# 23rd Internet Seminar "Evolutionary Equations"*

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# Contents

# 1 Introduction

This chapter is intended to give a brief introduction as well as a summary of the course to be presented throughout the semester. We shall highlight some of the main ideas and methods behind the theory and will also aim to provide some background on the main concept, which will be the central object of study in the forthcoming weeks: the notion of so-called

**Evolutionary Equations**

dating back to Picard in the seminal paper [Pic09]; see also [PM11, Chapter 6].
Another expression used to describe the same thing (and in order to distinguish the concept from *evolution equations*) is that of *evo-systems*. Before going into detail on what we think of when using the term evolutionary equations, we shall look into a seemingly similar class of equations first.

## 1.1 Evolution Equations

The term evolution equation is commonly referred to as a (partial) differential equation involving time. This is a well developed concept that can be found, for example, in the standard references [EN00; HP57; Paz83]. Before addressing a solution strategy for these kinds of problems we mention some examples. We shall revisit these examples again in the course later. One of the main examples of evolution equations, in the sense to be discussed in this section, is the heat equation in its second order form. More precisely,

$$\begin{cases} \partial_t\theta(t,x) = \Delta\theta(t,x), & (t,x) \in (0,\infty)\times\Omega, \\ \theta(0,x) = \theta_0(x), & x\in\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is some open set, and $\Delta = \sum_{j=1}^d \partial_j^2$ is the usual Laplacian carried out with respect to the '$x$-variables' or 'spatial variables', and $\theta_0$ is a given initial heat distribution and $\theta$ is the unknown (scalar-valued) heat distribution. The above heat equation is also accompanied with some boundary conditions for $\theta(t,x)$ which are required to be valid for all $t > 0$ and $x \in \partial\Omega$.
We shall explain one way of solving this problem. To this end we make a detour to the theory of ordinary differential equations. Let us consider an $n\times n$-matrix $A$ with entries from the field $\mathbb{K}$ of complex or real numbers, $\mathbb{C}$ or $\mathbb{R}$, and address the system of ordinary differential equations

$$\begin{cases} u'(t) = Au(t), & t>0, \\ u(0) = u_0 \end{cases}$$

for some given initial datum, $u_0 \in \mathbb{K}^n$. In this case, we know that there exists a unique solution. This solution can be computed with the help of the so-called matrix exponential

$$\mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^k}{k!} \in \mathbb{K}^{n\times n}$$

in the form

$$u(t) = \mathrm{e}^{tA} u_0.$$

As it turns out, this $u$ is continuously differentiable and $u$ satisfies the above equation. We note in particular that $\mathrm{e}^{tA}u_0 \to \mathrm{e}^{0A}u_0 = u_0$ as $t \to 0+$ and that $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$. In a way, to obtain the solution for the system of ordinary differential equations we need to construct $(\mathrm{e}^{tA})_{t\geqslant 0}$. This is the same idea behind the process for obtaining a solution for the aforementioned heat equation.
Indeed, given a suitable Banach space $X$ one aims to construct a so-called *$C_0$-semigroup,* $(T(t))_{t\geqslant 0}$, that is, for all $t \geqslant 0$, $T(t)$ is a bounded linear operator acting in $X$, $T(t) \in L(X)$, and the following conditions are satisfied

(a) semigroup law: $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$,

(b) strong continuity: for all $x \in X$, $\lim_{t\to 0+} T(t)x = x$.

For instance in the case of $X = L_2(\Omega)$, it is possible to construct such a family $(T(t))_{t\geqslant 0}$, written as $(\mathrm{e}^{t\Delta})_{t\geqslant 0}$, satisfying the just mentioned criteria. For every $\theta_0 \in L_2(\Omega)$ this $C_0$-semigroup provides a function $\theta\colon t \mapsto \mathrm{e}^{t\Delta}\theta_0 \in L_2(\Omega)$ which satisfies the above heat equation in a certain *generalised* sense. It is then an a posteriori question as to which additional conditions, for example on $\theta_0$, need to be imposed in order to assure that $\theta$ solves the above heat equation as it stands.
In the abstract setting of $C_0$-semigroups, the orbit $u\colon t \mapsto T(t)x$ for some $x \in X$ then satisfies the equation

$$\begin{cases} u'(t) = Bu(t), \quad t > 0, \\ u(0) = x \end{cases} \tag{1.1}$$

provided that $x \in \left\{ y \in X\,;\, By := \lim_{t\to 0+} \frac{1}{t}(T(t)y - y) \in X \text{ exists}\right\}$. It can be shown that $B$ is uniquely determined. $B$ is called the *generator* of the $C_0$-semigroup $(T(t))_{t\geqslant 0}$. The $C_0$-semigroup and the generator are in direct correspondence to each other. In applications, given some operator $B$ the task is to find a $C_0$-semigroup such that $Bx = \lim_{t\to 0+} \frac{1}{t}(T(t)x - x)$ for all $x \in X$ where either the left-hand side or the right-hand side is well-defined. In other words, the question is whether a given operator $B$ is actually the generator of a $C_0$-semigroup.
Note that in the case of the heat equation, the $C_0$-semigroup is also known as the *fundamental solution or Green's function* of the problem considered; in the abstract setting, $(T(t))_{t\geqslant 0}$ is the fundamental solution of (1.1).
As we have seen, $C_0$-semigroups focus on initial value problems. Moreover, the heat equation (as a partial differential equation) above is viewed as an *ordinary* differential equation with values in an *infinite-dimensional* state space $X$. While the left-hand side

of the equation is always of the same form, the complexity of the problem class is stored in the choice of $X$ and the operator $B$ (and, naturally, its domain of definition).
In the literature, explicit initial value problems like that of the ODE system or the heat equation are gathered under the umbrella term *evolution equation*. It has become customary to refer to any problem where $C_0$-semigroups play a fundamental role as an evolution equation. This is particularly the case when $X$ is infinite-dimensional. Then, arguably, the study of $C_0$-semigroups is the study of fundamental solutions (or abstract Green's functions) associated to a certain class of initial value problems for (partial) differential equations. A solution theory, that is, the proof for existence, uniqueness and continuous dependence on the data, is then contained in the construction of the fundamental solution in terms of the ingredients of the equation. More precisely, in the case of the ODE above, the fundamental solution is constructed in terms of $A$ and in case of the heat equation, the fundamental solution is constructed in terms of (a particular realisation of) $\Delta$ as an (unbounded) operator in $X$. We emphasise that there is some bias towards the temporal direction in the theory of $C_0$-semigroups in the sense that $C_0$-semigroups impose continuity in time while this is in general not assumed for the spatial directions.

## 1.2 Time-independent Problems

The construction of fundamental solutions is also a valuable method for obtaining a solution for time-independent problems, see, e.g., [Eva98]. To see this, let us consider Poisson's equation in $\mathbb{R}^3$: Given $f \in C_c^\infty(\mathbb{R}^3)$ we want to find a function $u\colon \mathbb{R}^3 \to \mathbb{R}$ with the property that

$$-\Delta u(x) = f(x) \quad (x \in \mathbb{R}^3).$$

It can be shown that $u$ given by

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \,\mathrm{d}y$$

is well-defined, twice continuously differentiable and satisfies Poisson's equation; cf. Exercise 1.3. Note that $x \mapsto \frac{1}{4\pi|x|}$ is also referred to as the *fundamental solution* or *Green's function* for Poisson's equation. The formula presented for $u$ is the *convolution* with the fundamental solution. The formula used to define $u$ also works for $f$ being merely bounded and measurable with compact support. In this case, however, the pointwise formula of Poisson's equation cannot be expected to hold anymore, simply because $f$ is well-defined only up to a set of measure 0. Thus, only a posteriori estimates under additional conditions on $f$ render $u$ to be twice continuously differentiable (say) with Poisson's equation holding for all $x \in \mathbb{R}^3$. However, similar to the semigroup setting, it is possible to *generalise* the meaning of $-\Delta u = f$. Then, again, the fundamental solution can be used to construct a solution for Poisson's equation for more general $f$.
The situation becomes different when we consider a boundary value problem instead of the problem above. More precisely, let $\Omega \subseteq \mathbb{R}^3$ be an open set and let $f \in L_2(\Omega)$. We

then need to ask whether there exists $u \in L_2(\Omega)$ such that

$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Notice that the task of just (mathematically) formulating this equation, let alone establishing a solution theory, is something that needs to be addressed. Indeed, we emphasise that it is unclear as to what $\Delta u$ is supposed to mean if $u \in L_2(\Omega)$, only. It turns out that the problem described is not well-posed in general. In particular – depending on the shape of $\Omega$ and the norms involved – it might, for instance, lack continuous dependence on the data, $f$.
In any case, the solution formula that we have used for the case when $\Omega = \mathbb{R}^3$ does not work anymore. Indeed, only particular shapes of $\Omega$ permit to construct a fundamental solution; see [Eva98, Section 2.2]. Despite this, when $\Omega$ is merely bounded, it is still possible to construct a solution, $u$, for the above problem. There are two key ingredients required for this approach. One is a clever application of Riesz's representation theorem for functionals in Hilbert spaces and the other one involves inventing 'suitable' interpretations of $\Delta u$ in $\Omega$ and $u = 0$ on $\partial\Omega$. Thus, the method of 'solving' Poisson's equation amounts to posing the correct question, which then can be addressed *without* invoking the fundamental solution. With this in mind, one could argue that the *setting* makes the problem solvable.

## 1.3 Evolution*ary* Equations

The central aim for evolutionary equations is to combine the rationales from both the $C_0$-semigroup theory and that from the time-independent case. That is to say, we wish to establish a setting that treats time-independent problems as well as time-dependent problems. At the same time we need to *generalise* solution concepts. We shall not aim to construct the fundamental solution in either the spatial or the temporal directions. The problem class will comprise of problems that can be written in the form

$$(\partial_t M(\partial_t) + A)\, U = F$$

where $U$ is the unknown and $F$ the known right-hand side. Furthermore, $A$ is an (unbounded, skew-selfadjoint) operator acting in some Hilbert space that is thought of as modelling spatial coordinates; $\partial_t$ is a realisation of the (time-)derivative operator and $M(\partial_t)$ is an analytic, bounded operator-valued function $M$, which is evaluated at the time derivative. In the course of the next chapters, we shall specfiy the definitions and how standard problems fit into this problem class.
Before going into greater depth on this approach, we would like to emphasise the key differences and similarities which arise when compared to the derivation of more traditional solution theories that we outlined above.
Since the solution theory for evolutionary equations will also encapsulate time-independent problems, we cannot just focus on initial value problems but rather on inhomogeneous problems. As we do not want to require the existence of any fundamental solution

we will also need to introduce a *generalisation* of the concept of a solution. Indeed, continuity in the form of the existence of a $C_0$-semigroup, or variants thereof, will neither be shown nor be expected. Moreoever, we shall see that both $\partial_t$ and $A$ are *unbounded* operators with $M(\partial_t)$ being a bounded operator. Thus, we need to make sense of the operator sum of the two unbounded operators $\partial_t M(\partial_t)$ and $A$, which, in general, cannot be realised as being onto but rather as having dense range, only.
A post-processing procedure will then ensure that for more regular right-hand sides, $F$, the solution $U$ will also be more regular. In some cases this will, for instance, amount to $U$ being continuous in the time variable. In this way, phrased in similar settings, $C_0$-semigroup theory may be viewed as a *regularity* theory for a subclass of evolutionary equations. We shall entirely confine ourselves within the Hilbert space case though. In this sense, the solution theory to be presented will be, in essence, an application of the projection theorem (similar to time-independent problems). In our case, however, there will not be as much of a regularity bias as there is in $C_0$-semigroup theory or in abstract ODEs with an (infinite-)dimensional state space. In fact, the projection theorem is applied in a Hilbert space, which combines both spatial and temporal variables.
The operator $M(\partial_t)$ is thought of as carrying all the 'complexity' of the model. This is different to $C_0$-semigroups, where this complexity is put on to the (domain of the) generator. What we mean by complexity will become more apparent when we discuss some examples.
Finally, let us stress that $A$ being 'skew-selfadjoint' is a way of implementing first order systems in our abstract setting. In fact, deviating from classical approaches, we shall focus on first order equations in *both* time *and* space. This is also another change in perspective when compared to classical approaches. As classical treatments might emphasise the importance of the Laplacian (and hence Poisson's equation) and variants thereof, evolutionary equations rather emphasise *Maxwell's equations* as the prototypical PDE. This change of point of view will be illustrated in the following section, where we address some classical examples.

## 1.4 Particular Examples and the Change of Perspective

Here we will focus on three examples. These examples will also be the first to be readdressed when we discuss the solution theory of evolutionary equations in a later chapter. In order to simplify the current presentation we will not consider boundary value problems but solely concentrate on problems posed on $\Omega = \mathbb{R}^3$. Furthermore, we shall dispose of any initial conditions.

### Maxwell's Equations

The prototypical evolutionary equation is the system provided by Maxwell's equations. Maxwell's equations consist of two equations describing an electro-magnetic field, $(E, H)$, subject to a given certain external current, $J$,

$$\partial_t \varepsilon E + \sigma E - \operatorname{curl} H = -J,$$

$$\partial_t \mu H + \operatorname{curl} E = 0.$$

We shall detail the properties of the material parameters $\varepsilon, \mu$, and $\sigma$ later on. For the time being it is safe to assume that they are non-negative real numbers and that they additionally satisfy that $\mu(\varepsilon+\sigma) > 0$. Now, in the setting of evolutionary equations, we gather the electro-magnetic field into one column vector and obtain

$$\left(\partial_t \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J \\ 0 \end{pmatrix}.$$

We shall see later that we obtain an evolutionary equation by setting

$$M(\partial_t) \coloneqq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \partial_t^{-1} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A \coloneqq \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}.$$

A formulation that fits well into the $C_0$-semigroup setting would be, for example,

$$\partial_t \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} -\sigma & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} -J \\ 0 \end{pmatrix},$$

provided that $\varepsilon > 0$. The inhomogeneous right-hand side $(-\frac{1}{\varepsilon}J, 0)$ can then be dealt with by means of the variation of constants formula, which is the incarnation of the convolution of $(-\frac{1}{\varepsilon}J, 0)$ with the fundamental solution in this time-dependent situation. Thus, in order to apply semigroup theory, the main task lies in showing that

$$\begin{pmatrix} -\frac{1}{\varepsilon}\sigma & \frac{1}{\varepsilon}\operatorname{curl} \\ -\frac{1}{\mu}\operatorname{curl} & 0 \end{pmatrix}$$

is the generator of a $C_0$-semigroup.

A different formulation needs to be put in place if $\varepsilon = 0$. The situation becomes even more complicated if $\varepsilon$ and $\sigma$ are bounded, non-negative, measurable functions of the spatial variable such that $\varepsilon + \sigma \geqslant c$ for some $c > 0$. In the setting of evolutionary equations, this problem, however, *can* be dealt with. Note that then one cannot expect $E$ to be continuous with respect to the temporal variable unless $J$ is smooth enough.

## Wave Equation

We shall discuss the scalar wave equation in a medium where the wave propagation speed is inhomogeneous in different directions of space. This is modelled by finding $u \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that, given a suitable forcing term $f \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ (again we skip initial values here), we have

$$\partial_t^2 u - \operatorname{div} a \operatorname{grad} u = f,$$

where $a = a^\top \in \mathbb{R}^{3\times 3}$ is positive definite; that is, $\langle \xi, a\xi \rangle_{\mathbb{R}^3} > 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$. In the context of evolutionary equations, we rewrite this as a first order problem in time *and* space. For this, we introduce $v \coloneqq \partial_t u$ and $q \coloneqq -a \operatorname{grad} u$ and obtain that

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Thus,

$$M(\partial_t) := \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and } A := \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$$

yield a corresponding formulation. A semigroup formulation would work again by multiplying through by $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

Let us mention briefly that it is also possible to rewrite the wave equation as a first order system in time only. For this, a standard ODE trick is used: one simply sticks with the additional variable $v = \partial_t u$ and obtains that

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \operatorname{div} a \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

In this formulation the 'complexity' of the model is contained in the operator

$$\begin{pmatrix} 0 & 1 \\ \operatorname{div} a \operatorname{grad} & 0 \end{pmatrix}.$$

One would then have to show that this operator is the generator of a $C_0$-semigroup.

## Heat Equation

We have already formulated the semigroup perspective of the heat equation

$$\partial_t \theta - \operatorname{div} a \operatorname{grad} \theta = Q,$$

in which we have added a heat source $Q$ and a conductivity $a = a^\top \in \mathbb{R}^{3\times 3}$ being positive definite. Here, again, we reformulate the heat equation as a first order system in time and space to end up (again setting $q := -a \operatorname{grad} \theta$) with

$$\left( \partial_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

In the context of evolutionary equations we then have that

$$M(\partial_t) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \partial_t^{-1} \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and } A := \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}.$$

The advantage of this reformulation is that it becomes easily comparable to the first order formulation of the wave equation outlined above. For instance it is now possible to easily consider mixed type problems of the form

$$\left( \partial_t \begin{pmatrix} 1 & 0 \\ 0 & (1-s)a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & s a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix},$$

with $s \colon \mathbb{R}^3 \to [0, 1]$ being an arbitrary measurable function. In fact, in the solution theory for evolutionary equations, this does not amount to any additional complication of the problem. Models of this type are particularly interesting in the context of so-called solid-fluid interaction, where the relations of a solid body and a flow of fluid surrounding it are addressed.

## 1.5 A Brief Outline of the Course

We now present an overview of the contents of the following chapters.

### Basics

In order to properly set the stage, we shall begin with some background operator theory in Banach and Hilbert spaces. We assume the readers to be acquainted with some knowledge on bounded linear operators, such as the uniform boundedness principle, and basic concepts in the topology of metric spaces, such as density and closure. The most important new material will be the adjoint of an operator, which need not be bounded anymore. In order to deal with this notion, we will consider relations rather than operators as they provide the natural setting for *unbounded* operators. Having finished this brief detour on operator theory, we will turn to a generalisation of Lebesgue spaces. More precisely, we will survey ideas from Lebesgue's integration theory for functions attaining values in an infinite-dimensional Banach space.

