsto \langle x,u\rangle + \langle y,v\rangle$.

Remark 2.2.1. Let $A \subseteq H_0 \times H_1$. Then we have

$$A^* = \{(u,v) \in H_1 \times H_0\,;\, \forall (x,y) \in A : \langle u,y\rangle = \langle v,x\rangle\}\,.$$

In particular, if A is a linear operator, we have

$$A^* = \{(u,v) \in H_1 \times H_0\,;\, \forall x \in \mathrm{dom}(A) : \langle u,Ax\rangle = \langle v,x\rangle\}\,.$$

Lemma 2.2.2. *Let $A \subseteq H_0 \times H_1$ be a relation. Then A^* is a linear relation. Moreover, we have*

$$A^* = -\left(\left(A^{\perp}\right)^{-1}\right) = \left((-A)^{-1}\right)^{\perp} = \left(-\left(A^{-1}\right)\right)^{\perp} = \left((-A)^{\perp}\right)^{-1} = \left(-\left(A^{\perp}\right)\right)^{-1}.$$

The proof of this lemma is left as Exercise 2.3.

Remark 2.2.3. Let $A \subseteq H_0 \times H_1$. Since A^* is the orthogonal complement of $-A^{-1}$, it follows immediately that A^* is closed. Moreover, $A^* = \left(\overline{A}\right)^*$ since $A^{\perp} = \left(\overline{A}\right)^{\perp}$.

Lemma 2.2.4. *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then*

$$A^{**} := (A^*)^* = \overline{A}.$$

Proof. We compute using Lemma 2.2.2

$$A^{**} = \left(\left(-\left(A^*\right)\right)^{-1}\right)^{\perp} = \left(\left(-\left(-\left(\left(A^{\perp}\right)^{-1}\right)\right)\right)^{-1}\right)^{\perp} = \left(A^{\perp}\right)^{\perp} = \overline{A}. \qquad \square$$

Theorem 2.2.5. *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then*

$$\mathrm{ran}(A)^{\perp} = \ker(A^*) \quad \textit{and} \quad \overline{\mathrm{ran}}(A^*) = \ker(\overline{A})^{\perp}.$$

Proof. Let $u \in \ker(A^*)$ and let $y \in \operatorname{ran}(A)$. Then we find $x \in \operatorname{dom}(A)$ such that $(x, y) \in A$. Moreover, note that $(u, 0) \in A^*$. Then, we compute

$$\langle y, u \rangle = \langle x, 0 \rangle = 0.$$

This equality shows that $\operatorname{ran}(A)^\perp \supseteq \ker(A^*)$. If on the other hand, $u \in \operatorname{ran}(A)^\perp$ then for all $(x, y) \in A$ we have that

$$0 = \langle y, u \rangle ,$$

which implies $(u, 0) \in A^*$ and hence $u \in \ker(A^*)$. The remaining equation follows from Lemma 2.2.4 together with the first equation applied to A^*. □

The following decomposition result is immediate from the latter theorem and will be used frequently throughout the text.

Corollary 2.2.6. *Let $A \subseteq H_0 \times H_1$ be a closed linear relation. Then*

$$H_1 = \overline{\operatorname{ran}}(A) \oplus \ker(A^*) \quad \text{and} \quad H_0 = \ker(A) \oplus \overline{\operatorname{ran}}(A^*).$$

We will now turn to the case where the adjoint relation is actually a linear operator.

Lemma 2.2.7. *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then A^* is a linear operator if and only if A is densely defined. If, in addition, A is a linear operator, then A is closable if and only if A^* is densely defined.*

