Exercises lecture 01

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Exercise 1.1: Let $\phi \in C(\mathbb{R},\mathbb{R})$. Assume that $\forall (t,s) \in \mathbb{R}^2, \phi(t+s) = \phi(t)\phi(s)$ and $\phi(0) = 1$. Show that there exists $\alpha \in \mathbb{R}$ such that $\forall t \in \mathbb{R}, \phi(t) = e^{\alpha t}$.

Proof. We provide an analysis-synthesis proof of the result using a density argument.
• **Analysis :** Let $\phi \in C(\mathbb{R},\mathbb{R})$ be such that $\phi(0) = 1$ and $\forall (t,s) \in \mathbb{R}^2, \phi(t+s) = \phi(t)\phi(s)$.
➢ We first remark that

$$\phi(1) = \phi\left(\frac{1}{2}+\frac{1}{2}\right) = \left(\phi\left(\frac{1}{2}\right)\right)^2 \geqslant 0.$$

Since

$$1 = \phi(0) = \phi(1-1) = \phi(1)\phi(-1),$$

we deduce that

$$\phi(1) > 0 \text{ and } \phi(-1) = \phi(1)^{-1}. \tag{1}$$

➢ Let $n \in \mathbb{N}$. Then

$$\phi(n) = \phi(\underbrace{1+1+...+1}_{n \text{ times}}) = \phi(1)^n.$$

In the same way, using (1), we have :

$$\phi(-n) = \phi(-1)^n = \phi(1)^{-n}.$$

Hence

$$\forall n \in \mathbb{Z}, \phi(n) = \phi(1)^n.$$

Let $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$. We remark that

$$\phi(1) = \phi\left(\frac{q}{q}\right) = \left(\phi\left(\frac{1}{q}\right)\right)^q,$$

so we deduce that

$$\phi\left(\frac{1}{q}\right) = \phi(1)^{\frac{1}{q}}.$$

Hence

$$\phi\left(\frac{p}{q}\right) = \left(\phi\left(\frac{1}{q}\right)\right)^p = (\phi(1))^{\frac{p}{q}}.$$

Thus, we have shown that if we set $\alpha = \ln(\phi(1)) \in \mathbb{R}$ (well defined by (1)) then :

$$\forall x \in \mathbb{Q}, \phi(x) = \phi(1)^x = e^{x\ln(\phi(1))} = e^{\alpha x}.$$

By density of $\mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$ and since the applications $t \mapsto \phi(t)$ and $t \mapsto e^{\alpha t}$ are both continuous on $\mathbb{R}$, we have :

$$\forall t \in \mathbb{R}, \phi(t) = e^{\alpha t}. \tag{2}$$

• **Synthesis :** Conversly, let ϕ be an application defined by (2). Then $\phi \in C(\mathbb{R},\mathbb{R})$ and satisfies

$$\phi(0) = e^0 = 1$$

and

$$\forall (t,s) \in \mathbb{R}^2, \phi(t+s) = e^{\alpha(t+s)} = e^{\alpha t + \alpha s} = e^{\alpha t}e^{\alpha s} = \phi(t)\phi(s).$$

□

Exercise 1.2 : Let $n \in \mathbb{N}$. Let $T \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ such that $\forall (t,s) \in \mathbb{R}^2, T(t+s) = T(t)T(s)$ and $T(0) = I_n$. Show that there exists $A \in M_n(\mathbb{R})$ such that $\forall t \in \mathbb{R}, T(t) = e^{tA}$.