### The Time Derivative

Banach space-valued (or rather Hilbert space-valued) integration theory will play a fundamental role in defining the time derivative as an unbounded, continuously invertible operator in a suitable Hilbert space. In order to obtain continuous invertibility, we have to introduce an exponential weighting function, which is akin to the exponential weight introduced in the space of continuous functions for a proof of the Picard–Lindelöf theorem. It is therefore natural to discuss the application of this operator to ordinary differential equations. In particular, we will present a Hilbert space solution theory for ordinary differential equations. Here, we will also have the opportunity to discuss ordinary differential equations with delay and memory. After this short detour, we will turn back to the time derivative operator and describe its spectrum. For this we introduce the so-called Fourier–Laplace transformation which transforms the time derivative into a multiplication operator. This unitary transformation will additionally serve to define (analytic and bounded) functions of the time derivative. This is absolutely essential for the formulation of evolutionary equations.

### Evolutionary Equations

Having finished the necessary preliminary work, we will then be in a position to provide the proper justification of the formulation and solution theory for evolutionary equations. We will accompany this solution theory not only with the three leading examples from above, but also with some more sophisticated equations. Amazingly, the considered space-time setting will allow us to discuss (time-)fractional differential equations, partial differential equations with delay terms and even a class of integro-differential equations. Withdrawing the focus on regularity with respect to the temporal variable, we are en passant able to generalise well-posedness conditions from the classical literature. However, we shall stick with the treatment of analytic operator-valued functions $M$ only.

Therefore, we will also include some arguments as to why this assumption seems to be *physically* meaningful. It will turn out that analyticity and causality are intimately related via both the so-called Paley–Wiener theorem and a representation theorem for time translation invariant causal operators.

## Initial Value Problems for Evolutionary Equations

As it has been outlined above, the focus of evolutionary equations is on inhomogeneous right-hand sides rather than on initial value problems. However, there is also the possibility to treat initial value problems with the approach discussed here. For this, we need to introduce extrapolation spaces. This then enables us to formulate initial value problems as inhomogeneous equations. We have to make a concession on the structure of the problem, however. In fact, we will focus on the case when $M(\partial_t) = M_0 + \partial_t^{-1} M_1$ for some bounded linear operators $M_0, M_1$ acting in the spatial variables alone. The initial condition will then read as $(M_0 U)(0+) = M_0 U_0$. Hence, one might argue that the initial condition $U(0+) = U_0$ is only assumed in a rather generalised sense. This is due to the fact that $M_0$ might be zero. However, for the case $A = 0$ we will also discuss the initial condition $U(0+) = U_0$, which amounts to a treatment of so-called differential-algebraic equations in both finite- and inifinite-dimensional state spaces.

## Properties of Solutions and Inhomogeneous Boundary Value Problems

Turning back to the case when $A \neq 0$ we will discuss qualitative properties of solutions of evolutionary equations. One of which will be exponential decay. We will identify a subclass of evolutionary equations where it is comparatively easy to show that if the right-hand side decays exponentially then so too must the solution. This is the proper replacement in our setting for the notion of exponential stability from $C_0$-semigroups. If the right-hand side is smooth enough we obtain that $U(t)$, the solution of the evolutionary equation at time $t$, decays exponentially if $t \to \infty$. Furthermore, we will frame inhomogeneous boundary value problems in the setting of evolutionary equations. The method will require a bit more on the regularity theory for evolutionary equations and a definition of suitable boundary values. In particular, we shall present a way of formulating classical inhomogeneous boundary value problems for domains without any boundary regularity.

## Properties of the Solution Operator

In the final part, we shall have another look at the advantages of the problem formulation. In fact, we will have a look at the notion of homogenisation of differential equations. In the problem formulation presented here, we shall analyse the continuity properties of the solution operator with respect to weak operator topology convergence of the operator $M(\partial_t)$. We will address an example for ordinary differential equations (when $A = 0$) and one for partial differential equations (when $A \neq 0$). It will turn out that the respective continuity properties are profoundly different from one another.

## 1.6 Comments

The focus presented here on the main notions behind evolutionary equations is mostly in order to properly motivate the theory and highlight the most striking differences in the philosophy. There are other solution concepts (and corresponding general settings) developed for partial differential equations; either time-dependent or without involving time.
There is an abundance of examples and additional concepts for $C_0$-semigroups for which we refer to the aforementioned standard treatments again. There is also a generalisation to problems that are second order in time, e.g., $u'' = Au$, where $u(0)$ and $u'(0)$ are given. This gives rise to cosine families of bounded linear operators which is another way of generalising the fundamental solution concept, see, for example, [Sov66].
The main focus of all of these equations is to address *initial value problems,* where the (first/second) time derivative of the unknown is explicit.
With a focus on static, that is, time-independent partial differential equations, the notion of Friedrichs systems is also concerned with a way of writing many PDEs from mathematical physics into a common form, see [Fri54; Fri58]. A time-dependent variant of constant coefficient Friedrichs systems are so-called symmetric-hyperbolic systems, see e.g. [BS07]. In these cases, whether the authors treat constant coefficients or not, the framework of evolutionary equations adds a profound amount of additional complexity by including the operator $M(\partial_t)$.
The treatment of time-dependent problems in space-time settings and addressing corresponding well-posedness properties of a sum of two unbounded operators has also been considered in [DG75] with elaborate conditions on the operators involved. In their studies, the flexibility introduced by the operator $M(\partial_t)$ in our setting is missing, thus the time derivative operator is not thought of having any variable coefficients attached to it.

## Exercises

**Exercise 1.1.** Let $\phi \in C(\mathbb{R}, \mathbb{R})$. Assume that $\phi(t+s) = \phi(t)\phi(s)$ for all $t, s \in \mathbb{R}$, $\phi(0) = 1$. Show that $\phi(t) = \mathrm{e}^{\alpha t}$ $(t \in \mathbb{R})$ for some $\alpha \in \mathbb{R}$.

**Exercise 1.2.** Let $n \in \mathbb{N}$, $T \colon \mathbb{R} \to \mathbb{R}^{n\times n}$ continuously differentiable such that $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}$, $T(0) = I$. Show that there exists $A \in \mathbb{R}^{n\times n}$ with the property that $T(t) = \mathrm{e}^{tA}$ $(t \in \mathbb{R})$.

**Exercise 1.3.** Show that $x \mapsto u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\,\mathrm{d}y$ satisfies Poisson's equation, given $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^3)$.

**Exercise 1.4.** Let $f \in C_{\mathrm{c}}^\infty(\mathbb{R})$. Define $u(t, x) \coloneqq f(x+t)$ for $x, t \in \mathbb{R}$. Show that $u$ satisfies the differential equation $\partial_t u = \partial_x u$ and $u(0, x) = f(x)$ for all $x \in \mathbb{R}$.

**Exercise 1.5.** Let $X, Y$ be Banach spaces, $(T_n)_{n\in\mathbb{N}}$ be a sequence in $L(X, Y)$, the set of bounded linear operators. If $\sup\{\|T_n\| \,;\, n \in \mathbb{N}\} = \infty$, show that there is $x \in X$ and a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\|T_{n_k} x\| \to \infty$.

**Exercise 1.6.** Let $n \in \mathbb{N}$. Denote by $\mathrm{GL}(n;\mathbb{K})$ the set of continuously invertible $n \times n$ matrices. Show that $\mathrm{GL}(n;\mathbb{K}) \subseteq \mathbb{K}^{n\times n}$ is open.

**Exercise 1.7.** Let $n \in \mathbb{N}$. Show that $\Phi\colon \mathrm{GL}(n;\mathbb{K}) \ni A \mapsto A^{-1} \in \mathbb{K}^{n\times n}$ is continuously differentiable. Compute $\Phi'$.

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# 2 Unbounded Operators

We will gather some information on operators in Banach and Hilbert spaces. Throughout this chapter let $X_0$, $X_1$, and $X_2$ be Banach spaces and $H_0$, $H_1$, and $H_2$ be Hilbert spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

## 2.1 Operators in Banach Spaces

The main difference of continuous linear operators, that is,

$$L(X_0, X_1) \coloneqq \left\{ B\colon X_0 \to X_1\,;\, B \text{ linear}, \|B\| \coloneqq \sup_{x \in X_0 \setminus \{0\}} \frac{\|Bx\|}{\|x\|} < \infty \right\}$$

(with the usual abbreviation $L(X_0) := L(X_0, X_0)$) and the set of uncontinuous or unbounded linear operators is that the latter only need to be defined on a subset of $X_0$. In order to define unbounded linear operators, we will first take a more general point of view and introduce (linear) relations. This perspective will turn out to be the natural setting later on.

**Definition.** A subset $A \subseteq X_0 \times X_1$ is called a *relation in* $X_0$ *and* $X_1$. We define the *domain*, *range* and *kernel of* $A$ as follows

$$\begin{aligned} \operatorname{dom}(A) &\coloneqq \{x \in X_0\,;\, \exists y \in X_1\colon (x, y) \in A\}\,, \\ \operatorname{ran}(A) &\coloneqq \{y \in X_1\,;\, \exists x \in X_0\colon (x, y) \in A\}\,, \\ \ker(A) &\coloneqq \{x \in X_0\,;\, (x, 0) \in A\}\,. \end{aligned}$$

The *image,* $A[M]$*, of a set* $M \subseteq X_0$ *under* $A$ is given by

$$A[M] \coloneqq \{y \in X_1\,;\, \exists x \in M\colon (x, y) \in A\}\,.$$

A relation $A$ is called *bounded*, if for all bounded $M \subseteq X_0$ the set $A[M] \subseteq X_1$ is bounded. For a given relation $A$ we define the *inverse relation*

$$A^{-1} \coloneqq \{(y, x) \in X_1 \times X_0\,;\, (x, y) \in A\}\,.$$

A relation $A$ is called *linear*, if $A \subseteq X_0 \times X_1$ is a linear subspace. A linear relation $A$ is called *linear operator* or just *operator from* $X_0$ *to* $X_1$, if

$$A[\{0\}] = \{y \in X_1\,;\, (0, y) \in A\} = \{0\}.$$

In this case, we also write

$$A\colon \operatorname{dom}(A) \subseteq X_0 \to X_1$$

to denote a linear operator from $X_0$ to $X_1$. Moreover, we shall write $Ax = y$ instead of $(x, y) \in A$ in this case. A linear operator $A$, which is not bounded, is called *unbounded.*

For completeness, we also define the sum, scalar multiples, and composition of relations.

**Definition.** Let $A \subseteq X_0 \times X_1$, $B \subseteq X_0 \times X_1$ and $C \subseteq X_1 \times X_2$ be relations, $\lambda \in \mathbb{K}$. Then we define

$$\begin{aligned} A + B &:= \{(x, y + w) \in X_0 \times X_1 \,;\, (x, y) \in A, (x, w) \in B\}, \\ \lambda A &:= \{(x, \lambda y) \in X_0 \times X_1 \,;\, (x, y) \in A\}, \\ CA &:= \{(x, z) \in X_0 \times X_2 \,;\, \exists y \in X_1 \colon (x, y) \in A, (y, z) \in C\}. \end{aligned}$$

For a relation $A \subseteq X_0 \times X_1$ we will use the abbreviation $-A := -1A$ (so that the minus sign only acts on the second component). We now proceed with topological notions for relations.

**Definition.** Let $A \subseteq X_0 \times X_1$ be a relation. $A$ is called *densely defined*, if $\operatorname{dom}(A)$ is dense in $X_0$. We call $A$ *closed*, if $A$ is a closed subset of the direct sum of the Banach spaces $X_0$ and $X_1$. If $A$ is a linear operator then we will call $A$ *closable*, whenever $\overline{A} \subseteq X_0 \times X_1$ is a linear operator.

**Proposition 2.1.1.** *Let $A \subseteq X_0 \times X_1$ be a relation, $C \in L(X_2, X_0)$ and $B \in L(X_0, X_1)$. Then the following statements hold.*

(a) *$A$ is closed if and only if $A^{-1}$ is closed. Moreover, we have $(\overline{A})^{-1} = \overline{A^{-1}}$.*

(b) *$A$ is closed if and only if $A + B$ is closed;*

(c) *if $A$ is closed, then $AC$ is closed.*

*Proof.* Statement (a) follows upon realising that $X_0 \times X_1 \ni (x, y) \mapsto (y, x) \in X_1 \times X_0$ is an isomorphism.
For statement (b), it suffices to show that the closedness of $A$ implies the same for $A + B$. Let $((x_n, y_n))_n$ be a sequence in $A + B$ convergent in $X_0 \times X_1$ to some $(x, y)$. Since $B \in L(X_0, X_1)$, it follows that $((x_n, y_n - Bx_n))_n$ in $A$ is convergent to $(x, y - Bx)$ in $X_0 \times X_1$. Since $A$ is closed, $(x, y - Bx) \in A$. Thus, $(x, y) \in A + B$.
For statement (c), let $((w_n, y_n))_n$ be a sequence in $AC$ convergent in $X_2 \times X_1$ to some $(w, y)$. Since $C$ is continuous, $(Cw_n)_n$ converges to $Cw$. Hence, $(Cw_n, y_n) \to (Cw, y)$ in $X_0 \times X_1$ and since $(Cw_n, y_n) \in A$ and $A$ is closed, it follows that $(Cw, y) \in A$. Equivalently, $(w, y) \in AC$, which yields closedness of $AC$. □

We shall gather some other elementary facts about closed operators in the following. We will make use of the following notion.

**Definition.** Let $A \colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then the *graph norm* of $A$ is defined by $\operatorname{dom}(A) \ni x \mapsto \|x\|_A := \sqrt{\|x\|^2 + \|Ax\|^2}$.

**Lemma 2.1.2.** *Let $A \colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then the following statements are equivalent:*

(i) *$A$ is closed.*

(ii) $\mathrm{dom}(A)$ *equipped with the graph norm is a Banach space.*

(iii) *For all* $(x_n)_n$ *in* $\mathrm{dom}(A)$ *convergent in* $X_0$ *such that* $(Ax_n)_n$ *is convergent in* $X_1$ *we have* $\lim_{n\to\infty} x_n \in \mathrm{dom}(A)$ *and* $A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n$.

*Proof.* For the equivalence (i)⇔(ii), it suffices to observe that $\mathrm{dom}(A) \ni x \mapsto (x, Ax) \in A$, where $\mathrm{dom}(A)$ is endowed with the graph norm, is an isomorphism. The equivalence (ii)⇔(iii) is an easy reformulation of the definition of closedness of $A \subseteq X_0 \times X_1$. □

Unless explicitly stated otherwise (e.g. in the form $\mathrm{dom}(A) \subseteq X_0$, where we regard $\mathrm{dom}(A)$ as a subspace of $X_0$), for closed operators $A$ we always consider $\mathrm{dom}(A)$ as a Banach space in its own right; that is, we shall regard it as being endowed with the graph norm.

**Lemma 2.1.3.** *Let* $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ *be a closed linear operator. Then* $A$ *is bounded if and only if* $\mathrm{dom}(A) \subseteq X_0$ *is closed.*

*Proof.* First of all note that boundedness of $A$ is equivalent to the fact that the graph norm and the $X_0$-norm on $\mathrm{dom}(A)$ are equivalent. Hence, the closedness and boundedness of $A$ implies that $\mathrm{dom}(A) \subseteq X_0$ is closed. On the other hand, the embedding

$$\iota\colon (\mathrm{dom}(A), \|\cdot\|_A) \hookrightarrow (\mathrm{dom}(A), \|\cdot\|_{X_0})$$

is continuous and bijective. Since the range is closed, the open mapping theorem implies that $\iota^{-1}$ is continuous. This yields the equivalence of the graph norm and the $X_0$-norm and, thus, the boundedness of $A$. □

For unbounded operators, obtaining a precise description of the domain may be difficult. However, there may be a subset of the domain which essentially (or approximately) describes the operator. This gives rise to the following notion of a core.

**Definition.** Let $A \subseteq X_0 \times X_1$. A set $D \subseteq \mathrm{dom}(A)$ is called a *core for* $A$ provided $\overline{A \cap (D \times X_1)} = \overline{A}$.

**Proposition 2.1.4.** *Let* $A \in L(X_0, X_1)$*, and* $D \subseteq X_0$ *a dense linear subspace. Then* $D$ *is a core for* $A$.

**Corollary 2.1.5.** *Let* $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ *be a densely defined, bounded linear operator. Then there exists a unique* $B \in L(X_0, X_1)$ *with* $B \supseteq A$*. In particular, we have* $B = \overline{A}$ *and*

$$\|B\| = \sup_{x\in \mathrm{dom}(A), x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

The proofs of Proposition 2.1.4 and Corollary 2.1.5 are asked for in Exercise 2.2.

## 2.2 Operators in Hilbert Spaces

Let us now focus on operators on Hilbert spaces. In this setting, we can additionally make use of scalar products $\langle\cdot,\cdot\rangle$, which in this course are considered to be linear in the second argument (and anti-linear in the first, in the case when $\mathbb{K}=\mathbb{C}$).
For a linear operator $A\colon \operatorname{dom}(A)\subseteq H_0\to H_1$ the graph norm of $A$ is induced by the scalar product

$$(x,y)\mapsto \langle x,y\rangle+\langle Ax,Ay\rangle\,,$$

known as the *graph scalar product of* $A$. If $A$ is closed then $\operatorname{dom}(A)$ (equipped with the graph norm) is a Hilbert space.
Of course, no presentation of operators in Hilbert spaces would be complete without the central notion of the adjoint operator. We wish to pose the adjoint within the relational framework just established. The definition is as follows.

**Definition.** For a relation $A\subseteq H_0\times H_1$ we define the *adjoint relation* $A^*$ by

$$A^*:=-\left(\left(A^{-1}\right)^{\perp}\right)\subseteq H_1\times H_0,$$

where the orthogonal complement is computed in the direct sum of the Hilbert spaces $H_1$ and $H_0$; that is, the set $H_1\times H_0$ endowed with the scalar product $\big((x,y),(u,v)\big)\mapsto \langle x,u\rangle_{H_1}+\langle y,v\rangle_{H_0}$.