Proof. For the first equivalence, it suffices to observe that

$$\{v \in H_0\,;\,(0, v) \in A^*\} = \operatorname{dom}(A)^\perp. \tag{2.1}$$

Indeed, A being densely defined is equivalent to having $\operatorname{dom}(A)^\perp = \{0\}$. Moreover, A^* is an operator if and only if $\{v \in H_0\,;\,(0, v) \in A^*\} = \{0\}$. Next, we show (2.1). For this, let $v \in H_0$. We have $v \in \operatorname{dom}(A)^\perp$ if and only if for all $x \in \operatorname{dom}(A)$ and $y \in H_1$ such that $(x, y) \in A$ it follows that

$$\langle x, v \rangle = 0 = \langle y, 0 \rangle ,$$

which is equivalent to having $(0, v) \in A^*$. For the remaining equivalence, we need to characterise $\overline{A}$ being an operator. Using Lemma 2.2.4 and the first equivalence, we deduce that $\overline{A} = (A^*)^*$ is a linear operator if and only if A^* is densely defined. □

Lemma 2.2.8. *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then $\overline{A} \in L(H_0, H_1)$ if and only if $A^* \in L(H_1, H_0)$. In either case, $\|A^*\| = \|\overline{A}\|$.*

Proof. Note that $\overline{A} \in L(H_0, H_1)$ implies that A is closable and densely defined. Thus, by Lemma 2.2.7, A^* is a densely defined, closed linear operator. For $u \in \operatorname{dom}(A^*)$ we compute using Lemma 2.2.4

$$\|A^* u\| = \sup_{x \in H_0 \setminus \{0\}} \frac{|\langle A^* u, x \rangle|}{\|x\|} = \sup_{x \in H_0 \setminus \{0\}} \frac{|\langle u, \overline{A} x \rangle|}{\|x\|} \leqslant \|\overline{A}\| \|u\|,$$

yielding $\|A^*\| \leqslant \|\overline{A}\|$. On the one hand, this implies that A^* is bounded, and on the other, since A^* is densely defined we deduce $A^* \in L(H_1, H_0)$ by Lemma 2.1.3. The other implication (and the other inequality) follows from the first one applied to A^* instead of A using $A^{**} = \overline{A}$. □

We end this section by defining some special classes of relations and operators.

Definition. Let H be a Hilbert space and $A \subseteq H \times H$ a linear relation. We call A *(skew-)Hermitian* if $A \subseteq A^*$ ($A \subseteq -A^*$). We say that A is *(skew-)symmetric* if A is (skew-)Hermitian and densely defined (so that A^* is a linear operator), and A is called *(skew-)selfadjoint* if $A = A^*$ ($A = -A^*$). Additionally, if A is densely defined, then we say that A is *normal* if $AA^* = A^*A$.

2.3 Computing the adjoint

In general it is a very difficult task to compute the adjoint of a given (unbounded) operator. There are, however, cases, where the adjoint of a sum or the product can be computed more readily. We start with the most basic case of bounded linear operators.

Proposition 2.3.1. *Let $A, B \in L(H_0, H_1), C \in L(H_2, H_0)$. Then $(A+B)^* = A^* + B^*$ and $(AC)^* = C^*A^*$.*

The latter results are special cases of more general statements to follow.

Theorem 2.3.2. *Let $A \subseteq H_0 \times H_1$ be a relation and $B \in L(H_0, H_1)$. Then $(A+B)^* = A^* + B^*$.*

Proof. Let $(u, v) \in H_1 \times H_0$. Then we compute

$$\begin{aligned}(u,v) \in A^* + B^* &\iff (u, v - B^*u) \in A^* \\ &\iff \forall\, (x,y) \in A\colon\ \langle y, u\rangle = \langle x, v - B^*u\rangle \\ &\iff \forall\, (x,y) \in A\colon\ \langle y + Bx, u\rangle = \langle x, v\rangle \\ &\iff \forall\, (x,z) \in A + B\colon\ \langle z, u\rangle = \langle x, v\rangle \\ &\iff (u,v) \in (A+B)^*\,.\end{aligned}$$

Note that for the first, third and fourth equivalence, we have used the fact that $B \in L(H_0, H_1)$ together with Lemma 2.2.8. □

Corollary 2.3.3. *Let $A \subseteq H_0 \times H_1$, $B \in L(H_0, H_1)$. If A is densely defined, then $A^* + B^*$ is a linear operator and $(A+B)^* = A^* + B^*$.*