Proof. Let $T \in C^1(\mathbb{R}, M_n(\mathbb{R}))$ satisfying :

$$\forall (t,s) \in \mathbb{R}^2, T(t+s) = T(t)T(s) \tag{3}$$

and $T(0) = I_n$.
We differentiate (3) with respect to t and find :

$$\forall (t,s) \in \mathbb{R}^2, T'(t+s) = T'(t)T(s).$$

In particular, for $t = 0$, we have, setting $A = T'(0) \in M_n(\mathbb{R})$:

$$\forall s \in \mathbb{R}, T'(s) = AT(s).$$

Thus T satisfies the Cauchy problem

$$\begin{cases} T' = AT\,, \\ T(0) = I_n\,. \end{cases}$$

Applying the linear Cauchy-Lipschitz theorem, we find that :

$$\forall s \in \mathbb{R}, T(s) = e^{sA}.$$

□

Exercise 1.3 : Let $f \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. We consider the application u from $\mathbb{R}^3$ to $\mathbb{R}$ defined by

$$\forall x \in \mathbb{R}^3, u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\, dy$$

Show that u satisfies Poisson's equation $\Delta u = -f$.

Proof. We can describe u as $u(x) = (f * g)(x)$ where $g(x) := \frac{1}{4\pi}\frac{1}{|x|}$. Since $g \in L^1_{loc}(\mathbb{R}^3)$ and $f \in C^2_c(\mathbb{R}^3)$, by the theory of convolution, then $u \in C^2(\mathbb{R}^3)$. Furthermore, we also have

$$\Delta u(x) = (\Delta f * g)(x) = \int_{\mathbb{R}^3} \Delta f(x-y) g(y)\,\mathrm{d}y\,.$$

Our goal is to prove that $\Delta u = f$. Since g blows up at 0, we will seperate the domain $\mathbb{R}^3$ into two domains : a neighborhood of 0 and its complement in $\mathbb{R}^3$. Let $\varepsilon > 0$, we have

$$\Delta u(x) = \int_{B(0;\varepsilon)} \Delta f(x-y)g(y)\,\mathrm{d}y + \int_{\mathbb{R}^3 \setminus B(0;\varepsilon)} \Delta f(x-y)g(y)\,\mathrm{d}y\,,$$

where $B(0;\varepsilon) := \{x \in \mathbb{R}^3 : |x| \leq \varepsilon\}$.
The integral on the ball $B(0;\varepsilon)$ can be estimated by

$$\begin{aligned} \left| \int_{B(0;\varepsilon)} \Delta f(x-y)g(y)\,\mathrm{d}y \right| &\leq \|\Delta f\|_{L^\infty(\mathbb{R}^3)} \int_{B(0;\varepsilon)} |g(y)|\,\mathrm{d}y \\ &= C\int_0^\varepsilon r\mathrm{d}r \\ &= C\frac{\varepsilon^2}{2}\,, \end{aligned}$$

where C is a constant. In the above estimate, we used spherical coordinates to compute the integral on the ball $B(0;\varepsilon)$.
Next, we will try to make the integral on the complement of $B(0;\varepsilon)$ come closer to $f(x)$ when ε is small enough. To do that, first, we need to use integration by parts (twice) to make f appear in the integral :

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B(0;\varepsilon)} \Delta f(x-y)g(y)\,\mathrm{d}y &= \int_{\partial B(0;\varepsilon)} g(y)\nabla f(x-y)\cdot N(y)\,\mathrm{d}S(y) - \int_{\mathbb{R}^3 \setminus B(0;\varepsilon)} \nabla_y f(x-y)\cdot \nabla g(y)\,\mathrm{d}y \\ &= \int_{\partial B(0;\varepsilon)} g(y)\nabla f(x-y)\cdot N(y)\,\mathrm{d}S(y) - \int_{\partial B(0;\varepsilon)} f(x-y)\nabla g(y)\cdot N(y)\,\mathrm{d}S(y) \\ &\quad + \int_{\mathbb{R}^3 \setminus B(0;\varepsilon)} f(x-y)\cdot \Delta g(y)\,\mathrm{d}y\,, \end{aligned}$$

where N is the inward pointing normal vector of $\partial B(0;\epsilon)$.
We have the following three estimates corresponding to the three above integrals :