*Remark* 2.2.1. Let $A\subseteq H_0\times H_1$. Then we have

$$A^*=\left\{(u,v)\in H_1\times H_0\,;\,\forall (x,y)\in A:\langle u,y\rangle_{H_1}=\langle v,x\rangle_{H_0}\right\}.$$

In particular, if $A$ is a linear operator, we have

$$A^*=\left\{(u,v)\in H_1\times H_0\,;\,\forall x\in \operatorname{dom}(A):\langle u,Ax\rangle_{H_1}=\langle v,x\rangle_{H_0}\right\}.$$

**Lemma 2.2.2.** *Let $A\subseteq H_0\times H_1$ be a relation. Then $A^*$ is a linear relation. Moreover, we have*

$$A^*=-\left(\left(A^{\perp}\right)^{-1}\right)=\left((-A)^{-1}\right)^{\perp}=\left(-\left(A^{-1}\right)\right)^{\perp}=\left((-A)^{\perp}\right)^{-1}=\left(-\left(A^{\perp}\right)\right)^{-1}.$$

The proof of this lemma is left as Exercise 2.3.

*Remark* 2.2.3. Let $A\subseteq H_0\times H_1$. Since $A^*$ is the orthogonal complement of $-A^{-1}$, it follows immediately that $A^*$ is closed. Moreover, $A^*=\left(\overline{A}\right)^*$ since $A^{\perp}=\left(\overline{A}\right)^{\perp}$.

**Lemma 2.2.4.** *Let $A\subseteq H_0\times H_1$ be a linear relation. Then*

$$A^{**}:=(A^*)^*=\overline{A}.$$

*Proof.* We compute using Lemma 2.2.2

$$A^{**}=\left(\left(-\left(A^*\right)\right)^{-1}\right)^{\perp}=\left(\left(-\left(-\left(\left(A^{\perp}\right)^{-1}\right)\right)\right)^{-1}\right)^{\perp}=\left(A^{\perp}\right)^{\perp}=\overline{A}.\qquad\square$$

**Theorem 2.2.5.** *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then*

$$\operatorname{ran}(A)^{\perp} = \ker(A^*) \quad \textit{and} \quad \overline{\operatorname{ran}}(A^*) = \ker(\overline{A})^{\perp}.$$

*Proof.* Let $u \in \ker(A^*)$ and let $y \in \operatorname{ran}(A)$. Then we find $x \in \operatorname{dom}(A)$ such that $(x, y) \in A$. Moreover, note that $(u, 0) \in A^*$. Then, we compute

$$\langle u, y \rangle_{H_1} = \langle 0, x \rangle_{H_0} = 0.$$

This equality shows that $\operatorname{ran}(A)^{\perp} \supseteq \ker(A^*)$. If on the other hand, $u \in \operatorname{ran}(A)^{\perp}$ then for all $(x, y) \in A$ we have that

$$0 = \langle u, y \rangle_{H_1},$$

which implies $(u, 0) \in A^*$ and hence $u \in \ker(A^*)$. The remaining equation follows from Lemma 2.2.4 together with the first equation applied to $A^*$. $\square$

The following decomposition result is immediate from the latter theorem and will be used frequently throughout the text.

**Corollary 2.2.6.** *Let $A \subseteq H_0 \times H_1$ be a closed linear relation. Then*

$$H_1 = \overline{\operatorname{ran}}(A) \oplus \ker(A^*) \quad \textit{and} \quad H_0 = \ker(A) \oplus \overline{\operatorname{ran}}(A^*).$$

We will now turn to the case where the adjoint relation is actually a linear operator.

**Lemma 2.2.7.** *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then $A^*$ is a linear operator if and only if $A$ is densely defined. If, in addition, $A$ is a linear operator, then $A$ is closable if and only if $A^*$ is densely defined.*

*Proof.* For the first equivalence, it suffices to observe that

$$A^*[\{0\}] = \operatorname{dom}(A)^{\perp}. \tag{2.1}$$

Indeed, $A$ being densely defined is equivalent to having $\operatorname{dom}(A)^{\perp} = \{0\}$. Moreover, $A^*$ is an operator if and only if $A^*[\{0\}] = \{0\}$. Next, we show (2.1). For this, apply Theorem 2.2.5 to the linear relation $A^{-1}$. One obtains $(\operatorname{ran} A^{-1})^{\perp} = \ker(A^{-1})^*$. Hence, $(\operatorname{dom}(A))^{\perp} = \ker(A^*)^{-1} = A^*[\{0\}]$, which is (2.1). For the remaining equivalence, we need to characterise $\overline{A}$ being an operator. Using Lemma 2.2.4 and the first equivalence, we deduce that $\overline{A} = (A^*)^*$ is a linear operator if and only if $A^*$ is densely defined. $\square$

*Remark* 2.2.8. Note that the statement "$A^*$ is an operator if $A$ is densely defined" asserted in Lemma 2.2.7 is also true for *any* relation. For this, it suffices to observe that (2.1) is true for any relation $A \subseteq H_0 \times H_1$. Indeed, let $A \subseteq H_0 \times H_1$ be a relation; define $B \coloneqq \operatorname{lin} A$. Then $\operatorname{dom}(B) = \operatorname{lin}\operatorname{dom}(A)$. Also, we have

$$A^* = -(A^{\perp})^{-1} = -(B^{\perp})^{-1} = B^*.$$

With these preparations, we can write

$$\operatorname{dom}(A)^{\perp} = (\operatorname{lin}\operatorname{dom}(A))^{\perp} = \operatorname{dom}(B)^{\perp} = B^*[\{0\}] = A^*[\{0\}],$$

where we used that (2.1) holds for linear relations.

**Lemma 2.2.9.** *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then $\overline{A} \in L(H_0, H_1)$ if and only if $A^* \in L(H_1, H_0)$. In either case, $\|A^*\| = \|\overline{A}\|$.*

*Proof.* Note that $\overline{A} \in L(H_0, H_1)$ implies that $A$ is closable and densely defined. Thus, by Lemma 2.2.7, $A^*$ is a densely defined, closed linear operator. For $u \in \operatorname{dom}(A^*)$ we compute using Lemma 2.2.4

$$\|A^*u\| = \sup_{x \in H_0 \setminus \{0\}} \frac{|\langle A^*u, x\rangle|}{\|x\|} = \sup_{x \in H_0 \setminus \{0\}} \frac{|\langle u, \overline{A}x\rangle|}{\|x\|} \leqslant \|\overline{A}\| \|u\|,$$

yielding $\|A^*\| \leqslant \|\overline{A}\|$. On the one hand, this implies that $A^*$ is bounded, and on the other, since $A^*$ is densely defined we deduce $A^* \in L(H_1, H_0)$ by Lemma 2.1.3. The other implication (and the other inequality) follows from the first one applied to $A^*$ instead of $A$ using $A^{**} = \overline{A}$. $\square$

We end this section by defining some special classes of relations and operators.

**Definition.** Let $H$ be a Hilbert space and $A \subseteq H \times H$ a linear relation. We call $A$ *(skew-)Hermitian* if $A \subseteq A^*$ ($A \subseteq -A^*$). We say that $A$ is *(skew-)symmetric* if $A$ is (skew-)Hermitian and densely defined (so that $A^*$ is a linear operator), and $A$ is called *(skew-)selfadjoint* if $A = A^*$ ($A = -A^*$). Additionally, if $A$ is densely defined, then we say that $A$ is *normal* if $AA^* = A^*A$.

## 2.3 Computing the Adjoint

In general it is a very difficult task to compute the adjoint of a given (unbounded) operator. There are, however, cases, where the adjoint of a sum or the product can be computed more readily. We start with the most basic case of bounded linear operators.

**Proposition 2.3.1.** *Let $A, B \in L(H_0, H_1), C \in L(H_2, H_0)$. Then $(A + B)^* = A^* + B^*$ and $(AC)^* = C^*A^*$.*

The latter results are special cases of more general statements to follow.

**Theorem 2.3.2.** *Let $A \subseteq H_0 \times H_1$ be a relation and $B \in L(H_0, H_1)$. Then $(A + B)^* = A^* + B^*$.*

*Proof.* Let $(u, v) \in H_1 \times H_0$. Then we compute

$$\begin{aligned}
(u, v) \in A^* + B^* &\iff (u, v - B^*u) \in A^* \\
&\iff \forall\, (x, y) \in A\colon\ \langle y, u\rangle_{H_1} = \langle x, v - B^*u\rangle_{H_0} \\
&\iff \forall\, (x, y) \in A\colon\ \langle y + Bx, u\rangle_{H_1} = \langle x, v\rangle_{H_0} \\
&\iff \forall\, (x, z) \in A + B\colon\ \langle z, u\rangle_{H_1} = \langle x, v\rangle_{H_0} \\
&\iff (u, v) \in (A + B)^*.
\end{aligned}$$

Note that for the first, third and fourth equivalence, we have used the fact that $B \in L(H_0, H_1)$ together with Lemma 2.2.9. $\square$

**Corollary 2.3.3.** *Let $A \subseteq H_0 \times H_1$, $B \in L(H_0, H_1)$. If $A$ is densely defined, then $A^* + B^*$ is an operator and $(A+B)^* = A^* + B^*$.*

**Theorem 2.3.4.** *Let $A \subseteq H_0 \times H_1$ be a closed linear relation and $C \in L(H_2, H_0)$. Then $(AC)^* = \overline{C^* A^*}$.*

*Proof.* We first show that $AC \subseteq (C^*A^*)^*$. For this, let $(w, y) \in AC$. Then $(Cw, y) \in A$. Hence, for all $(u, z) \in C^*A^*$; that is, for all $(u, v) \in A^*$ and $z = C^*v$, we compute

$$\langle u, y\rangle_{H_1} = \langle v, Cw\rangle_{H_0} = \langle C^*v, w\rangle_{H_2} = \langle z, w\rangle_{H_2},$$

which implies that $AC \subseteq (C^*A^*)^*$. Next, let $(w, y) \in (C^*A^*)^*$. Then for all $(u, v) \in A^*$ and $z = C^*v$ we obtain

$$\langle u, y\rangle_{H_1} = \langle z, w\rangle_{H_2} = \langle C^*v, w\rangle_{H_2} = \langle v, Cw\rangle_{H_0}.$$

Thus, we obtain $(Cw, y) \in A^{**} = \overline{A} = A$. Thus, $(w, y) \in AC$. Hence,

$$AC = (C^*A^*)^*,$$

which yields the assertion by adjoining this equation. □

**Corollary 2.3.5.** *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. Then $\left(\overline{A}C\right)^* = \overline{C^*A^*}$.*

*Proof.* The result follows upon realising that $A^* = A^{***} = \left(\overline{A}\right)^*$. □

**Corollary 2.3.6.** *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. If $\overline{A}C$ is densely defined, then $C^*A^*$ is a closable linear operator with $\overline{C^*A^*} = \left(\overline{A}C\right)^*$.*

*Remark* 2.3.7. (a) Note that if $B \in L(H_1, H_2)$ and $A \subseteq H_0 \times H_1$ linear, then $\left(B\overline{A}\right)^* = A^*B^*$. Indeed, this follows from Theorem 2.3.4 applied to $A^*$ and $B$ instead of $A$ and $C^*$, respectively, since then we obtain $(A^*B^*)^* = \overline{B^{**}A^{**}} = \overline{B\overline{A}}$. Computing adjoints on both sides again and using that $A^*B^*$ is closed by Proposition 2.1.1, we get the assertion.

(b) We note here that in Corollary 2.3.5 and Corollary 2.3.6 we $\overline{A}C$ cannot be replaced by $\overline{AC}$ and encourage the reader to find a counterexample for $A$ being a closable linear operator. We also refer to [Pic13] for a counterexample due to J. Epperlein.

We have already seen that $A^* = \overline{A}^*$. We can even restrict $A$ to a core and still obtain the same adjoint.

**Proposition 2.3.8.** *Let $A \subseteq H_0 \times H_1$ be a linear relation, $D \subseteq \operatorname{dom}(A)$ a linear subspace. Then $D$ is a core for $A$ if and only if $(A \cap (D \times H_1))^* = A^*$.*

*Proof.* We set $A|_D := A \cap (D \times H_1)$. Then

$$D \text{ core} \iff \overline{A|_D} = \overline{A} \iff \overline{A|_D}^{\perp} = \overline{A}^{\perp} \iff A|_D{}^{\perp} = A^{\perp} \iff A|_D^* = A^*. \qquad \square$$

## 2.4 The Spectrum and Resolvent Set

In this section, we focus on operators acting on a single Banach space. As such, throughout this section let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $A\colon \operatorname{dom}(A) \subseteq X \to X$ be a closed linear operator.

**Definition.** The set

$$\rho(A) \coloneqq \left\{\lambda \in \mathbb{K}\,;\,(\lambda - A)^{-1} \in L(X)\right\}$$

is called the *resolvent set* of $A$. We define

$$\sigma(A) \coloneqq \mathbb{K} \setminus \rho(A)$$

to be the *spectrum* of $A$.

We state and prove some elementary properties of the spectrum and the resolvent set. We shall see natural examples for $A$ which satisfy that $\sigma(A) = \mathbb{K}$ or $\sigma(A) = \varnothing$ later on. For a metric space $(X, d)$, we will write $B\left(x, r\right) = \{y \in X\,;\, d(x, y) < r\}$ for the open ball around $x$ of radius $r$ and $B\left[x, r\right] = \{y \in X\,;\, d(x, y) \leqslant r\}$ for the closed ball.

**Proposition 2.4.1.** *If $\lambda, \mu \in \rho(A)$, then the* resolvent identity *holds. That is*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)\,(\lambda - A)^{-1}\,(\mu - A)^{-1}\,.$$

*Moreover, the set $\rho(A)$ is open. More precisely, if $\lambda \in \rho(A)$ then $B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right) \subseteq \rho(A)$ and for $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$ we have*

$$\left\|(\mu - A)^{-1}\right\| \leqslant \frac{\left\|(\lambda - A)^{-1}\right\|}{1 - |\lambda - \mu|\left\|(\lambda - A)^{-1}\right\|}.$$

*The mapping $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in L(X)$ is analytic.*

*Proof.* For the first assertion, we let $\lambda, \mu \in \rho(A)$ and compute

$$\begin{aligned}(\lambda - A)^{-1} - (\mu - A)^{-1} &= (\lambda - A)^{-1}\big((\mu - A) - (\lambda - A)\big)(\mu - A)^{-1}\\ &= (\lambda - A)^{-1}(\mu - \lambda)(\mu - A)^{-1}\\ &= (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}.\end{aligned}$$

Next, let $\lambda \in \rho(A)$ and $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$. Then

$$\left\|(\lambda - \mu)(\lambda - A)^{-1}\right\| < 1.$$

Hence, $1 - (\lambda - \mu)(\lambda - A)^{-1}$ admits an inverse in $L(X)$ satisfying

$$\left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right)^{-1} = \sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda - A)^{-1}\right)^{k}. \tag{2.2}$$

We claim that $\mu \in \rho(A)$. For this, we compute

$$\mu - A = \lambda - A - (\lambda - \mu) = (\lambda - A)\left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right).$$

Since $\left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right)$ is an isomorphism in $L(X)$, we deduce that the right-hand side admits a continuous inverse if and only if the left-hand side does. As $\lambda \in \rho(A)$, we thus infer $\mu \in \rho(A)$. The estimate follows from (2.2). Indeed, we have

$$\begin{aligned}\left\|(\mu - A)^{-1}\right\| &\leqslant \left\|(\lambda - A)^{-1}\right\| \left\|\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda - A)^{-1}\right)^k\right\| \\ &\leqslant \left\|(\lambda - A)^{-1}\right\| \sum_{k=0}^{\infty} \left\|(\lambda - \mu)(\lambda - A)^{-1}\right\|^k = \frac{\left\|(\lambda - A)^{-1}\right\|}{1 - \left\|(\lambda - \mu)(\lambda - A)^{-1}\right\|}.\end{aligned}$$

For the final claim of the present proposition, we observe that

$$(\mu - A)^{-1} = \left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right)^{-1} (\lambda - A)^{-1} = \sum_{k=0}^{\infty} (\lambda - \mu)^k \left((\lambda - A)^{-1}\right)^{k+1},$$

which is an operator norm convergent power series expression for the resolvent at $\mu$ about $\lambda$. Thus, analyticity follows. □

We shall consider multiplication operators in $L_2(\mu)$ next. For a measurable function $V\colon \Omega \to \mathbb{R}$ we will use the notation $[V \leqslant c] := V^{-1}\left[(-\infty, c]\right]$ for some constant $c \in \mathbb{R}$ (and similarly for relational symbols other than $\leqslant$).