Theorem 2.3.4. *Let $A \subseteq H_0 \times H_1$ be a closed linear relation and $C \in L(H_2, H_0)$. Then $(AC)^* = \overline{C^*A^*}$.*

Proof. We first show that $AC \subseteq (C^*A^*)^*$. For this, let $(w, y) \in AC$. Then $(Cw, y) \in A$. Hence, for all $(u, z) \in C^*A^*$ or for all $(u, v) \in A^*$ and $z = C^*v$, we compute

$$\langle u, y\rangle = \langle v, Cw\rangle = \langle C^*v, w\rangle = \langle z, w\rangle ,$$

which implies that $AC \subseteq (C^*A^*)^*$. Next, let $(w, y) \in (C^*A^*)^*$. Then for all $(u, v) \in A^*$ and $z = C^*v$ we obtain

$$\langle u, y\rangle = \langle z, w\rangle = \langle C^*v, w\rangle = \langle v, Cw\rangle .$$

Thus, we obtain $(Cw, y) \in A^{**} = \overline{A} = A$. Thus, $(w, y) \in AC$. Hence,

$$AC = (C^*A^*)^* ,$$

which yields the assertion by adjoining this equation. □

Corollary 2.3.5. *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. Then $\left(\overline{A}C\right)^* = \overline{C^*A^*}$.*

Proof. The result follows upon realising that $A^* = A^{***} = \left(\overline{A}\right)^*$. □

Corollary 2.3.6. *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. If $\overline{A}C$ is densely defined, then C^*A^* is a closable linear operator with $\overline{C^*A^*} = \left(\overline{A}C\right)^*$.*

Remark 2.3.7. (a) Note that if $B \in L(H_1, H_2)$ and $A \subseteq H_0 \times H_1$ linear, then $\left(B\overline{A}\right)^* = A^*B^*$. Indeed, this follows from Theorem 2.3.4 applied to A^* and B instead of A and C^*, respectively, since then we obtain $(A^*B^*)^* = \overline{B^{**}A^{**}} = \overline{B\overline{A}}$. Computing adjoints on both sides again and using that A^*B^* is closed by Proposition 2.1.1, we get the assertion.
(b) We note here that $\overline{A}C$ cannot be replaced by $\overline{AC}$ and encourage the reader to find a counterexample for A being a closable linear operator. We also refer to [5] for a counterexample due to J. Epperlein.

We have already seen that $A^* = \overline{A}^*$. We can even restrict A to a core and still obtain the same adjoint.

Proposition 2.3.8. *Let $A \subseteq H_0 \times H_1$ be a linear relation, $D \subseteq \operatorname{dom}(A)$ a linear subspace. Then D is a core for A if and only if $(A \cap (D \times H_1))^* = A^*$.*

Proof. Assume that D is a core for A. Then $\overline{A \cap (D \times H_1)} \supseteq A$ and therefore we have

$$A^* \subseteq (A \cap (D \times H_1))^* = \left(\overline{A \cap (D \times H_1)}\right)^* \subseteq A^*.$$

On the other hand, assume that $(A \cap (D \times H_1))^* = A^*$. Then we obtain

$$\overline{A \cap (D \times H_1)} = (A \cap (D \times H_1))^{**} = A^{**} = \overline{A},$$

which yields the assertion. □

2.4 The spectrum and resolvent set

In this section, we focus on operators acting on a single Banach space. As such, throughout this section let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $A\colon \operatorname{dom}(A) \subseteq X \to X$ be a closed linear operator.

Definition. The set

$$\rho(A) \coloneqq \left\{\lambda \in \mathbb{K}\,;\, (\lambda - A)^{-1} \in L(X)\right\}$$

is called the *resolvent set* of A. We define

$$\sigma(A) \coloneqq \mathbb{K} \setminus \rho(A)$$

to be the *spectrum* of A.