1.
$$\int_{\partial B(0;\varepsilon)} g(y)\nabla f(x-y)\cdot N(y)\,\mathrm{d}S(y) \le \|\nabla f\|_{L^\infty(\mathbb{R}^3)} \int_{\partial B(0;\varepsilon)} g(y)\,\mathrm{d}S(y) \le C\varepsilon\,.$$

2. Since $\nabla g(y) = \frac{-1}{4\pi}\frac{y}{|y|^3}$ and $N(y) = -\frac{y}{|y|} = -\frac{y}{\varepsilon}$ on $\partial B(0;\varepsilon)$, it leads to
$$\int_{\partial B(0;\varepsilon)} f(x-y)\nabla g(y)\cdot N(y)\,\mathrm{d}S(y) = \frac{1}{4\pi\epsilon^2}\int_{\partial B(0;\varepsilon)} f(x-y)\,\mathrm{d}S(y) = \frac{1}{|\partial B(x;\varepsilon)|}\int_{\partial B(x;\varepsilon)} f(y)\,\mathrm{d}S(y)\,.$$
Here $\partial B(0;\varepsilon)$ denotes the surface area of the ball $B(0;\varepsilon)$. By the continuity of f at x, we deduce that
$$\lim_{\varepsilon\to 0}\frac{1}{|\partial B(x;\varepsilon)|}\int_{\partial B(x;\varepsilon)} f(y)\,\mathrm{d}S(y) = f(x)\,.$$

3. For $y \neq 0$, by straightward computation,
$$\begin{aligned}
\Delta g(y) &= \frac{1}{4\pi}\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial y_3^2}\right)\frac{1}{|y|}\\
&= \frac{1}{4\pi}\left(\frac{\partial}{\partial y_1}\left(\frac{-y_1}{|y|^3}\right)+\frac{\partial}{\partial y_2}\left(\frac{-y_2}{|y|^3}\right)+\frac{\partial}{\partial y_3}\left(\frac{-y_3}{|y|^3}\right)\right)\\
&= \frac{1}{4\pi}\left(\frac{-|y|^3+3y_1^2|y|}{|y|^6}+\frac{-|y|^3+3y_2^2|y|}{|y|^6}+\frac{-|y|^3+3y_3^2|y|}{|y|^6}\right)\\
&= \frac{1}{4\pi}\left(\frac{-3}{|y|^3}+\frac{3}{|y|^3}\right)\\
&= 0\,.
\end{aligned}$$
Therefore, we obtain
$$\int_{\mathbb{R}^3\setminus B(0;\varepsilon)} f(x-y)\cdot\Delta g(y)\,\mathrm{d}y = 0\,.$$

For the above reasons, by taking the limit $\varepsilon \to 0$, we can conclude that
$$\Delta u(x) = -f(x)\,.$$

□

Exercise 1.4 : Let $f \in C_c^\infty(\mathbb{R},\mathbb{R})$. We consider the application u from $\mathbb{R}^2$ to $\mathbb{R}$ defined by
$$\forall (t,x)\in\mathbb{R}^2, u(t,x) = f(x+t).$$
Show that u satisfies the differential equation $\partial_t u = \partial_x u$ and show that $\forall x \in \mathbb{R}, u(0,x) = f(x)$.

Proof. u and f have the same regularity.
Using the chain rule, we have :
$$\forall (t,x)\in\mathbb{R}^2, \partial_t u(t,x) = 1\times f'(x+t) = \partial_x u(t,x).$$
We also have :
$$\forall x\in\mathbb{R}, u(0,x) = f(x+0) = f(x).$$

□

Exercise 1.5 : Let X and Y be two Banach spaces. Let $(T_n)_{n\in\mathbb{N}} \in L(X,Y)^{\mathbb{N}}$. We assume that
$$\sup_{n\in\mathbb{N}}\|T_n\|_{L(X,Y)} = +\infty.$$
Show that there exists $x \in X$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\|T_{n_k}x\|_Y \underset{k\to+\infty}{\longrightarrow} +\infty$.