**Theorem 2.4.2.** *Let $(\Omega, \Sigma, \mu)$ be a measure space and $V : \Omega \to \mathbb{K}$ a measurable function. Then the operator*

$$\begin{aligned}V(\mathrm{m})\colon \operatorname{dom}(V(\mathrm{m})) \subseteq L_2(\mu) &\to L_2(\mu) \\ f &\mapsto \left(\omega \mapsto V(\omega)f(\omega)\right),\end{aligned}$$

*with* $\operatorname{dom}(V(\mathrm{m})) := \left\{f \in L_2(\mu)\,;\, \left(\omega \mapsto V(\omega)f(\omega)\right) \in L_2(\mu)\right\}$ *satisfies the following properties:*

(a) *$V(\mathrm{m})$ is densely defined and closed.*

(b) *$(V(\mathrm{m}))^* = V^*(\mathrm{m})$ where $V^*(\omega) = V(\omega)^*$ for all $\omega \in \Omega$ (here $V(\omega)^*$ denotes the complex conjugate of $V(\omega)$).*

(c) *If $V$ is $\mu$-almost everywhere bounded, then $V(\mathrm{m})$ is continuous. Moreover, we have $\|V(\mathrm{m})\|_{L(L_2(\mu))} \leqslant \|V\|_{L_\infty(\mu)}$.*

(d) *If $V \neq 0$ $\mu$-a.e. then $V(\mathrm{m})$ is injective and $V(\mathrm{m})^{-1} = \frac{1}{V}(\mathrm{m})$, where*

$$\frac{1}{V}(\omega) := \begin{cases} \frac{1}{V(\omega)}, & V(\omega) \neq 0, \\ 0, & V(\omega) = 0, \end{cases}$$

*for all $\omega \in \Omega$.*

*Proof.* For the whole proof we let $\Omega_n := [|V| \leqslant n]$ and put $\mathbb{1}_n := \mathbb{1}_{\Omega_n}$.
(a) We first show that $V(\mathrm{m})$ is densely defined. Let $f \in L_2(\mu)$. Then, we have for all $n \in \mathbb{N}$ that $\mathbb{1}_n f \in \mathrm{dom}(V(\mathrm{m}))$. From $\Omega = \bigcup_n \Omega_n$ and $\Omega_n \subseteq \Omega_{n+1}$ it follows that $\mathbb{1}_n f \to f$ in $L_2(\mu)$ as $n \to \infty$.
Next, we confirm that $V(\mathrm{m})$ is closed. Let $(f_k)_k$ in $\mathrm{dom}(V(\mathrm{m}))$ convergent in $L_2(\mu)$ with $(V(\mathrm{m})f_k)_k$ be convergent in $L_2(\mu)$. Denote the respective limits by $f$ and $g$. It is clear that for all $n \in \mathbb{N}$ we have $\mathbb{1}_n f_k \to \mathbb{1}_n f$ as $k \to \infty$. Also, we have

$$\mathbb{1}_n g = \lim_{k\to\infty} \mathbb{1}_n V(\mathrm{m}) f_k = \lim_{k\to\infty} V(\mathrm{m})(\mathbb{1}_n f_k) = V(\mathrm{m})(\mathbb{1}_n f) = \mathbb{1}_n V f.$$

Hence, $g = Vf$ $\mu$-almost everywhere and since $g \in L_2(\mu)$, we have that $f \in \mathrm{dom}(V(\mathrm{m}))$.
(b) It is easy to see that $V^*(\mathrm{m}) \subseteq V(\mathrm{m})^*$. For the other inclusion, we let $u \in \mathrm{dom}(V(\mathrm{m})^*)$. Then, for all $f \in L_2(\mu)$ and $n \in \mathbb{N}$ we have $\mathbb{1}_n f \in \mathrm{dom}(V(\mathrm{m}))$ and, hence,

$$\langle f, \mathbb{1}_n V^* u\rangle = \int_{\Omega_n} f^* V^* u \,\mathrm{d}\mu = \langle V(\mathrm{m})(\mathbb{1}_n f), u\rangle = \langle \mathbb{1}_n f, V(\mathrm{m})^* u\rangle = \langle f, \mathbb{1}_n V(\mathrm{m})^* u\rangle .$$

It follows that $\mathbb{1}_n V^* u = \mathbb{1}_n V(\mathrm{m})^* u$ for all $n \in \mathbb{N}$. Thus, $\Omega = \bigcup_n \Omega_n$ implies $V^* u = V(\mathrm{m})^* u$ and therefore $u \in \mathrm{dom}(V^*(\mathrm{m}))$ and $V^*(\mathrm{m})u = V(\mathrm{m})^* u$.
(c) If $|V| \leqslant \kappa$ $\mu$-almost everywhere for some $\kappa \geqslant 0$, then for all $f \in L_2(\mu)$ we have $|V(\omega) f(\omega)| \leqslant \kappa |f(\omega)|$ for $\mu$-almost every $\omega \in \Omega$. Squaring and integrating this inequality yields boundedness of $V(\mathrm{m})$ and the asserted inequality.
(d) Assume that $V \neq 0$ $\mu$-a.e. and $V(\mathrm{m})f = 0$. Then, $f(\omega) = 0$ for $\mu$-a.e. $\omega \in \Omega$, which implies $f = 0$ in $L_2(\mu)$. Moreover, if $V(\mathrm{m})f = g$ for $f, g \in L_2(\mu)$, then for $\mu$-a.e. $\omega \in \Omega$ we deduce that $f(\omega) = \frac{1}{V}(\omega) g(\omega)$, which shows $\frac{1}{V}(\mathrm{m}) \supseteq V(\mathrm{m})^{-1}$. If on the other hand $g \in \mathrm{dom}\left(\frac{1}{V}(\mathrm{m})\right)$, then a similar computation reveals that $\frac{1}{V}(\mathrm{m}) g \in \mathrm{dom}(V(\mathrm{m}))$ and $V(\mathrm{m}) \frac{1}{V}(\mathrm{m}) g = g$. □

The spectrum of $V(\mathrm{m})$ from the latter example can be computed once we consider a less general class of measure spaces. For later use, we provide a characterisation of these measure spaces first.

**Proposition 2.4.3.** *Let $(\Omega, \Sigma, \mu)$ be a measure space. Then the following statements are equivalent:*

(i) *$(\Omega, \Sigma, \mu)$ is* semi-finite, *that is, for every $A \in \Sigma$ with $\mu(A) = \infty$, there exists $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that $B \subseteq A$.*

(ii) *For all measurable $V \colon \Omega \to \mathbb{K}$ with $V(\mathrm{m}) \in L(L_2(\mu))$, we have $V \in L_\infty(\mu)$ and $\|V\|_{L_\infty(\mu)} \leqslant \|V(\mathrm{m})\|_{L(L_2(\mu))}$.*

*Proof.* (i)⇒(ii): Let $\varepsilon > 0$ and $A_\varepsilon := [|V| \geqslant \|V(\mathrm{m})\|_{L(L_2(\mu))} + \varepsilon]$. Assume that $\mu(A_\varepsilon) > 0$. Since $(\Omega, \Sigma, \mu)$ is semi-finite we find $B_\varepsilon \subseteq A_\varepsilon$ such that $0 < \mu(B_\varepsilon) < \infty$. Define $f := \mu(B_\varepsilon)^{-1/2} \mathbb{1}_{B_\varepsilon} \in L_2(\mu)$ with $\|f\|_{L_2(\mu)} = 1$. Consequently, we obtain

$$\|V(\mathrm{m})\|_{L(L_2(\mu))} \geqslant \|V(\mathrm{m}) f\|_{L_2(\mu)} \geqslant \|V(\mathrm{m})\|_{L(L_2(\mu))} + \varepsilon,$$

which yields a contradiction, and hence (ii).
(ii)⇒(i): Assume that $(\Omega, \Sigma, \mu)$ is not semi-finite. Then we find $A \in \Sigma$ with $\mu(A) = \infty$ such that for each $B \subseteq A$ measurable, we have $\mu(B) \in \{0, \infty\}$. Then $V := \mathbb{1}_A$ is bounded and measurable with $\|V\|_{L_\infty(\mu)} = 1$. However, $V(\mathrm{m}) = 0$. Indeed, if $f \in L_2(\mu)$ then $[f \neq 0] = \bigcup_{n\in\mathbb{N}}[|f|^2 \geqslant n^{-1}]$. Thus,

$$[V(\mathrm{m})f \neq 0] = [f \neq 0] \cap A = \bigcup_{n\in\mathbb{N}} [|f|^2 \geqslant n^{-1}] \cap A.$$

Since $\mu([|f|^2 \geqslant n^{-1}]) < \infty$ as $f \in L_2(\mu)$, we infer $\mu([|f|^2 \geqslant n^{-1}] \cap A) = 0$ by the property assumed for $A$. Thus, $\mu([V(\mathrm{m})f \neq 0]) = 0$ implying $V(\mathrm{m}) = 0$. Hence, $\|V(\mathrm{m})\|_{L(L_2(\mu))} = 0 < 1 = \|V\|_{L_\infty(\mu)}$. □

A straightforward consequence of Theorem 2.4.2 (c) and Proposition 2.4.3 is the following.

**Proposition 2.4.4.** *Let $(\Omega, \Sigma, \mu)$ be a semi-finite measure space, $V\colon \Omega \to \mathbb{K}$ measurable and bounded. Then $\|V(\mathrm{m})\|_{L(L_2(\mu))} = \|V\|_{L_\infty(\mu)}$.*

**Theorem 2.4.5.** *Let $(\Omega, \Sigma, \mu)$ be a semi-finite measure space and let $V\colon \Omega \to \mathbb{K}$ be measurable. Then*

$$\sigma\left(V(\mathrm{m})\right) = \operatorname{ess-ran} V := \left\{\lambda \in \mathbb{K}\,;\, \forall \varepsilon > 0\colon \mu\left([|\lambda - V| < \varepsilon]\right) > 0\right\}.$$

*Proof.* Let $\lambda \in \operatorname{ess-ran} V$. For all $n \in \mathbb{N}$ we find $B_n \in \Sigma$ with non-zero, but finite measure such that $B_n \subseteq \left[|\lambda - V| < \frac{1}{n}\right]$. We define $f_n := \sqrt{\frac{1}{\mu(B_n)}}\mathbb{1}_{B_n} \in L_2(\mu)$. Then $\|f_n\|_{L_2(\mu)} = 1$ and

$$|V(\omega)f_n(\omega)| \leqslant |V(\omega) - \lambda||f_n(\omega)| + |\lambda||f_n(\omega)| \leqslant \left(\frac{1}{n} + |\lambda|\right)|f_n(\omega)|$$

for $\omega \in \Omega$, which shows that $(f_n)_n$ is in $\operatorname{dom}(V(\mathrm{m}))$. A similar estimate, on the other hand, shows that

$$\|(V(\mathrm{m}) - \lambda)\, f_n\|_{L_2(\mu)} \to 0 \quad (n \to \infty).$$

Thus, $(V(\mathrm{m}) - \lambda)^{-1}$ cannot be continuous as $\|f_n\|_{L_2(\mu)} = 1$ for all $n \in \mathbb{N}$.
Let now $\lambda \in \mathbb{K}\backslash\operatorname{ess-ran} V$. Then there exists $\varepsilon > 0$ such that $N := [|\lambda - V| < \varepsilon]$ is a $\mu$-null set. In particular, $\lambda - V \neq 0$ $\mu$-a.e. Hence, $(\lambda - V(\mathrm{m}))^{-1} = \frac{1}{\lambda - V}(\mathrm{m})$ is a linear operator. Since, $\left|\frac{1}{\lambda - V}\right| \leqslant 1/\varepsilon$ $\mu$-almost everywhere, we deduce that $(\lambda - V(\mathrm{m}))^{-1} \in L(L_2(\mu))$ and hence, $\lambda \in \rho(V(\mathrm{m}))$. □

We conclude this chapter by stating that multiplication operators as discussed in Theorem 2.4.2, Proposition 2.4.3, Proposition 2.4.4, and Theorem 2.4.5 are *the* prototypical example for normal operators. It is also important to note that, as we have seen in Theorem 2.4.2, a multiplication operator in $L_2(\mu)$ is self-adjoint if and only if $V$ assumes values in the real numbers, only.

## 2.5 Comments

The material presented in this lecture is basic textbook knowledge. We shall thus refer to the monographs [Kat95; Wei80]. Note that spectral theory for self-adjoint operators is a classical topic in functional analysis. For a glimpse on further theory of linear relations we exemplarily refer to [Are61; BTW16; Cro98]. The restriction in Proposition 2.4.4 and Theorem 2.4.5 to semi-finite measure spaces is not very severe. In fact, if $(\Omega, \Sigma, \mu)$ was not semi-finite, it is possible to construct a semi-finite measure space $(\Omega_{\mathrm{loc}}, \Sigma_{\mathrm{loc}}, \mu_{\mathrm{loc}})$ such that $L_p(\mu)$ is isometrically isomorphic to $L_p(\mu_{\mathrm{loc}})$, see [VV17, Section 2].

## Exercises

**Exercise 2.1.** Let $A \subseteq X_0 \times X_1$ be an unbounded linear operator. Show that for every linear operator $B \subseteq X_0 \times X_1$ with $B \supseteq A$ and $\mathrm{dom}(B) = X_0$, we have that $B$ is not closed.

**Exercise 2.2.** Prove Proposition 2.1.4 and Corollary 2.1.5. Hint: One might use that bounded linear relations are always operators.

**Exercise 2.3.** Prove Lemma 2.2.2.

**Exercise 2.4.** Let $A\colon \mathrm{dom}(A) \subseteq H_0 \to H_0$ be a closed and densely defined linear operator. Show that for all $\lambda \in \mathbb{K}$ we have

$$\lambda \in \rho(A) \iff \lambda^* \in \rho(A^*).$$

**Exercise 2.5.** Let $U \subseteq H_0 \times H_1$ satisfy $U^{-1} = U^*$. Show that $U \in L(H_0, H_1)$ and that $U$ is *unitary*, that is, $U$ is onto and for all $x \in H_0$ we have $\|Ux\|_{H_1} = \|x\|_{H_0}$.

**Exercise 2.6.** Let $\delta\colon C[0,1] \subseteq L_2(0,1) \to \mathbb{K}, f \mapsto f(0)$, where $C[0,1]$ denotes the set of $\mathbb{K}$-valued continuous functions on $[0,1]$. Show that $\delta$ is not closable. Compute $\overline{\delta}$.

**Exercise 2.7.** Let $C \subseteq \mathbb{C}$ be closed. Provide a Hilbert space $H$ and a densely defined closed linear operator $A$ on $H$ such that $\sigma(A) = C$.

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# 3 The Time Derivative

It is the aim of this chapter to define a derivative operator on a suitable $L_2$-space, which will be used as the derivative with respect to the temporal variable in our applications. As we want to deal with Hilbert space-valued functions, we start by introducing the concept of Bochner–Lebesgue spaces, which generalises the classical scalar-valued $L_p$-spaces to the Banach space-valued case.

## 3.1 Bochner–Lebesgue Spaces

Throughout, let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $X$ a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We are aiming to define the spaces $L_p(\mu; X)$ for $1 \leqslant p \leqslant \infty$. This is the space of (equivalence classes of) measurable functions attaining values in $X$, which are $p$-integrable (if $p < \infty$), or essentially bounded (if $p = \infty$) with respect to the measure $\mu$. We begin by defining the space of simple functions on $\Omega$ with values in $X$ and the notion of Bochner-measurability.

**Definition.** For a function $f\colon \Omega \to X$ and $x \in X$ we set

$$A_{f,x} \coloneqq f^{-1}[\{x\}].$$

A function $f\colon \Omega \to X$ is called *simple* if $f[\Omega]$ is finite and for each $x \in X \setminus \{0\}$ the set $A_{f,x}$ belongs to $\Sigma$ and has finite measure. We denote the set of simple functions by $S(\mu; X)$. A function $f\colon \Omega \to X$ is called *Bochner-measurable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that

$$f_n(\omega) \to f(\omega) \quad (n \to \infty)$$

for $\mu$-a.e. $\omega \in \Omega$.

*Remark* 3.1.1. (a) For a simple function $f$ we have

$$f = \sum_{x\in X} x \cdot \mathbb{1}_{A_{f,x}},$$

where the sum is actually finite, since $\mathbb{1}_{A_{f,x}} = 0$ for all $x \notin f[\Omega]$.

(b) If $X = \mathbb{K}$, then a function is Bochner-measurable if and only if it has a $\mu$-measurable representative.

(c) It is easy to check that $S(\mu; X)$ is a vector space and an $S(\mu; \mathbb{K})$-module; that is, for $f \in S(\mu; X)$ and $g \in S(\mu; \mathbb{K})$ we have $g \cdot f \in S(\mu; X)$.

(d) If $f\colon \Omega \to X$ is Bochner-measurable, then $\|f(\cdot)\|_X\colon \Omega \to \mathbb{R}$ is Bochner-measurable. Indeed, since

$$\|f(\cdot)\|_X = \lim_{n\to\infty} \|f_n(\cdot)\|_X$$

for $\mu$-a.e. $\omega \in \Omega$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$, it suffices to show that $\|f_n(\cdot)\|_X$ is simple for all $n \in \mathbb{N}$. The latter follows since $A_{f,x} \cap A_{f,y} = \varnothing$ for $x \neq y$ and thus

$$\|f_n(\cdot)\|_X = \sum_{x\in f_n[\Omega]} \|x\|_X \cdot \mathbb{1}_{A_{f,x}}$$

is a real-valued simple function.

(e) If one deals with arbitrary measure spaces, the definition of simple functions has to be weakened by allowing the sets $A_{f,x}$ to have infinite measure. However, since in the applications to follow we only work with weighted Lebesgue measures, we restrict ourselves to $\sigma$-finite measure spaces.

Next, we state and prove the celebrated Theorem of Pettis, which characterises Bochner-measurabilty in terms of weak measurability. In what follows, let $X' := L(X, \mathbb{K})$ denote the dual space of $X$.