We state and prove some elementary properties of the spectrum and the resolvent set. We shall see natural examples for A which satisfy that $\sigma(A) = \mathbb{K}$ or $\sigma(A) = \varnothing$ later on. For a metric space (X, d), we will write $B\left(x, r\right) = \{y \in X\,;\, d(x, y) < r\}$ for the open ball around x of radius r and $B\left[x, r\right] = \{y \in X\,;\, d(x, y) \leqslant r\}$ for the closed ball.

Proposition 2.4.1. *If $\lambda, \mu \in \rho(A)$, then the* resolvent identity *holds. That is*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)\,(\lambda - A)^{-1}\,(\mu - A)^{-1}\,.$$

Moreover, the set $\rho(A)$ is open. More precisely, if $\lambda \in \rho(A)$ then $B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right) \subseteq \rho(A)$ and for $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$ we have

$$\left\|(\mu - A)^{-1}\right\| \leqslant \frac{\left\|(\lambda - A)^{-1}\right\|}{1 - |\lambda - \mu|\left\|(\lambda - A)^{-1}\right\|}.$$

The mapping $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in L(X)$ is analytic.

Proof. For the first assertion, we let $\lambda, \mu \in \rho(A)$ and compute

$$\begin{aligned}(\lambda - A)^{-1} - (\mu - A)^{-1} &= (\lambda - A)^{-1}\left((\mu - A) - (\lambda - A)\right)(\mu - A)^{-1} \\ &= (\lambda - A)^{-1}\,(\mu - \lambda)\,(\mu - A)^{-1} \\ &= (\mu - \lambda)\,(\lambda - A)^{-1}\,(\mu - A)^{-1}\,.\end{aligned}$$

Next, let $\lambda \in \rho(A)$ and $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$. Then

$$\left\|(\lambda - \mu)\,(\lambda - A)^{-1}\right\| < 1.$$

Hence, $1-(\lambda-\mu)(\lambda-A)^{-1}$ admits an inverse in $L(X)$ satisfying

$$\left(1-(\lambda-\mu)(\lambda-A)^{-1}\right)^{-1}=\sum_{k=0}^{\infty}\left((\lambda-\mu)(\lambda-A)^{-1}\right)^{k}. \tag{2.2}$$

We claim that $\mu\in\rho(A)$. For this, we compute

$$\mu-A=\lambda-A-(\lambda-\mu)=(\lambda-A)\left(1-(\lambda-\mu)(\lambda-A)^{-1}\right).$$

Since $\left(1-(\lambda-\mu)(\lambda-A)^{-1}\right)$ is an isomorphsim in $L(X)$, we deduce that the right-hand side admits a continuous inverse if and only if the left-hand side does. As $\lambda\in\rho(A)$, we thus infer $\mu\in\rho(A)$. The estimate follows from (2.2). Indeed, we have

$$\begin{aligned}\left\|(\mu-A)^{-1}\right\| &\leqslant \left\|(\lambda-A)^{-1}\right\|\left\|\sum_{k=0}^{\infty}\left((\lambda-\mu)(\lambda-A)^{-1}\right)^{k}\right\| \\ &\leqslant \left\|(\lambda-A)^{-1}\right\|\sum_{k=0}^{\infty}\left\|(\lambda-\mu)(\lambda-A)^{-1}\right\|^{k}=\frac{\left\|(\lambda-A)^{-1}\right\|}{1-\left\|(\lambda-\mu)(\lambda-A)^{-1}\right\|}.\end{aligned}$$

For the final claim of the present proposition, we observe that

$$(\mu-A)^{-1}=\left(1-(\lambda-\mu)(\lambda-A)^{-1}\right)^{-1}(\lambda-A)^{-1}=\sum_{k=0}^{\infty}(\lambda-\mu)^{k}\left((\lambda-A)^{-1}\right)^{k+1},$$

which is an operator norm convergent power series expression for the resolvent at μ about λ. Thus, analyticity follows. □

We shall consider multiplication operators in $L_2(\mu)$ next. For a measurable function $V\colon\Omega\to\mathbb{R}$ we will use the notation $[V\leqslant c]:=V^{-1}\big[(-\infty,c]\big]$ for some constant $c\in\mathbb{R}$ (and similarly for relational symbols other than $\leqslant$).