Proof. The result to prove is exactly the contrapositive of Banach-Steinhauss theorem. □

Exercise 1.6 : Let $n \in \mathbb{N}$. We denote by $GL_n(\mathbb{K})$ the set of invertible matrices of order n.
Show that $GL_n(\mathbb{K})$ is an open subset of $M_n(\mathbb{K})$.

Proof. We remark that :

$$GL_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^*).$$

Since the application $\det : M_n(\mathbb{K}) \to \mathbb{K}$ is continuous on $M_n(\mathbb{K})$ because polynomial in the coefficients of the matrix looking at the definition

$$\forall M = (m_{i,j})_{1\leqslant i,j\leqslant n} \in M_n(\mathbb{K}), \det(M) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}$$

and $\mathbb{K}^*$ is an open subset of $\mathbb{K}$, we deduce that $GL_n(\mathbb{K})$ is an open set of $M_n(\mathbb{K})$. □

Exercise 1.7 : Let $n \in \mathbb{N}$. We consider the application Φ from $GL_n(\mathbb{K})$ to $M_n(\mathbb{K})$ defined by

$$\forall A \in GL_n(\mathbb{K}), \Phi(A) = A^{-1}.$$

Show that Φ is continuously differentiable and compute its differential application at every point of $GL_n(\mathbb{K})$.

Proof. We equip $M_n(\mathbb{K})$ with a norm of algebra $\| \cdot \|$.
• ➢ We first compute the differential of Φ at I_n.
Let $H \in M_n(\mathbb{K})$ be such that $\| H \| < 1$. Then the series $\sum_{(k\geqslant 0)} (-1)^k H^k$ is absolutely convergent in the Banach space $(M_n(\mathbb{K}), \| \cdot \|)$ so convergent. We denote by S its sum.
Then S satisfies :

$$S(I_n + H) = I_n = (I_n + H)S.$$

So $I_n + H \in GL_n(\mathbb{K})$ and $(I_n + H)^{-1} = S$.
Thus, we deduce that Φ is differentiable at I_n and $d\Phi(I_n)H = -H$ because

$$\Phi(I_n + H) = (I_n + H)^{-1} = \sum_{k=0}^{+\infty} (-1)^k H^k = I_n - H + O\left(\| H \|^2\right) = \Phi(I_n) - H + o\left(\| H \|\right). \tag{4}$$

➢ We now compute the differential of Φ at a general point in $GL_n(\mathbb{K})$.
Let $A \in GL_n(\mathbb{K})$. Let $H \in M_n(\mathbb{K})$ such that $A + H \in GL_n(\mathbb{K})$ (which is possible since $GL_n(\mathbb{K})$ is open) and $\| A^{-1}H \| < 1$.
Then using (4), we have :

$$\Phi(A + H) = (A + H)^{-1} = \left(A(I_n + A^{-1}H)\right)^{-1} = (I_n + A^{-1}H)^{-1}A^{-1} = \underbrace{A^{-1}}_{=\Phi(A)} - A^{-1}HA^{-1} + o\left(\| H \|\right).$$

Thus, the application Φ is differentiable at A and we have :

$$d\Phi(A)H = -A^{-1}HA^{-1}.$$

• Let $(A, B) \in \left(GL_n(\mathbb{K})\right)^2$. Let $H \in M_n(\mathbb{K})$. Then :

$$\begin{aligned} \| d\Phi(A)H - d\Phi(B)H \| &= \| A^{-1}HA^{-1} - B^{-1}HB^{-1} \| \\ &= \| \left(A^{-1} - B^{-1}\right) HA^{-1} + B^{-1}H\left(A^{-1} - B^{-1}\right) \| \\ &\leqslant \| A^{-1} - B^{-1} \| \left(\| A^{-1} \| + \| B^{-1} \|\right) \| H \| \end{aligned}$$

So

$$\| d\Phi(A) - d\Phi(B) \|_{L(M_n(\mathbb{K}))} \underset{\| A-B \| \to 0}{\longrightarrow} 0$$

i.e. Φ is of class C^1. □