**Theorem 3.1.2** (Theorem of Pettis)**.** *Let $f : \Omega \to X$. Then $f$ is Bochner-measurable if and only if*

(a) *$f$ is* weakly Bochner-measurable*; that is, $x' \circ f : \Omega \to \mathbb{K}$ is Bochner-measurable for each $x' \in X'$, and*

(b) *$f$ is* almost separably-valued*; that is, $\overline{\operatorname{lin} f[\Omega \setminus N_0]}$ is separable for some $N_0 \in \Sigma$ with $\mu(N_0) = 0$.*

*Proof.* If $f$ is Bochner-measurable, then clearly it is weakly Bochner-measurable. Further, as $f$ is the almost everywhere limit of simple functions, it is almost separably-valued, since each simple function attains values in a finite dimensional subspace of $X$.
Assume now conversely that $f$ satisfies (a) and (b). We define $Y := \overline{\operatorname{lin} f[\Omega \setminus N_0]}$, which is a separable Banach space by (b). Thus, there exists a sequence $(x'_n)_{n\in\mathbb{N}}$ in $X'$ such that

$$\|y\| = \sup_{n\in\mathbb{N}} |x'_n(y)| \quad (y \in Y).$$

Since for each $n \in \mathbb{N}$ the function $g_n := |x'_n \circ f|$ is Bochner-measurable by (a) and Remark 3.1.1(d), we find a $\mu$-nullset $N_n$ and a measurable function $\widetilde{g}_n : \Omega \to \mathbb{R}$ such that $g_n = \widetilde{g}_n$ on $\Omega \setminus N_n$ by Remark 3.1.1(b). Then $\sup_{n\in\mathbb{N}} \widetilde{g}_n(\cdot)$ is measurable and

$$\|f(\omega)\| = \sup_{n\in\mathbb{N}} \widetilde{g}_n(\omega) \quad (\omega \in \Omega \setminus N),$$

where $N := \bigcup_{n\in\mathbb{N}_0} N_n$, which shows that $\|f(\cdot)\|$ is Bochner-measurable. Let $\varepsilon > 0$, $(y_n)_{n\in\mathbb{N}}$ a dense sequence in $Y$. Applying the previous argument to the function $f_k(\cdot) := f(\cdot) - y_k$ for $k \in \mathbb{N}$ we infer that $\|f_k(\cdot)\|$ is Bochner measurable and hence, there is a

$\mu$-nullset $N_k'$ and a measurable funtion $\widetilde{f_k} : \Omega \to \mathbb{R}$ such that $\|f_k\| = \widetilde{f_k}$ on $\Omega \setminus N_k'$. Consequently, the sets

$$E_k := [\widetilde{f_k} \leqslant \varepsilon] := \Big\{\omega \in \Omega\,;\, |\widetilde{f_k}(\omega)| \leqslant \varepsilon\Big\} \quad (k \in \mathbb{N})$$

are measurable. Moreover, by the density of $\{y_n\,;\, n \in \mathbb{N}\}$ in $Y$, we get that $\Omega \setminus N' \subseteq \bigcup_{k\in\mathbb{N}} E_k$ with $N' := \bigcup_{k=1}^{\infty} N_k' \cup N_0$. Setting $F_1 := E_1$ and $F_{n+1} = E_{n+1} \setminus \bigcup_{k=1}^{n} F_k$ for $n \in \mathbb{N}$, we obtain a sequence of pairwise disjoint measurable sets $(F_n)_{n\in\mathbb{N}}$ with $\Omega \setminus N' \subseteq \bigcup_{n\in\mathbb{N}} F_n$. We set

$$g := \sum_{k=1}^{\infty} y_k \mathbb{1}_{F_k}$$

and obtain $\|f(\omega) - g(\omega)\| \leqslant \varepsilon$ for each $\omega \in \Omega \setminus N'$. Hence, if $g$ is Bochner-measurable, then $f$ is Bochner-measurable as well. For showing the Bochner-measurability of $g$, let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable sets such that $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$ and $\mu(\Omega_k) < \infty$ for each $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we set

$$g_n := \sum_{k,j=1}^{n} y_k \mathbb{1}_{F_k \cap \Omega_j}.$$

Then $(g_n)_{n\in\mathbb{N}}$ is a sequence of simple functions with $g_n \to g$ pointwise as $n \to \infty$ and thus, $g$ is Bochner-measurable. □

**Corollary 3.1.3.** *Let $f_n, f : \Omega \to X$ for $n \in \mathbb{N}$. Moreover, assume that $f_n$ is Bochner-measurbale for each $n \in \mathbb{N}$ and $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for $\mu$-almost every $\omega \in \Omega$. Then $f$ is Bochner-measurable*

*Proof.* By Theorem 3.1.2 we find $\mu$-nullsets $N_n \in \Sigma$ such that $X_n := \overline{\operatorname{lin} f_n[\Omega \setminus N_n]}$ is separable. Moreover, we find a $\mu$-nullset $N' \in \Sigma$ such that

$$f_n(\omega) \to f(\omega) \quad (n \to \infty, \omega \in \Omega \setminus N').$$

We set $N := \bigcup_{n\in\mathbb{N}} N_n \cup N'$. Then

$$\overline{\operatorname{lin} f[\Omega \setminus N]} \subseteq \overline{\operatorname{lin} \bigcup_{n\in\mathbb{N}} X_n}$$

and thus, $f$ is almost separably-valued. Moreover, for $x' \in X'$ we have that

$$(x' \circ f)(\omega) = \lim_{n\to\infty} (x' \circ f_n)(\omega) \quad (\omega \in \Omega \setminus N')$$

and since all functions $x' \circ f_n$ are $\mu$-measurable outside a $\mu$-nullset, so is $x' \circ f$ and thus, $x' \circ f$ is Bochner-measurable, see Remark 3.1.1(b). Thus, Theorem 3.1.2 yields the Bochner-measurability of $f$. □

**Definition** (Bochner–Lebesgue spaces)**.** For $p \in [1, \infty]$ we define

$$\mathcal{L}_p(\mu; X) := \{ f \colon \Omega \to X\, ;\, f \text{ Bochner-measurable, } \|f(\cdot)\|_X \in \mathcal{L}_p(\mu) \} ,$$

as well as

$$L_p(\mu; X) := {}^{\mathcal{L}_p(\mu; X)}\!\big/\!_{\sim},$$

where $\sim$ denotes the usual equivalence relation of equality $\mu$-almost everywhere.

**Proposition 3.1.4.** *Let $p \in [1, \infty]$. Then*

$$\|f\|_p := \begin{cases} \left( \int_\Omega \|f(\omega)\|_X^p \, \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname{ess-sup}_{\omega \in \Omega} \|f(\omega)\|_X, & \text{if } p = \infty, \end{cases}$$

*defines a seminorm on $\mathcal{L}_p(\mu; X)$ where $\|f\|_p = 0$ if and only if $f = 0$ $\mu$-a.e. Consequently, $\|\cdot\|_p$ defines a norm on $L_p(\mu; X)$. Moreover, $(L_p(\mu; X), \|\cdot\|_p)$ is a Banach space and if $X = H$ is a Hilbert space, then so too is $L_2(\mu; H)$ with the scalar product given by*

$$\langle f, g \rangle_2 := \int\limits_\Omega \langle f(\omega), g(\omega) \rangle_H \, \mathrm{d}\mu(\omega) \quad (f, g \in L_2(\mu; H)).$$

*Proof.* We just show the completeness of $L_p(\mu; X)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L_p(\mu; X)$ such that $\sum_{n=1}^\infty \|f_n\|_p < \infty$. We set

$$g_n(\omega) := \|f_n(\omega)\|_X \quad (n \in \mathbb{N}, \omega \in \Omega).$$

Then $(g_n)_{n \in \mathbb{N}}$ is a sequence in $L_p(\mu)$ such that $\sum_{n=1}^\infty \|g_n\|_p < \infty$. By the completeness of $L_p(\mu)$ we infer that

$$g := \sum_{n=1}^\infty g_n$$

exists and is an element in $L_p(\mu)$. In particular, $g(\omega) < \infty$ for $\mu$-a.e. $\omega \in \Omega$ and thus,

$$\sum_{n=1}^\infty \|f_n(\omega)\|_X = \sum_{n=1}^\infty g_n(\omega) < \infty$$

for $\mu$-a.e. $\omega \in \Omega$. By the completeness of $X$ we can define

$$f(\omega) := \sum_{n=1}^\infty f_n(\omega)$$

for $\mu$-a.e. $\omega \in \Omega$. Note that $f$ is Bochner-measurable by Corollary 3.1.3. We need to prove that $f \in L_p(\mu; X)$ and that $\sum_{n=1}^k f_n \to f$ in $L_p(\mu; X)$ as $k \to \infty$. For this, it suffices to prove that

$$\sum_{n=k}^\infty f_n \in L_p(\mu; X) \text{ and } \sum_{n=k}^\infty f_n \to 0 \text{ in } L_p(\mu; X) \text{ as } k \to \infty. \tag{3.1}$$

Indeed, this would imply both that $f - \sum_{n=1}^{k} f_n \in L_p(\mu; X)$ and the desired convergence result. We prove (3.1) for $p < \infty$ and $p = \infty$ separately.
First, let $p = \infty$. For each $n \in \mathbb{N}$ we have $f_n \in L_\infty(\mu; X)$ and thus $\|f_n(\omega)\|_X \leqslant \|f_n\|_\infty$ for all $\omega \in \Omega \setminus N_n$ and some null set $N_n \subseteq \Omega$. We set $N := \bigcup_{n=1}^{\infty} N_n$, which is again a null set. For $k \in \mathbb{N}$ and $\omega \in \Omega \setminus N$ we then estimate

$$\left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \leqslant \sum_{n=k}^{\infty} \|f_n(\omega)\|_X \leqslant \sum_{n=k}^{\infty} \|f_n\|_\infty,$$

which yields (3.1).
Now, let $p < \infty$. For $k \in \mathbb{N}$ we estimate

$$\begin{aligned}
\left( \int_\Omega \left( \left\| \sum_{n=k}^{\infty} f_n(\omega) \right\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} &\leqslant \left( \int_\Omega \left( \sum_{n=k}^{\infty} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&= \left( \int_\Omega \lim_{m\to\infty} \left( \sum_{n=k}^{m} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&= \lim_{m\to\infty} \left( \int_\Omega \left( \sum_{n=k}^{m} \|f_n(\omega)\|_X \right)^p \mathrm{d}\mu(\omega) \right)^{\frac{1}{p}} \\
&\leqslant \lim_{m\to\infty} \sum_{n=k}^{m} \|f_n\|_p = \sum_{n=k}^{\infty} \|f_n\|_p,
\end{aligned}$$

where we have used monotone convergence in the third line. This estimate yields (3.1). □

We now want to define an $X$-valued integral for functions in $L_1(\mu; X)$; the so-called Bochner-integral. To do this, we need the following density result.

**Lemma 3.1.5.** *The space $S(\mu; X)$ is dense in $L_p(\mu; X)$ for $p \in [1, \infty)$.*

*Proof.* Let $f \in L_p(\mu; X)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$ for some null set $N \subseteq \Omega$. W.l.o.g. we may assume that $\|f_n(\cdot)\|_X$ and $\|f(\cdot)\|_X$ are measurable on $\Omega \setminus N$ for each $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we define the set

$$I_n := \{\omega \in \Omega \setminus N\,;\, \|f_n(\omega)\|_X \leqslant 2\|f(\omega)\|_X\} \in \Sigma,$$

and set $\widetilde{f_n} := f_n \mathbb{1}_{I_n}$. Then $\widetilde{f_n} \in S(\mu; X)$ and we claim that $\widetilde{f_n}(\omega) \to f(\omega)$ for all $\omega \in \Omega \setminus N$. Indeed, if $f(\omega) = 0$ then $\widetilde{f_n}(\omega) = 0$ also and the claim follows. If $f(\omega) \neq 0$, then there is some $n_0 \in \mathbb{N}$ such that $\|f_n(\omega)\|_X \leqslant 2\|f(\omega)\|_X$ for $n \geq n_0$, and hence $\omega \in \bigcap_{n\geqslant n_0} I_n$. Consequently $\widetilde{f_n}(\omega) = f_n(\omega) \to f(\omega)$. By dominated convergence, it now follows that

$$\int_\Omega \left\| \widetilde{f_n}(\omega) - f(\omega) \right\|_X^p \mathrm{d}\mu(\omega) \to 0 \quad (n \to \infty),$$

which proves the claim. $\square$

**Proposition 3.1.6.** *The mapping*[1]

$$\int_\Omega \mathrm{d}\mu\colon S(\mu;X) \subseteq L_1(\mu;X) \to X$$
$$f \mapsto \sum_{x\in X} x\cdot \mu(A_{f,x})$$

*is linear and continuous, and thus has a unique continuous linear extension to* $L_1(\mu;X)$, *called the* Bochner-integral. *Moreover,*

$$\left\|\int_\Omega f\,\mathrm{d}\mu\right\|_X \leqslant \|f\|_1 \quad (f\in L_1(\mu;X)),$$

*and for* $A\in\Sigma, f\in L_1(\mu;X)$ *we set*

$$\int_A f\,\mathrm{d}\mu := \int_\Omega f\cdot \mathbb{1}_A\,\mathrm{d}\mu.$$

*Proof.* We first show linearity. Let $f,g\in S(\mu;X)$ and $\lambda\in\mathbb{K}$. We then compute

$$\begin{aligned}
\int_\Omega (\lambda f+g)\,\mathrm{d}\mu &= \sum_{x\in X} x\cdot\mu(A_{\lambda f+g,x}) = \sum_{x\in X} x\cdot\mu\left(\bigcup_{y\in X}(A_{f,y}\cap A_{g,x-\lambda y})\right)\\
&= \sum_{x\in X}\sum_{y\in X} x\cdot\mu(A_{f,y}\cap A_{g,x-\lambda y})\\
&= \sum_{y\in X}\sum_{x\in X} \lambda y\cdot\mu(A_{f,y}\cap A_{g,x-\lambda y}) + \sum_{x\in X}\sum_{y\in X}(x-\lambda y)\,\mu(A_{f,y}\cap A_{g,x-\lambda y})\\
&= \lambda\sum_{y\in X} y\cdot\mu\left(A_{f,y}\cap\bigcup_{x\in X}A_{g,x-\lambda y}\right) + \sum_{z\in X} z\cdot\mu\left(\bigcup_{y\in X}A_{f,y}\cap A_{g,z}\right)\\
&= \lambda\int_\Omega f\,\mathrm{d}\mu + \int_\Omega g\,\mathrm{d}\mu,
\end{aligned}$$

where we have used the fact that all occuring unions and sums are finite. In order to prove continuity, let $f\in S(\mu;X)$. We estimate

$$\begin{aligned}
\left\|\int_\Omega f\,\mathrm{d}\mu\right\|_X &= \left\|\sum_{x\in f[\Omega]} x\cdot\mu(A_{f,x})\right\|_X \leqslant \sum_{x\in f[\Omega]}\|x\|_X\mu(A_{f,x}) = \int_\Omega\sum_{x\in f[\Omega]}\|x\|_X\mathbb{1}_{A_{f,x}}\,\mathrm{d}\mu\\
&= \int_\Omega \|f(\cdot)\|_X\,\mathrm{d}\mu = \|f\|_1.
\end{aligned}$$

The remaining assertions now follow from Lemma 3.1.5 by continuous extension (see Corollary 2.1.5). $\square$

[1] Note that the sum is indeed finite and all summands are well-defined if we set $0_X\cdot\infty = 0_X$.

The next proposition tells us how the Bochner-integral of a function behaves if we compose the function with a bounded or closed linear operator first.

**Proposition 3.1.7.** *Let $f \in L_1(\mu; X)$, $Y$ a Banach space.*

(a) *Let $B \in L(X, Y)$. Then $B \circ f \in L_1(\mu; Y)$ and*
$$\int_\Omega B \circ f \,\mathrm{d}\mu = B \int_\Omega f \,\mathrm{d}\mu.$$

(b) *If $X_0 \subseteq X$ is a closed subspace and $f(\omega) \in X_0$ for $\mu$-a.e. $\omega \in \Omega$, then $\int_\Omega f \,\mathrm{d}\mu \in X_0$.*

(c) *(Theorem of Hille) Let $A\colon \operatorname{dom}(A) \subseteq X \to Y$ be a closed linear operator and assume that $f(\omega) \in \operatorname{dom}(A)$ for $\mu$-a.e. $\omega \in \Omega$ and that $A \circ f \in L_1(\mu; Y)$. Then $\int_\Omega f \,\mathrm{d}\mu \in \operatorname{dom}(A)$ and*
$$A \int_\Omega f \,\mathrm{d}\mu = \int_\Omega A \circ f \,\mathrm{d}\mu.$$

*Proof.* (a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $S(\mu; X)$ such that $f_n \to f$ $\mu$-a.e. Then $B \circ f_n \in S(\mu; Y)$ since
$$(B \circ f_n)\,[\Omega] = B\big[f_n[\Omega]\big]$$
is finite and for $y \in (B \circ f_n)[\Omega] \setminus \{0\}$ we have that
$$A_{B\circ f_n, y} = \bigcup_{x \in B^{-1}[\{y\}] \cap f_n[\Omega]} A_{f_n, x} \in \Sigma,$$
and thus, $\mu(A_{B\circ f_n, y}) < \infty$. Due to the continuity of $B$ we have that $B \circ f_n \to B \circ f$ $\mu$-a.e., hence $B \circ f$ is Bochner-measurable. Moreover, $\|B \circ f(\cdot)\|_Y \leqslant \|B\| \|f(\cdot)\|_X$, which yields that $B \circ f \in L_1(\mu; Y)$. By continuity of both $B$ and $\int_\Omega \mathrm{d}\mu$, it suffices to check the interchanging property for any $f \in S(\mu; X)$ alone. However, this is clear, since for a simple function $f$
$$B \circ f = B\left(\sum_{x\in X} x \cdot \mathbb{1}_{A_{f,x}}\right) = \sum_{x \in X} Bx \cdot \mathbb{1}_{A_{f,x}},$$
where the sum is actually finite and hence,
$$\begin{aligned}\int_\Omega B \circ f \,\mathrm{d}\mu &= \int_\Omega \sum_{x\in X} Bx \cdot \mathbb{1}_{A_{f,x}} \,\mathrm{d}\mu = \sum_{x\in X} \int_\Omega Bx \cdot \mathbb{1}_{A_{f,x}} \,\mathrm{d}\mu \\ &= \sum_{x\in X} Bx \cdot \mu(A_{f,x}) = B\left(\sum_{x\in X} x \cdot \mu(A_{f,x})\right) = B \int_\Omega f \,\mathrm{d}\mu,\end{aligned}$$
where in the third equality we have used that $Bx \cdot \mathbb{1}_{A_{f,x}}$ is a simple function.

(b) Let $x' \in X'$ with $x'|_{X_0} = 0$. It follows from (a) that

$$x'\left(\int_\Omega f \,\mathrm{d}\mu\right) = \int_\Omega x' \circ f \,\mathrm{d}\mu = 0,$$

and since $x'$ was arbitrary, it follows that $\int_\Omega f \,\mathrm{d}\mu \in X_0$ from the Theorem of Hahn–Banach.