Example 2.4.2. Let (Ω,Σ,μ) be a measure space and $V\colon\Omega\to\mathbb{K}$ a measurable function. Then the operator

$$\begin{aligned}V(\mathrm{m})\colon\ \operatorname{dom}(V(\mathrm{m}))\subseteq L_2(\mu)&\to L_2(\mu)\\ f&\mapsto\big(\omega\mapsto V(\omega)f(\omega)\big),\end{aligned}$$

where $\operatorname{dom}(V(\mathrm{m})):=\big\{f\in L_2(\mu)\,;\,\big(\omega\mapsto V(\omega)f(\omega)\big)\in L_2(\mu)\big\}$, is densely defined and closed. Moreover, $(V(\mathrm{m}))^{*}=V^{*}(\mathrm{m})$ where $V^{*}(\omega)=V(\omega)^{*}$ for all $\omega\in\Omega$ (here $V(\omega)^{*}$ denotes the complex conjugate of $V(\omega)$). If V is μ-almost everywhere bounded, then $V(\mathrm{m})$ is continuous. If $V\neq0$ μ-a.e. then V is injective and $V(\mathrm{m})^{-1}=\frac{1}{V}(\mathrm{m})$, where

$$\frac{1}{V}(\omega):=\begin{cases}\frac{1}{V(\omega)}, & V(\omega)\neq0,\\ 0, & V(\omega)=0,\end{cases}$$

for all $\omega\in\Omega$.

Let us now prove these statements. We first show that $V(\mathrm{m})$ is densely defined. For this, consider $\Omega_n := [|V| \leqslant n]$ and put $\mathbb{1}_n := \mathbb{1}_{\Omega_n}$. Let $f \in L_2(\mu)$. Then, we have for all $n \in \mathbb{N}$ that $\mathbb{1}_n f \in \operatorname{dom}(V(\mathrm{m}))$. From $\Omega = \bigcup_n \Omega_n$ and $\Omega_n \subseteq \Omega_{n+1}$ it follows that $\mathbb{1}_n f \to f$ in $L_2(\mu)$ as $n \to \infty$.
Next, we confirm that $V(\mathrm{m})$ is closed. Let $(f_k)_k$ in $\operatorname{dom}(V(\mathrm{m}))$ convergent in $L_2(\mu)$ with $(V(\mathrm{m})f_k)_k$ be convergent in $L_2(\mu)$. Denote the respective limits by f and g. It is clear that for all $n \in \mathbb{N}$ we have $\mathbb{1}_n f_k \to \mathbb{1}_n f$ as $k \to \infty$. Also, we have

$$\mathbb{1}_n g = \lim_{k\to\infty} \mathbb{1}_n V(\mathrm{m}) f_k = \lim_{k\to\infty} V(\mathrm{m}) \mathbb{1}_n f_k = V(\mathrm{m}) \mathbb{1}_n f = V \mathbb{1}_n f.$$

Hence, $g = Vf$ μ-almost everywhere and since $g \in L_2(\mu)$, we have that $f \in \operatorname{dom}(V(\mathrm{m}))$.
We will compute the adjoint of $V(\mathrm{m})$ next. For this, we observe that $V^*(\mathrm{m}) \subseteq V(\mathrm{m})^*$. For the other inclusion, we let $u \in \operatorname{dom}(V(\mathrm{m})^*)$. Then, for all $f \in \operatorname{dom}(V(\mathrm{m}))$ and $n \in \mathbb{N}$, we compute

$$\langle V(\mathrm{m})f, \mathbb{1}_n u\rangle = \langle \mathbb{1}_n V(\mathrm{m})f, u\rangle = \langle V(\mathrm{m})\mathbb{1}_n f, u\rangle = \langle \mathbb{1}_n f, V(\mathrm{m})^* u\rangle = \langle f, \mathbb{1}_n V(\mathrm{m})^* u\rangle .$$

This shows that $\mathbb{1}_n u \in \operatorname{dom}(V(\mathrm{m})^*)$ and $V(\mathrm{m})^* \mathbb{1}_n u = \mathbb{1}_n V(\mathrm{m})^* u$. Furthermore, we also get that

$$\langle V(\mathrm{m})f, \mathbb{1}_n u\rangle = \langle f, V^*(\mathrm{m})\mathbb{1}_n u\rangle$$

for all $f \in \operatorname{dom}(V(\mathrm{m}))$ and $n \in \mathbb{N}$, which gives $V^*(\mathrm{m})\mathbb{1}_n u = V(\mathrm{m})^* \mathbb{1}_n u = \mathbb{1}_n V(\mathrm{m})^* u$. Since $V^*(\mathrm{m})$ is closed, we infer that $u \in \operatorname{dom}(V^*(\mathrm{m}))$ and $V^*(\mathrm{m})u = V(\mathrm{m})^* u$.
If $|V| \leqslant \kappa$ μ-almost everywhere for some $\kappa \geqslant 0$, then for all $f \in L_2(\mu)$ we have $|V(\omega)f(\omega)| \leqslant \kappa |f(\omega)|$ for μ-almost every $\omega \in \Omega$. Squaring and integrating this inequality yields boundedness of $V(\mathrm{m})$.
Assume that $V \neq 0$ μ-a.e. and $V(\mathrm{m})f = 0$. Then, $f(\omega) = 0$ for μ-a.e. $\omega \in \Omega$, which implies $f = 0$ in $L_2(\mu)$. Moreover, if $V(\mathrm{m})f = g$ for $f, g \in L_2(\mu)$, then for μ-a.e. $\omega \in \Omega$ we deduce that $f(\omega) = \frac{1}{V}(\omega) g(\omega)$, which shows $\frac{1}{V}(\mathrm{m}) \supseteq V(\mathrm{m})^{-1}$. If on the other hand $g \in \operatorname{dom}\left(\frac{1}{V}(\mathrm{m})\right)$, then a similar computation reveals that $\frac{1}{V}(\mathrm{m})g \in \operatorname{dom}(V(\mathrm{m}))$ and $V(\mathrm{m})\frac{1}{V}(\mathrm{m})g = g$.

The spectrum of $V(\mathrm{m})$ from the latter example can be computed once we consider a less general class of measure spaces.

Example 2.4.3. Let (Ω, Σ, μ) be a measure space with the property that for every $A \in \Sigma$ with $\mu(A) = \infty$, there exists $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that $B \subseteq A$ (such measure spaces are called *semi-finite*). Let $V \colon \Omega \to \mathbb{K}$ be a measurable function. Then we have

$$\sigma\left(V(\mathrm{m})\right) = \operatorname{ess-ran} V := \{\lambda \in \mathbb{K}\,;\, \forall \varepsilon > 0 \colon \mu\left([|\lambda - V| < \varepsilon]\right) > 0\}.$$

Let $\lambda \in \operatorname{ess-ran} V$. For all $n \in \mathbb{N}$ we find $B_n \in \Sigma$ with non-zero, but finite measure such that $B_n \subseteq \left[|\lambda - V| < \frac{1}{n}\right]$. We define $f_n := \sqrt{\frac{1}{\mu(B_n)}}\mathbb{1}_{B_n} \in L_2(\mu)$. Then $\|f_n\|_{L_2(\mu)} = 1$ and

$$|V(\omega)f_n(\omega)| \leqslant |V(\omega) - \lambda||f_n(\omega)| + |\lambda||f_n(\omega)| \leqslant \left(\frac{1}{n} + |\lambda|\right)|f_n(\omega)|$$

for $\omega \in \Omega$, which shows that $(f_n)_n$ is in $\mathrm{dom}(V(\mathrm{m}))$. A similar estimate, on the other hand, shows that