(c) Consider the space $L_1(\mu; X \times Y)$. By assumption, it follows that

$$(f, A \circ f) \in L_1(\mu; X \times Y).$$

However, $(f, A \circ f)(\omega) = (f(\omega), (A \circ f)(\omega)) \in A \subseteq X \times Y$ for $\mu$-a.e. $\omega \in \Omega$, and since $A$ is closed we can use (b) to derive that

$$\int_\Omega (f, A \circ f) \,\mathrm{d}\mu \in A. \tag{3.2}$$

Let $\pi_1, \pi_2$ be the projection from $X \times Y$ to $X$ and $Y$, respectively. It then follows from part (a) that

$$\pi_1\left(\int_\Omega (f, A \circ f) \,\mathrm{d}\mu\right) = \int_\Omega \pi_1(f, A \circ f) \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu,$$

and analogously for $\pi_2$. Using these equalities we derive from (3.2) that $\int_\Omega f \,\mathrm{d}\mu \in \operatorname{dom}(A)$ and that $A \int_\Omega f \,\mathrm{d}\mu = \int_\Omega A \circ f \,\mathrm{d}\mu$. $\square$

As a consequence of the latter proposition, we derive the fundamental theorem of calculus for Banach space-valued functions.

**Corollary 3.1.8** (fundamental theorem of calculus). *Let $a, b \in \mathbb{R}, a < b$ and consider the measure space $([a,b], \mathcal{B}([a,b]), \lambda)$, where $\mathcal{B}([a,b])$ denotes the Borel-$\sigma$-algebra of $[a,b]$ and $\lambda$ is the Lebesgue measure. Let $f\colon [a,b] \to X$ be continuously differentiable.*[2] *Then*

$$f(b) - f(a) = \int_{[a,b]} f' \,\mathrm{d}\lambda.$$

*Proof.* Note first of all that continuous functions are Bochner-measurable (which can be easily seen using Theorem 3.1.2). Thus, the integral on the right-hand side is well-defined. Let $\varphi \in X'$. Then $\varphi \circ f\colon [a,b] \to \mathbb{K}$ is continuously differentiable, and $(\varphi \circ f)'(t) = (\varphi \circ f')(t)$. Using Proposition 3.1.7 (a) together with the fundamental theorem of calculus for the scalar-valued case we get

$$\varphi\left(\int_{[a,b]} f' \,\mathrm{d}\lambda\right) = \int_{[a,b]} (\varphi \circ f') \,\mathrm{d}\lambda = \varphi(f(b)) - \varphi(f(a)) = \varphi(f(b) - f(a)).$$

Since this holds for all $\varphi \in X'$, the assertion follows from the Theorem of Hahn–Banach. $\square$

[2] By this we mean that $f$ is continuous on $[a,b]$, continuously differentiable on $(a,b)$ and $f'$ has a continuous extension to $[a,b]$.

We conclude this section with a density result, which will be useful throughout the course.

**Lemma 3.1.9.** *Let $\mathcal{D} \subseteq L_2(\mu)$ be total in $L_2(\mu)$ and $H$ a Hilbert space. Then the set $\{\varphi(\cdot)x\,;\, x \in H,\, \varphi \in \mathcal{D}\}$ is total in $L_2(\mu; H)$.*

*Proof.* By Lemma 3.1.5, we know that $S(\mu; X)$ is dense in $L_2(\mu; H)$. Thus, it suffices to approximate $\mathbb{1}_A x$ for some $A \in \Sigma$ and $x \in H$. For this, however, take a sequence $(\phi_n)_n$ in the linear hull of $\mathcal{D}$ with $\phi_n \to \mathbb{1}_A$ in $L_2(\mu)$ as $n \to \infty$. Then

$$\|\mathbb{1}_A x - \phi_n x\|_{L_2(\mu;H)} = \|x\|_H \|\mathbb{1}_A - \phi_n\|_{L_2(\mu)} \to 0 \quad (n \to \infty).$$

Thus, the claim follows. □

**Lemma 3.1.10.** *Let $\mathcal{D} \subseteq L_2(\mu)$ be total in $L_2(\mu)$, $H$ a Hilbert space, $D_0 \subseteq H$ total in $H$. Then $\{\varphi(\cdot)x\,;\, x \in D_0, \varphi \in \mathcal{D}\}$ is total in $L_2(\mu; H)$.*

*Proof.* The proof follows upon realising that the set $\{\varphi(\cdot)x\,;\, x \in D_0,\, \varphi \in \mathcal{D}\}$ is total in the set $\{\varphi(\cdot)x\,;\, x \in H,\, \varphi \in \mathcal{D}\}$. From here we just apply Lemma 3.1.9. □

## 3.2 The Time Derivative as a Normal Operator

Now let $H$ be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $\nu \in \mathbb{R}$ and $p \in [1, \infty)$ we define the measure

$$\mu_{p,\nu}(A) := \int_A \mathrm{e}^{-p\nu t}\,\mathrm{d}\lambda(t)$$

for $A$ in the Borel-$\sigma$-algebra, $\mathcal{B}(\mathbb{R})$, of $\mathbb{R}$. As our underlying Hilbert space for the time derivative we set

$$L_{2,\nu}(\mathbb{R}; H) := L_2(\mu_{2,\nu}; H).$$

In the same way we define

$$L_{p,\nu}(\mathbb{R}; H) := L_p(\mu_{p,\nu}; H)$$

for $p \in [1, \infty)$. If $H = \mathbb{K}$ we abbreviate $L_{p,\nu}(\mathbb{R}) := L_{p,\nu}(\mathbb{R}; \mathbb{K})$.
Our aim is to define the time derivative on $L_{2,\nu}(\mathbb{R}; H)$. For this, we define the integral as an operator, which for $\nu \neq 0$ turns out to be one-to-one and bounded. Then we introduce the time derivative as the inverse of this integral. The reason for doing it that way is to easily get a formula for the adjoint for the time derivative using the boundedness of the integral.
We start our considerations with the definition of convolution operators in $L_{2,\nu}(\mathbb{R}; H)$.

**Lemma 3.2.1.** *Let $k \in L_{1,\nu}(\mathbb{R})$. We define the convolution operator*

$$k*\colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H)$$

*by*

$$(k * f)(t) := \int_{\mathbb{R}} k(s) f(t-s)\,\mathrm{d}s,$$

*which exists for a.e. $t \in \mathbb{R}$. Then, $k*$ is linear and bounded with $\|k*\| \leqslant \|k\|_{L_{1,\nu}(\mathbb{R})}$.*

*Proof.* We first prove that $s \mapsto k(s)f(t-s) \in L_1(\mathbb{R};H)$ for a.e. $t \in \mathbb{R}$. The Bochner-measurability is clear since $k$ and $f$ are both Bochner-measurable. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \|k(s)f(t-s)\|_H \,\mathrm{d}s \right)^2 \mathrm{e}^{-2\nu t} \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} |k(s)|^{\frac{1}{2}} \mathrm{e}^{-\frac{\nu}{2}s} \|f(t-s)\|_H \mathrm{e}^{-\nu(t-s)} \,\mathrm{d}s \right)^2 \mathrm{d}t \\
&\leqslant \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |k(s)| \mathrm{e}^{-\nu s} \,\mathrm{d}s \right) \left( \int_{\mathbb{R}} |k(s)| \mathrm{e}^{-\nu s} \|f(t-s)\|_H^2 \mathrm{e}^{-2\nu(t-s)} \,\mathrm{d}s \right) \mathrm{d}t \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})} \int_{\mathbb{R}} |k(s)| \int_{\mathbb{R}} \|f(t-s)\|^2 \mathrm{e}^{-2\nu(t-s)} \,\mathrm{d}t \, \mathrm{e}^{-\nu s} \,\mathrm{d}s \\
&= \|k\|_{L_{1,\nu}(\mathbb{R})}^2 \|f\|_{L_{2,\nu}(\mathbb{R};H)}^2,
\end{aligned}$$

which on the one hand proves that

$$\int_{\mathbb{R}} \|k(s)f(t-s)\|_H \,\mathrm{d}s < \infty$$

for a.e. $t \in \mathbb{R}$ and on the other hand shows the norm estimate, once we have shown the Bochner-measurability of $k * f$. For proving the latter, we apply Theorem 3.1.2. Since $f$ is Bochner-measurable, we find a nullset $N$ such that $H_0 \coloneqq \overline{\operatorname{lin} f[\mathbb{R} \setminus N]}$ is separable. Hence, for almost every $t \in \mathbb{R}$ we have

$$(k * f)(t) = \int_{\mathbb{R}} k(s)f(t-s) \,\mathrm{d}s = \int_{\mathbb{R}\setminus N} k(t-s)f(s) \,\mathrm{d}s \in H_0$$

by Proposition 3.1.7 (b). Thus, $k * f$ is almost separably-valued. Moreover, for $x' \in H'$ we have by Proposition 3.1.7 (a)

$$x' \circ (k * f) = k * (x' \circ f)$$

almost everywhere and thus, the weak Bochner-measurability follows from the fact that the convolution of two measurable scalar-valued functions is measurable. Since the linearity of $k*$ is clear the proof is done. $\square$

**Definition.** For $\nu \neq 0$ we define the operator

$$I_\nu \colon L_{2,\nu}(\mathbb{R};H) \to L_{2,\nu}(\mathbb{R};H)$$

by

$$I_\nu \coloneqq \begin{cases} \mathbb{1}_{[0,\infty)}*, & \text{if } \nu > 0, \\ -\mathbb{1}_{(-\infty,0]}*, & \text{if } \nu < 0. \end{cases}$$

Note that, by Lemma 3.2.1, $I_\nu$ is bounded with $\|I_\nu\| \leqslant \frac{1}{|\nu|}$.

*Remark* 3.2.2. For $\nu > 0$, $f \in L_{2,\nu}(\mathbb{R}, H)$ we have

$$I_\nu f(t) = \mathbb{1}_{[0,\infty)} * f(t) = \int_0^\infty f(t-s)\,\mathrm{d}s = \int_{-\infty}^t f(s)\,\mathrm{d}s \quad (t \in \mathbb{R} \text{ a.e.}).$$

Analogously, for $\nu < 0$, $f \in L_{2,\nu}(\mathbb{R}, H)$ we have

$$I_\nu f(t) = -\int_t^\infty f(s)\,\mathrm{d}s \quad (t \in \mathbb{R} \text{ a.e.}).$$

**Proposition 3.2.3.** *Let $\nu \neq 0$. Then $I_\nu$ is one-to-one and $C_\mathrm{c}^1(\mathbb{R}; H)$, the space of continuously differentiable, compactly supported functions on $\mathbb{R}$ with values in $H$, is in the range of $I_\nu$.*

*Proof.* We just prove the assertion for the case when $\nu > 0$. Let $f \in L_{2,\nu}(\mathbb{R}; H)$ satisfy $I_\nu f = 0$. In particular, we obtain for all $t \in \mathbb{R} \setminus N$ that $0 = I_\nu f(t) = \int_{-\infty}^t f(s)\,\mathrm{d}s$ for some Lebesgue null set, $N \subseteq \mathbb{R}$. Then for $a, b \in \mathbb{R} \setminus N$ with $a < b$ and $x \in H$ we have that

$$\begin{aligned}
\left\langle f, \mathrm{e}^{2\nu(\cdot)} \mathbb{1}_{[a,b]} \cdot x \right\rangle_{L_{2,\nu}(\mathbb{R};H)} &= \int_\mathbb{R} \left\langle f(t), \mathrm{e}^{2\nu t} \mathbb{1}_{[a,b]}(t) \cdot x \right\rangle_H \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\
&= \left\langle \int_a^b f(t)\,\mathrm{d}t, x \right\rangle_H \\
&= \langle (I_\nu f)(b) - (I_\nu f)(a), x \rangle_H = 0.
\end{aligned}$$

Thus $f = 0$. Indeed, since $\mathbb{R} \setminus N$ is dense in $\mathbb{R}$, $\left\{ \mathrm{e}^{2\nu(\cdot)} \mathbb{1}_{[a,b]} \,;\, a, b \in \mathbb{R} \setminus N \right\}$ is total in $L_{2,\nu}(\mathbb{R})$. Hence, $\left\{ \mathrm{e}^{2\nu(\cdot)} \mathbb{1}_{[a,b]} \cdot x \,;\, a, b \in \mathbb{R} \setminus N,\, x \in H \right\}$ is total in $L_{2,\nu}(\mathbb{R}; H)$ by Lemma 3.1.9. This proves the injectivity of $I_\nu$. Moreover, if $\varphi \in C_\mathrm{c}^1(\mathbb{R}; H)$ then by Corollary 3.1.8 we have

$$\varphi(t) = \int_{-\infty}^t \varphi'(s)\,\mathrm{d}s = \left(I_\nu \varphi'\right)(t) \quad (t \in \mathbb{R} \text{ a.e.}). \qquad \square$$

**Definition.** For $\nu \neq 0$ we define the *time derivative*, $\partial_{t,\nu}$, on $L_{2,\nu}(\mathbb{R}; H)$ by

$$\partial_{t,\nu} \coloneqq I_\nu^{-1}.$$

Note that by Lemma 3.2.1 and Proposition 3.2.3, $\partial_{t,\nu}$ is a closed linear operator for which $C_\mathrm{c}^1(\mathbb{R}; H) \subseteq \operatorname{dom}(\partial_{t,\nu})$. Since

$$C_\mathrm{c}^1(\mathbb{R}; H) \supseteq \operatorname{lin}\left\{ \varphi \cdot x \,;\, \varphi \in C_\mathrm{c}^1(\mathbb{R}),\, x \in H \right\}$$

we infer that $\partial_{t,\nu}$ is densely defined by Lemma 3.1.9 and Exercise 3.2. Moreover, since $I_\nu \varphi' = \varphi$ for $\varphi \in C_\mathrm{c}^1(\mathbb{R}; H)$ we get that

$$\partial_{t,\nu} \varphi = \varphi';$$

that is, $\partial_{t,\nu}$ extends the classical derivative of continuously differentiable functions. We shall discuss the actual domain of $\partial_{t,\nu}$ in the next chapter.

**Proposition 3.2.4.** *Let $\nu \neq 0$. Then $\mathcal{D}_H \coloneqq \operatorname{lin}\{\varphi \cdot x\,;\, \varphi \in C_c^\infty(\mathbb{R}),\, x \in H\}$ is a core for $\partial_{t,\nu}$. Here, $C_c^\infty(\mathbb{R})$ denotes the space of smooth functions on $\mathbb{R}$ with compact support.*

*Proof.* We first prove that

$$\{\varphi'\,;\, \varphi \in C_c^\infty(\mathbb{R})\} \tag{3.3}$$

is dense in $L_{2,\nu}(\mathbb{R})$. As $C_c^\infty(\mathbb{R})$ is dense in $L_{2,\nu}(\mathbb{R})$ (see Exercise 3.2), it suffices to approximate functions in $C_c^\infty(\mathbb{R})$. For this, let $f \in C_c^\infty(\mathbb{R})$. We now define

$$\varphi_n(t) \coloneqq \begin{cases} \int_{-\infty}^t f(s) - f(s-n)\,\mathrm{d}s & \text{if } \nu > 0, \\ \int_{-\infty}^t f(s) - f(s+n)\,\mathrm{d}s & \text{if } \nu < 0 \end{cases} \quad (t \in \mathbb{R}, n \in \mathbb{N}).$$

Then $\varphi_n \in C_c^\infty(\mathbb{R})$ for each $n \in \mathbb{N}$ and

$$\varphi_n'(t) = \begin{cases} f(t) - f(t-n) & \text{if } \nu > 0, \\ f(t) - f(t+n) & \text{if } \nu < 0 \end{cases} \quad (t \in \mathbb{R}, n \in \mathbb{N}).$$

Consequently,

$$\begin{aligned} \|\varphi_n' - f\|_{L_{2,\nu}(\mathbb{R})}^2 &= \begin{cases} \int_{\mathbb{R}} |f(t-n)|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t & \text{if } \nu > 0, \\ \int_{\mathbb{R}} |f(t+n)|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t & \text{if } \nu < 0 \end{cases} \\ &= \|f\|_{L_{2,\nu}(\mathbb{R})}^2 \mathrm{e}^{-2|\nu| n} \to 0 \quad (n \to \infty), \end{aligned}$$

which shows the density of (3.3) in $L_{2,\nu}(\mathbb{R})$. By Lemma 3.1.9 we have that

$$\{\varphi' \cdot x\,;\, \varphi \in C_c^\infty(\mathbb{R}),\, x \in H\}$$

is total in $L_{2,\nu}(\mathbb{R}; H)$ and so $\partial_{t,\nu}[\mathcal{D}_H]$ is dense in $L_{2,\nu}(\mathbb{R}; H)$. Now let $f \in \operatorname{dom}(\partial_{t,\nu})$ and $\varepsilon > 0$. By what we have shown above there exists some $\varphi \in \mathcal{D}_H$ such that

$$\|\partial_{t,\nu}\varphi - \partial_{t,\nu} f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \varepsilon.$$

Since $\partial_{t,\nu}^{-1} = I_\nu$ is bounded with $\|\partial_{t,\nu}^{-1}\| \leqslant \frac{1}{|\nu|}$, the latter implies that

$$\|\varphi - f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \frac{\varepsilon}{|\nu|},$$

and hence, $\mathcal{D}_H$ is indeed a core for $\partial_{t,\nu}$. □

**Corollary 3.2.5.** *For $\nu \in \mathbb{R}$ the mapping*

$$\begin{aligned} \exp(-\nu \mathrm{m}) : L_{2,\nu}(\mathbb{R}; H) &\to L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto \mathrm{e}^{-\nu t} f(t)) \end{aligned}$$

*is unitary, and for $\nu, \mu \neq 0$ one has*

$$\exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu) \exp(-\nu \mathrm{m})^{-1} = \exp(-\mu \mathrm{m})(\partial_{t,\mu} - \mu) \exp(-\mu \mathrm{m})^{-1}.$$

*Proof.* The proof is left as Exercise 3.5. □

By Corollary 3.2.5 we can now define $\partial_{t,0}$. Let $\nu \neq 0$. Then

$$\partial_{t,0} \coloneqq \exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu) \exp(-\nu \mathrm{m})^{-1}.$$

Note that in view of Corollary 3.2.5, the assertion of Proposition 3.2.4 now also holds for $\nu = 0$.

Finally, we want to compute the adjoint of $\partial_{t,\nu}$.

**Corollary 3.2.6.** *Let $\nu \in \mathbb{R}$. The adjoint of $\partial_{t,\nu}$ is given by*

$$\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

*In particular, $\partial_{t,\nu}$ is a normal operator with $\operatorname{Re} \partial_{t,\nu} \coloneqq \frac{1}{2}\left(\overline{\partial_{t,\nu} + \partial_{t,\nu}^*}\right) = \nu$, and $\partial_{t,0}$ is skew-selfadjoint.*

*Proof.* Let $\nu \neq 0$ first. Integrating by parts, one obtains

$$\begin{aligned}\int_{\mathbb{R}} \langle \partial_{t,\nu}\varphi(t), \psi(t)\rangle \, \mathrm{e}^{-2\nu t} \, \mathrm{d}t &= \int_{\mathbb{R}} \big\langle \varphi'(t), \psi(t) \big\rangle \, \mathrm{e}^{-2\nu t} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \big\langle \varphi(t), -\psi'(t) + 2\nu\psi(t) \big\rangle \, \mathrm{e}^{-2\nu t} \, \mathrm{d}t\end{aligned}$$

for $\varphi, \psi \in C_{\mathrm{c}}^\infty(\mathbb{R}; H)$. Since $C_{\mathrm{c}}^\infty(\mathbb{R}; H)$ is a core for $\partial_{t,\nu}$ by Proposition 3.2.4, the latter shows

$$\partial_{t,\nu} \subseteq -\partial_{t,\nu}^* + 2\nu.$$

Since we know that $\partial_{t,\nu}$ is onto, it suffices to prove that $-\partial_{t,\nu}^* + 2\nu$ is one-to-one, since this would imply equality in the latter operator inclusion. For doing so, we apply Theorem 2.2.5 to compute

$$\ker(-\partial_{t,\nu}^* + 2\nu) = \operatorname{ran}(-\partial_{t,\nu} + 2\nu)^\perp.$$

Moreover, we have that $-\partial_{t,\nu} + 2\nu$ is unitarily equivalent to $-\partial_{t,-\nu}$ by Corollary 3.2.5 and since $\partial_{t,-\nu}$ is onto, so is $-\partial_{t,\nu} + 2\nu$ and thus $\ker(-\partial_{t,\nu}^* + 2\nu) = L_{2,\nu}(\mathbb{R}; H)^\perp = \{0\}$, which yields the assertion.

The case $\nu = 0$ follows directly from the definition of $\partial_{t,0}$. □

## 3.3 Comments

Standard references for Bochner integration and related results are [Are+11; DU77]. Considering the derivative operator in an exponentially weighted space goes back (at least) to Morgenstern [Mor52], where ordinary differential equations were considered in a classical setting. In fact, we shall return to this observation in the next chapter when we devote our study to some implications of the already developed concepts on ordinary and delay differential equations.

A first occurence of the derivative operator in exponentially weighted $L^2$-spaces can be found in [Pic89], where a corresponding spectral theorem has been focussed on. We will prove in a later chapter that the spectral representation of the time-derivative as a multiplication operator can be realised by a shifted variant of the Fourier transformation – the so-called Fourier–Laplace transformation.
In an applied context, the time derivative operator discussed here has been introduced in [Pic09].

## Exercises

**Exercise 3.1.** A sequence $(\varphi_n)_n$ in $C_c^\infty(\mathbb{R}^d)$ is called a *$\delta$-sequence,* if

(a) $\varphi_n \geqslant 0$ for $n \in \mathbb{N}$,

(b) $\operatorname{spt}\varphi_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]^d$ for $n \in \mathbb{N}$,

(c) $\int_{\mathbb{R}^d} \varphi_n = 1$ for $n \in \mathbb{N}$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{spt}\varphi \subseteq [-1,1]^d$, $\varphi \geqslant 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Prove that $(\varphi_n)_n$ given by $\varphi_n(x) := n^d\varphi(nx)$ for $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ defines a $\delta$-sequence. Moreover, give an example for such a function $\varphi$.

**Exercise 3.2.** It is well-known that $\{\mathbb{1}_I\,;\, I \text{ } d\text{-dimensional bounded interval}\}$ is total in $L_2(\mathbb{R}^d)$.

(a) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $f \in L_2(\mathbb{R}^d)$. Define as usual $f * \varphi := \Big(x \mapsto \int_{\mathbb{R}^d} f(x-y)\varphi(y)\,\mathrm{d}y\Big)$. Prove that $f * \varphi \in C^\infty(\mathbb{R}^d)$ with $\partial^\alpha(f*\varphi) = f * \partial^\alpha\varphi$ for all $\alpha \in \mathbb{N}_0^d$, where $\partial^\alpha\varphi = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}\varphi$. Moreover, prove that $\operatorname{spt} f*\varphi \subseteq \operatorname{spt} f + \operatorname{spt}\varphi$.

(b) Let $(\varphi_n)_n$ be a $\delta$-sequence and $f \in L_2(\mathbb{R}^d)$. Show that $f * \varphi_n \to f$ in $L_2(\mathbb{R}^d)$ as $n \to \infty$.
Hint: Prove that $\mathbb{1}_I * \varphi_n \to \mathbb{1}_I$ in $L_2(\mathbb{R}^d)$ for all $d$-dimensional intervals and use that $\|f*\varphi_n\|_2 \leqslant \|f\|_2$ (see also Lemma 3.2.1).

(c) Prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$.

**Exercise 3.3.** Let $a < b$, $X_0, X_1, X_2$ be Banach spaces, $f\colon (a,b) \to X_0$ and $g\colon (a,b) \to X_1$ both continuously differentiable, $\ell\colon X_0 \times X_1 \to X_2$ bilinear and continuous. Prove that $h\colon (a,b) \to X_2$ given by

$$h(t) := \ell(f(t), g(t)) \quad (t \in (a,b))$$

is continuously differentiable with

$$h'(t) = \ell(f'(t), g(t)) + \ell(f(t), g'(t)) \quad (t \in (a,b)).$$

If $f, f', g, g'$ have continuous extensions to $[a,b]$, prove the integration by parts formula:

$$\int_a^b \ell(f'(t), g(t))\,\mathrm{d}t = \ell(f(b), g(b)) - \ell(f(a), g(a)) - \int_a^b \ell(f(t), g'(t))\,\mathrm{d}t.$$

**Exercise 3.4.** For $\nu \neq 0$, show that $\|I_\nu\| = \frac{1}{|\nu|}$.

**Exercise 3.5.** Prove Corollary 3.2.5.

**Exercise 3.6.** Let $\nu \in \mathbb{R}$ and $H$ be a complex Hilbert space. Prove that $\sigma(\partial_{t,\nu}) \subseteq \{\mathrm{i}t + \nu\,;\, t \in \mathbb{R}\}$, where $\partial_{t,0}$ is defined in Corollary 3.2.6.
Hint: For $f \in \operatorname{dom}(\partial_{t,\nu}), z \in \mathbb{C}$ compute $\operatorname{Re}\langle (z - \partial_{t,\nu})f, f\rangle_{L_{2,\nu}(\mathbb{R};H)}$ by using Corollary 3.2.6. For proving the surjectivity of $z - \partial_{t,\nu}$ for a suitable $z$, use the formula

$$\overline{\operatorname{ran}}(z - \partial_{t,\nu}) = \ker(z^* - \partial_{t,\nu}^*)^\perp.$$

Remark: Later we will see that, actually, $\sigma(\partial_{t,\nu}) = \{\mathrm{i}t + \nu\,;\, t \in \mathbb{R}\}$.

**Exercise 3.7.** Consider the differential equation

$$\left(\partial_{t,\nu}^2 - 1\right) u = \mathbb{1}_{[-1,1]}.$$

Since $\partial_{t,\nu}^2 - 1 = (\partial_{t,\nu} - 1)(\partial_{t,\nu} + 1)$, it follows by Exercise 3.6 that there is a unique $u \in L_{2,\nu}(\mathbb{R})$ solving this equation if $\nu \notin \{-1, 1\}$. Compute these solutions.
Hint: For $u \in \operatorname{dom}(\partial_{t,\nu})$ use the fact that $u$ is necessarily continuous (which we shall establish in the next lecture).

## References


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[DU77] J. Diestel and J. J. Uhl Jr. *Vector measures.* With a foreword by B. J. Pettis, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977, pp. xiii+322.

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[Pic09] R. Picard. 'A structural observation for linear material laws in classical mathematical physics'. In: *Mathematical Methods in the Applied Sciences* 32 (2009), pp. 1768–1803.

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# 4 Ordinary Differential Equations

In this lecture, we discuss a first application of the time derivative operator constructed in the previous lecture. More precisely, we analyse well-posedness of ordinary differential equations and will at the same time provide a Hilbert space proof of the classical Picard–Lindelöf theorem. We shall furthermore see that the abstract theory developed here also allows for more general differential equations to be considered. In particular, we will have a look at so-called delay differential equations with finite or infinite delay; neutral differential equations are considered in the exercises section.
We start with some information on the time derivative and its domain.

## 4.1 The Domain of $\partial_{t,\nu}$ and the Sobolev Embedding Theorem

Let $H$ be a Hilbert space. Readers familiar with the notion of Sobolev spaces might have already realised that the domain of $\partial_{t,\nu}$ can be described as $L_{2,\nu}(\mathbb{R};H)$-functions with distributional derivative lying in $L_{2,\nu}(\mathbb{R};H)$. In order to stress this, we include the following result. Later on, we have the opportunity to have a more detailed look at Sobolev spaces in more general contexts.

**Proposition 4.1.1.** *Let $\nu \in \mathbb{R}$ and $f, g \in L_{2,\nu}(\mathbb{R};H)$. Then the following conditions are equivalent:*

(i) *$f \in \operatorname{dom}(\partial_{t,\nu})$ and $\partial_{t,\nu} f = g$.*

(ii) *For all $\phi \in C_c^\infty(\mathbb{R})$ we have*
$$-\int_{\mathbb{R}} \phi' f = \int_{\mathbb{R}} \phi g.$$

*Proof.* Assume that $f \in \operatorname{dom}(\partial_{t,\nu})$. By Proposition 3.2.4 and Corollary 3.2.6, we have that $\mathcal{D}_H = \operatorname{lin}\{\varphi \cdot x\,;\, \varphi \in C_c^\infty(\mathbb{R}),\, x \in H\} \subseteq \operatorname{dom}(\partial_{t,\nu}^*)$ (which also holds for $\nu = 0$) and

$$\langle \partial_{t,\nu} f, \psi \cdot x\rangle_{L_{2,\nu}} = \left\langle f, \left(-\psi' + 2\nu\psi\right) \cdot x\right\rangle_{L_{2,\nu}}$$

for all $x \in H$ and $\psi \in C_c^\infty(\mathbb{R})$. Hence, we obtain for all $\psi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \left(-\psi' + 2\nu\psi\right) f \mathrm{e}^{-2\nu\cdot} = \int_{\mathbb{R}} \psi \partial_{t,\nu} f \mathrm{e}^{-2\nu\cdot};$$

putting $\phi := \mathrm{e}^{-2\nu\cdot}\psi$ and using that multiplication by $\mathrm{e}^{-2\nu\cdot}$ is a bijection on $C_c^\infty(\mathbb{R})$, we deduce the claimed formula with $g = \partial_{t,\nu} f$.

On the other hand, the equation involving $g$ applied to $\phi = \mathrm{e}^{-2\nu\cdot}\psi$ for $\psi \in C_c^\infty(\mathbb{R})$ implies that

$$\int_{\mathbb{R}} \left(-\psi' + 2\nu\psi\right) f\mathrm{e}^{-2\nu\cdot} = \int_{\mathbb{R}} \psi g\mathrm{e}^{-2\nu\cdot}.$$

Testing this equation with $x \in H$ yields

$$\langle g, \psi \cdot x\rangle_{L_{2,\nu}} = \left\langle f, \left(-\psi' + 2\nu\psi\right) \cdot x\right\rangle_{L_{2,\nu}} = \left\langle f, \left(-\partial_{t,\nu}\psi \cdot x + 2\nu\psi \cdot x\right)\right\rangle_{L_{2,\nu}}.$$

Since $\mathcal{D}_H$ is dense in $\operatorname{dom}(\partial_{t,\nu})$ by Proposition 3.2.4, we infer that

$$\langle g, h\rangle_{L_{2,\nu}} = \langle f, (-\partial_{t,\nu}h + 2\nu h)\rangle_{L_{2,\nu}}$$

for all $h \in \operatorname{dom}(\partial_{t,\nu})$. Now, Corollary 3.2.6, yields

$$\langle g, h\rangle_{L_{2,\nu}} = \left\langle f, \partial_{t,\nu}^* h\right\rangle_{L_{2,\nu}} \quad (h \in \operatorname{dom}(\partial_{t,\nu}^*)).$$

Thus, $f \in \operatorname{dom}(\partial_{t,\nu}^{**}) = \operatorname{dom}(\partial_{t,\nu})$ and $\partial_{t,\nu} f = g$. □

The next result confirms that functions in the domain of $\partial_{t,\nu}$ are continuous. This result was announced in Exercise 3.7 and is known as the Sobolev embedding theorem. Here, we make use of the explicit form of the domain of $\partial_{t,\nu}$ as being the range space of the integral operator $I_\nu$. We define

$$C_\nu(\mathbb{R}; H) := \left\{ f \colon \mathbb{R} \to H\,;\, f \text{ continuous}, \ \|f\|_{\nu,\infty} := \sup_{t\in\mathbb{R}} \left\|\mathrm{e}^{-\nu t} f(t)\right\|_H < \infty \right\}$$

and regard it as being endowed with the obvious norm.

**Theorem 4.1.2** (Sobolev embedding theorem)**.** *Let $\nu \in \mathbb{R}$. Then every $f \in \operatorname{dom}(\partial_{t,\nu})$ has a continuous representative, and the mapping*

$$\operatorname{dom}(\partial_{t,\nu}) \ni f \mapsto f \in C_\nu(\mathbb{R}; H)$$

*is continuous.*

*Proof.* We restrict ourselves to the case when $\nu > 0$; the remaining cases can be proved by invoking Corollary 3.2.5. Let $f \in \operatorname{dom}(\partial_{t,\nu})$. By definition, we find $g \in L_{2,\nu}(\mathbb{R}; H)$ such that $f = \partial_{t,\nu}^{-1} g = I_\nu g$. Then for all $t \in \mathbb{R}$ we compute

$$\begin{aligned}\int_{-\infty}^{t} \|g(\tau)\| \,\mathrm{d}\tau &= \int_{-\infty}^{t} \|g(\tau)\|\mathrm{e}^{-\nu\tau}\mathrm{e}^{\nu\tau} \,\mathrm{d}\tau \leqslant \sqrt{\int_{-\infty}^{t} \|g(\tau)\|^2 \mathrm{e}^{-2\nu\tau} \,\mathrm{d}\tau} \sqrt{\int_{-\infty}^{t} \mathrm{e}^{2\nu\tau} \,\mathrm{d}\tau} \\ &\leqslant \|\partial_{t,\nu} f\|_{L_{2,\nu}} \sqrt{\frac{1}{2\nu}} \mathrm{e}^{\nu t}.\end{aligned}$$

Thus, $g$ is integrable on $(-\infty, t]$ for all $t \in \mathbb{R}$ and dominated convergence implies that

$$f = \left(t \mapsto \int_{-\infty}^{t} g(s) \,\mathrm{d}s\right)$$

is continuous. Moreover, for $t \in \mathbb{R}$ we obtain

$$\|f(t)\| \leqslant \int_{-\infty}^{t} \|g(\tau)\| \, \mathrm{d}\tau \leqslant \|\partial_{t,\nu} f\|_{L_{2,\nu}} \sqrt{\frac{1}{2\nu}} \mathrm{e}^{\nu t}$$

which yields the claimed continuity. □

**Corollary 4.1.3.** *For all $f \in \operatorname{dom}(\partial_{t,\nu})$, we have that $\left\|\mathrm{e}^{-\nu t} f(t)\right\|_H \to 0$ as $t \to \pm\infty$.*

The proof is left as Exercise 4.2.

## 4.2 The Picard–Lindelöf Theorem

The prototype of the Picard–Lindelöf theorem will be formulated for so-called uniformly Lipschitz continuous functions.
We first need a preparation.

**Definition.** Let $X$ be a Banach space. Then we define

$$S_{\mathrm{c}}(\mathbb{R}; X) := \{f : \mathbb{R} \to X \,;\, f \,\text{simple}, \operatorname{spt} f \,\text{compact}\}$$

to be the set of *simple functions from $\mathbb{R}$ to $X$ with compact support.*

**Lemma 4.2.1.** *Let $X$ be a Banach space and $\nu, \eta \in \mathbb{R}$. Then $S_{\mathrm{c}}(\mathbb{R}; X)$ is dense in $L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}, X)$; that is, for all $f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$ there exists $(f_n)_n$ in $S_{\mathrm{c}}(\mathbb{R}; X)$ such that $f_n \to f$ in both $L_{2,\nu}(\mathbb{R}; X)$ and $L_{2,\eta}(\mathbb{R}; X)$. In particular, $S_{\mathrm{c}}(\mathbb{R}, X)$ is dense in $L_{2,\nu}(\mathbb{R}; X)$.*

*Proof.* Let $f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$. Then for all $n \in \mathbb{N}$ we have that $\mathbb{1}_{[-n,n]} f \in L_{2,\nu}(\mathbb{R}; X) \cap L_{2,\eta}(\mathbb{R}; X)$ and $\mathbb{1}_{[-n,n]} f \to f$ in $L_{2,\nu}(\mathbb{R}; X)$ and in $L_{2,\eta}(\mathbb{R}; X)$ as $n \to \infty$. For $n \in \mathbb{N}$ let $(\widetilde{f}_{n,k})_k$ be in $S(\mu_{2,\nu}; X)$ such that $\widetilde{f}_{n,k} \to \mathbb{1}_{[-n,n]} f$ in $L_{2,\nu}(\mathbb{R}; X)$ as $k \to \infty$. We put $f_{n,k} := \mathbb{1}_{[-n,n]} \widetilde{f}_{n,k} \in S_{\mathrm{c}}(\mathbb{R}; X)$. Then $f_{n,k} \to \mathbb{1}_{[-n,n]} f$ in $L_{2,\nu}(\mathbb{R}; X)$ and in $L_{2,\eta}(\mathbb{R}; X)$ as $k \to \infty$. □

In order to define the notion of uniformly Lipschitz continuous functions, we first need the Lipschitz semi-norm.