$$\|(V(\mathrm{m}) - \lambda)\, f_n\|_{L_2(\mu)} \to 0 \quad (n \to \infty).$$

Thus, $(V(\mathrm{m}) - \lambda)^{-1}$ cannot be continuous as $\|f_n\|_{L_2(\mu)} = 1$ for all $n \in \mathbb{N}$.
Let now $\lambda \in \mathbb{K}\backslash\text{ess-ran}\, V$. Then there exists $\varepsilon > 0$ such that $N := [|\lambda - V| < \varepsilon]$ is a μ-null set. In particular, $\lambda - V \neq 0$ μ-a.e. Hence, $(\lambda - V(\mathrm{m}))^{-1} = \frac{1}{\lambda - V}(\mathrm{m})$ is a linear operator. Since, $\left|\frac{1}{\lambda - V}\right| \leqslant 1/\varepsilon$ μ-almost everywhere, we deduce that $(\lambda - V(\mathrm{m}))^{-1} \in L(L_2(\mu))$ and hence, $\lambda \in \rho(V(\mathrm{m}))$.

We conclude this lecture by stating that multiplication operators as discussed in Example 2.4.2 and Example 2.4.3 are *the* prototypical example for normal operators. It is also important to note that, as we have seen in Example 2.4.2, a multiplication operator in $L_2(\mu)$ is self-adjoint if and only if V assumes values in the real numbers, only.

2.5 Comments

The material presented in this lecture is basic textbook knowledge. We shall thus refer to the monographs [4, 6]. Note that spectral theory for self-adjoint operators is a classical topic in functional analysis. For a glimpse on further theory of linear relations we exemplarily refer to [1, 2, 3].

Exercises

Exercise 2.1. Let $A \subseteq X_0 \times X_1$ be an unbounded linear operator. Show that for every linear operator $B \subseteq X_0 \times X_1$ with $B \supseteq A$ and $\mathrm{dom}(B) = X_0$, we have that B is not closed.

Exercise 2.2. Prove Proposition 2.1.4 and Corollary 2.1.5.

Exercise 2.3. Prove Lemma 2.2.2.

Exercise 2.4. Let $A\colon \mathrm{dom}(A) \subseteq H_0 \to H_0$ be a closed and densely defined linear operator. Show that for all $\lambda \in \mathbb{K}$ we have

$$\lambda \in \rho(A) \iff \lambda^* \in \rho(A^*).$$

Exercise 2.5. Let $U \subseteq H_0 \times H_1$ satisfy $U^{-1} = U^*$. Show that $U \in L(H_0, H_1)$ and that U is *unitary*, that is, U is onto and for all $x \in H_0$ we have $\|Ux\|_{H_1} = \|x\|_{H_0}$.

Exercise 2.6. Let $\delta\colon C\,[0,1] \subseteq L_2(0,1) \to \mathbb{K}, f \mapsto f(0)$, where $C\,[0,1]$ denotes the set of $\mathbb{K}$-valued continuous functions on $[0,1]$. Show that δ is not closable. Compute $\overline{\delta}$.

Exercise 2.7. Let $C \subseteq \mathbb{C}$ be closed. Provide a Hilbert space H and a densely defined closed linear operator A on H such that $\sigma(A) = C$.

References

[1] R. Arens. 'Operational calculus of linear relations'. In: *Pacific J. Math.* 11 (1961), pp. 9–23.

[2] T. Berger, C. Trunk and H. Winkler. 'Linear relations and the Kronecker canonical form'. In: *Linear Algebra Appl.* 488 (2016), pp. 13–44.

[3] R. Cross. *Multivalued linear operators*. Vol. 213. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1998, pp. x+335.

[4] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995, pp. xxii+619.

[5] R. Picard. 'Mother operators and their descendants'. In: *J. Math. Anal. Appl.* 403.1 (2013). with an extension by S. Trostorff and M. Waurick in arXiv:1203.6762, pp. 54–62.

[6] J. Weidmann. *Linear operators in Hilbert spaces*. Vol. 68. Graduate Texts in Mathematics. Translated from the German by Joseph Szücs. Springer-Verlag, New York-Berlin, 1980, pp. xiii+402.