**Definition.** Let $X_0, X_1$ be normed spaces, and $F \colon X_0 \to X_1$ Lipschitz continuous. Then

$$\|F\|_{\mathrm{Lip}} := \sup_{\substack{x,y \in X_0 \\ x \neq y}} \frac{\|F(x) - F(y)\|}{\|x - y\|}$$

is the *Lipschitz semi-norm* of $F$.

**Definition.** Let $H_0, H_1$ be Hilbert spaces, $\mu \in \mathbb{R}$. Then a function $F \colon S_{\mathrm{c}}(\mathbb{R}; H_0) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ is called *uniformly Lipschitz continuous*, if for all $\nu \geqslant \mu$ we have that $F$ considered in $L_{2,\nu}(\mathbb{R}; H_0) \times L_{2,\nu}(\mathbb{R}; H_1)$ is Lipschitz continuous, and for the unique Lipschitz continuous extensions $F^\nu$, $\nu \geqslant \mu$, we have that

$$\sup_{\nu \geqslant \mu} \|F^\nu\|_{\mathrm{Lip}} < \infty.$$

*Remark* 4.2.2. Another way to introduce uniformly Lipschitz continuous mappings is the following. Let $H_0, H_1$ be Hilbert spaces, $\mu \in \mathbb{R}$. Let $(F^\nu)_{\nu\geqslant\mu}$ be a family of Lipschitz continuous mappings $F^\nu\colon L_{2,\nu}(\mathbb{R};H_0) \to L_{2,\nu}(\mathbb{R};H_1)$ such that

$$\sup_{\nu\geqslant\mu} \|F^\nu\|_{\mathrm{Lip}} < \infty$$

and the mappings are consistent in the sense that for all $\nu,\eta \geqslant \mu$ and $f \in L_{2,\nu}(\mathbb{R};H_0) \cap L_{2,\eta}(\mathbb{R};H_0)$ we have

$$F^\nu(f) = F^\eta(f).$$

Then, for $\nu \geqslant \mu$ and $f \in S_{\mathrm{c}}(\mathbb{R};H_0)$ we have $F^\nu(f) \in \bigcap_{\eta\geqslant\mu} L_{2,\eta}(\mathbb{R};H_1)$ and $F^\nu|_{S_{\mathrm{c}}(\mathbb{R};H_0)}$ is uniformly Lipschitz continuous.

**Theorem 4.2.3** (Picard–Lindelöf – Hilbert space version). *Let $H$ be a Hilbert space, $\mu \in \mathbb{R}$ and $F\colon S_{\mathrm{c}}(\mathbb{R};H) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R};H)$ uniformly Lipschitz continuous, with $L := \sup_{\nu\geqslant\mu}\|F^\nu\|_{\mathrm{Lip}}$. Then for all $\nu > \max\{L,\mu\}$ the equation*

$$\partial_{t,\nu} u_\nu = F^\nu(u_\nu)$$

*admits a unique solution $u_\nu \in \mathrm{dom}(\partial_{t,\nu})$. Furthermore, for all $\nu > \max\{L,\mu\}$ the following properties hold:*

(a) *If $F^\nu(u_\nu)$ is continuous in a neighborhood of $a \in \mathbb{R}$, then $u_\nu$ is continuously differentiable in a neighborhood of $a$.*

(b) *For all $a \in \mathbb{R}$, $\mathbb{1}_{(-\infty,a]}u_\nu$ is the unique fixed point $v \in L_{2,\nu}(\mathbb{R};H)$ of $\mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1}F^\nu$, that is, $v$ uniquely solves*

$$v = \mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1}F^\nu(v).$$

(c) *For all $\eta \geqslant \nu$ we have that $u_\nu = u_\eta$.*

(d) *For all $f \in L_{2,\nu}(\mathbb{R};H)$ the equation*

$$\partial_{t,\nu} v = F^\nu(v) + f$$

*admits a unique solution $v_{\nu,f} \in \mathrm{dom}(\partial_{t,\nu})$, and if $f,g \in L_{2,\nu}(\mathbb{R};H)$ satisfy $f = g$ on $(-\infty,a]$ for some $a \in \mathbb{R}$, then $v_{\nu,f} = v_{\nu,g}$ on $(-\infty,a]$.*

*Proof of Theorem 4.2.3 – first part.* Define $\Phi\colon L_{2,\nu}(\mathbb{R};H) \to L_{2,\nu}(\mathbb{R};H)$ by

$$\Phi(u) = \partial_{t,\nu}^{-1}F^\nu(u).$$

Since $\left\|\partial_{t,\nu}^{-1}\right\| \leqslant \frac{1}{\nu}$ and $\nu > L$ it follows that $\Phi$ is a contraction and thus admits a unique fixed point, which by definition solves the equation in question. Moreover, we have that $u_\nu = \Phi(u_\nu) = \partial_{t,\nu}^{-1}F^\nu(u_\nu) \in \mathrm{dom}(\partial_{t,\nu})$.
Differentiability of $u_\nu$ as in (a) follows from Exercise 4.1 and the continuity of $F^\nu(u_\nu)$.
For the unique existence asserted in (d), note that the unique existence of $v_{\nu,f}$ follows from the above considerations after realising that $\Phi(v) := \partial_{t,\nu}^{-1}F^\nu(v) + \partial_{t,\nu}^{-1}f$ defines a contraction in $L_{2,\nu}(\mathbb{R};H)$. For the remaining statements in (d) and the statements in (b) and (c), we need some prerequisites. □

**Definition.** Let $H_0, H_1$ be Hilbert spaces, and $F\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ Lipschitz continuous. Then, $F$ is called *causal* if for all $a \in \mathbb{R}$ and all $f, g \in L_{2,\nu}(\mathbb{R}; H_0)$ with $f = g$ on $(-\infty, a]$, we have that $F(f) = F(g)$ on $(-\infty, a]$.

*Remark* 4.2.4. In the following, we will frequently make use of the following easy estimates: Let $\nu \in \mathbb{R}$, $a \in \mathbb{R}$. If $f \in L_{2,\nu}(\mathbb{R}; H)$ with $\operatorname{spt} f \subseteq (-\infty, a]$ then $f \in \bigcap_{\eta \leqslant \nu} L_{2,\eta}(\mathbb{R}; H)$ and

$$\|f\|_{L_{2,\eta}(\mathbb{R};H)} \leqslant \mathrm{e}^{(\nu-\eta)a}\|f\|_{L_{2,\nu}(\mathbb{R};H)} \quad (\eta \leqslant \nu).$$

Likewise, if $\operatorname{spt} f \subseteq [a, \infty)$, we get $f \in \bigcap_{\rho \geqslant \nu} L_{2,\rho}(\mathbb{R}; H)$ with

$$\|f\|_{L_{2,\rho}(\mathbb{R};H)} \leqslant \mathrm{e}^{(\nu-\rho)a}\|f\|_{L_{2,\nu}(\mathbb{R};H)} \quad (\rho \geqslant \nu).$$

**Lemma 4.2.5.** *Let $H_0, H_1$ be Hilbert spaces, $\mu \in \mathbb{R}$, $F\colon S_{\mathrm{c}}(\mathbb{R}; H_0) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ uniformly Lipschitz continuous. Then the following statements hold:*

(a) *$F^\nu$ is causal for all $\nu \geqslant \mu$.*

(b) *The mapping $\partial_{t,\nu}^{-1} F^\nu$ is causal if $\nu \geqslant \max\{\mu, 0\}$ and $\nu \neq 0$.*

(c) *For all $\nu \geqslant \eta \geqslant \mu$, we have that $F^\nu = F^\eta$ on $L_{2,\nu}(\mathbb{R}; H_0) \cap L_{2,\mu}(\mathbb{R}; H_0)$.*

*Proof.* (a) We divide the proof into three steps.
(i) Let $\nu \geqslant \mu$. In order to show causality of $F^\nu$, we first note that it suffices to have $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$ for all $f, g \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ with $f = g$ on $(-\infty, a]$. Indeed, let $f, g \in L_{2,\nu}(\mathbb{R}; H)$ with $f = g$ on $(-\infty, a]$ for some $a \in \mathbb{R}$. By Lemma 4.2.1 we find $(f_n)_n$ and $(\widetilde{g}_n)_n$ in $S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f_n \to f$ and $\widetilde{g}_n \to g$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Next, $\mathbb{1}_{(-\infty,a]} f_n \to \mathbb{1}_{(-\infty,a]} f = \mathbb{1}_{(-\infty,a]} g$ as $n \to \infty$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Thus, putting $g_n \coloneqq \mathbb{1}_{(-\infty,a]} f_n + \mathbb{1}_{(a,\infty)} \widetilde{g}_n$ for all $n \in \mathbb{N}$ we obtain that $g_n \to g$ in $L_{2,\nu}(\mathbb{R}; H_0)$. Since $F^\nu(f_n) = F^\nu(g_n)$ on $(-\infty, a]$ for all $n \in \mathbb{N}$ and $F^\nu\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ is continuous, taking the limit $n \to \infty$ yields $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$.
(ii) Let $a \in \mathbb{R}$, $c \geqslant 0$ and $f \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f = 0$ on $(-\infty, a]$, $g \in \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H_1)$ such that $\|g\|_{L_{2,\nu}(\mathbb{R};H_1)} \leqslant c\|f\|_{L_{2,\nu}(\mathbb{R};H_0)}$ for all $\nu \geqslant \mu$. Then

$$\int_{-\infty}^{a} \|g(t)\|_{H_1}^2 \mathrm{e}^{2\nu(a-t)}\,\mathrm{d}t \leqslant \int_{\mathbb{R}} \|g(t)\|_{H_1}^2 \mathrm{e}^{2\nu(a-t)}\,\mathrm{d}t \leqslant c^2 \int_{a}^{\infty} \|f(t)\|_{H_0}^2 \mathrm{e}^{2\nu(a-t)}\,\mathrm{d}t \to 0$$

as $\nu \to \infty$. Since $\mathrm{e}^{2\nu(a-t)} \to \infty$ as $\nu \to \infty$ for all $t < a$, the monotone convergence theorem implies $g = 0$ on $(-\infty, a]$.
(iii) Let $f, g \in S_{\mathrm{c}}(\mathbb{R}; H_0)$ such that $f = g$ on $(-\infty, a]$ for some $a \in \mathbb{R}$. Then $f - g = 0$ on $(-\infty, a]$. Since $F$ is uniformly Lipschitz continuous, with $L \coloneqq \sup_{\nu \geqslant \mu} \|F^\nu\|_{\mathrm{Lip}}$ we obtain $\|F^\nu(f) - F^\nu(g)\|_{L_{2,\nu}(\mathbb{R};H_1)} \leqslant L\|f - g\|_{L_{2,\nu}(\mathbb{R};H_0)}$ for all $\nu \geqslant \mu$. By (ii) we conclude $F^\nu(f) = F^\nu(g)$ on $(-\infty, a]$ for all $\nu \geqslant \mu$, which by (i) yields the assertion.
The statement in (b) directly follows from (a). Note that $\partial_{t,\nu}^{-1} F^\nu$ is uniformly Lipschitz continuous only for $\nu > 0$.
Let us prove (c). Since $F^\nu(f) = F(f) = F^\eta(f)$ for $f \in S_{\mathrm{c}}(\mathbb{R}; H_0)$, the set $S_{\mathrm{c}}(\mathbb{R}; H_0)$ is dense in $L_{2,\nu}(\mathbb{R}; H_0) \cap L_{2,\mu}(\mathbb{R}; H_0)$ by Lemma 4.2.1, and $F^\nu$ and $F^\eta$ are Lipschitz-continuous, we obtain the assertion. $\square$

*Proof of Theorem 4.2.3 – second part.* The remaining part in (d): Let $f, g \in L_{2,\nu}(\mathbb{R}; H)$ with $f = g$ on $(-\infty, a]$. Since $\nu > L \geqslant 0$, we compute using Lemma 4.2.5(b) and causality of $\partial_{t,\nu}^{-1}$ that

$$\begin{aligned}
\mathbb{1}_{(-\infty,a]} v_{\nu,f} &= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu} \left(v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} f \\
&= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu} \left(\mathbb{1}_{(-\infty,a]} v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} f \\
&= \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu} \left(\mathbb{1}_{(-\infty,a]} v_{\nu,f}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g.
\end{aligned}$$

The same computation also yields that

$$\mathbb{1}_{(-\infty,a]} v_{\nu,g} = \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu} \left(\mathbb{1}_{(-\infty,a]} v_{\nu,g}\right) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g.$$

It is easy to see that $u \mapsto \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}(u) + \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} \mathbb{1}_{(-\infty,a]} g$ defines a contraction in $L_{2,\nu}(\mathbb{R}; H)$. Hence, the contraction mapping principle implies that $\mathbb{1}_{(-\infty,a]} v_{\nu,f} = \mathbb{1}_{(-\infty,a]} v_{\nu,g}$.
The statement in (b) follows from the fact that $u \mapsto \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} F^{\nu}(u)$ defines a contraction and Lemma 4.2.5(b).
For the proof of (c), we observe that for all $n \in \mathbb{N}$, we have $\mathbb{1}_{(-\infty,n]} u_{\eta} \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$. Hence, by (b) and Lemma 4.2.5(c), it follows that

$$\mathbb{1}_{(-\infty,n]} u_{\eta} = \mathbb{1}_{(-\infty,n]} \partial_{t,\eta}^{-1} F^{\eta} \left(\mathbb{1}_{(-\infty,n]} u_{\eta}\right) = \mathbb{1}_{(-\infty,n]} \partial_{t,\nu}^{-1} F^{\nu} \left(\mathbb{1}_{(-\infty,n]} u_{\eta}\right).$$

As $\mathbb{1}_{(-\infty,n]} u_{\nu}$ satisfies the same fixed point equation, we deduce $\mathbb{1}_{(-\infty,n]} u_{\eta} = \mathbb{1}_{(-\infty,n]} u_{\nu}$ for all $n \in \mathbb{N}$, which yields the assertion. □

As a first application of Theorem 4.2.3 we state and prove the classical version of the Theorem of Picard–Lindelöf.

**Theorem 4.2.6** (Picard–Lindelöf – classical version)**.** *Let $H$ be a Hilbert space, $\Omega \subseteq \mathbb{R} \times H$ be open, $f\colon \Omega \to H$ continuous, $(t_0, x_0) \in \Omega$. Assume there exists $L \geqslant 0$ such that for all $(t, x), (t, y) \in \Omega$ we have*

$$\|f(t,x) - f(t,y)\| \leqslant L\|x - y\|.$$

*Then, there exists $\delta > 0$ such that the initial value problem*

$$\begin{cases} u'(t) = f(t, u(t)) \quad (t \in (t_0, t_0 + \delta)), \\ u(t_0) = x_0, \end{cases} \tag{4.1}$$

*admits a unique continuously differentiable solution, $u\colon [t_0, t_0 + \delta] \to H$, which satisfies $(t, u(t)) \in \Omega$ for all $t \in [t_0, t_0 + \delta]$.*

*Proof.* First of all we observe that we may assume, without loss of generality, that $x_0 = 0$. Indeed, to solve the initial value problem

$$\begin{cases} v'(t) = f(t, v(t) + x_0) \quad (t \in (t_0, t_0 + \delta)), \\ v(t_0) = 0, \end{cases}$$

for a continuously differentiable $v\colon [t_0, t_0 + \delta] \to H$ is equivalent to solving the problem in Theorem 4.2.6 for $u$ by setting $u = v + \mathbb{1}_{[t_0,t_0+\delta]}x_0$. Appropriately shifting the time coordinate, we may also assume that $t_0 = 0$.
Thus, let $(0, 0) \in \Omega$. Let also $[0, \delta'] \times B\,[0, \varepsilon] \subseteq \Omega$ for some $\delta', \varepsilon > 0$. Denote by $P\colon H \to H$ the orthogonal projection onto $B\,[0, \varepsilon]$. By Exercise 4.4, $P$ is Lipschitz continuous with Lipschitz semi-norm bounded by 1. We then define

$$\begin{aligned} F\colon S_{\mathrm{c}}(\mathbb{R}; H) &\to \bigcap_{\nu\geqslant 0} L_{2,\nu}(\mathbb{R}; H) \\ g &\mapsto \big(t \mapsto \mathbb{1}_{[0,\delta')}(t)f(t, P(g(t)))\big) \end{aligned}$$

and will prove that $F$ is well-defined and uniformly Lipschitz continuous. Since the mapping $t \mapsto \mathbb{1}_{[0,\delta')}(t)f(t, 0)$ is supported on $[0, \delta']$, we obtain for $\nu \geqslant 0$ that $F(0) \in L_{2,\nu}(\mathbb{R}; H)$. Moreover, for $\nu \geqslant 0$ and $g, h \in S_{\mathrm{c}}(\mathbb{R}; H)$ we estimate

$$\begin{aligned} &\|F(g) - F(h)\|^2_{L_{2,\nu}(\mathbb{R};H)} \\ &= \int_{\mathbb{R}} \|F(g)(t) - F(h)(t)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_0^{\delta'} \|f(t, P(g(t))) - f(t, P(h(t)))\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &\leqslant L^2 \int_0^{\delta'} \|P(g(t)) - P(h(t))\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant L^2 \int_0^{\delta'} \|g(t) - h(t)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \\ &\leqslant L^2 \|g - h\|^2_{L_{2,\nu}(\mathbb{R};H)}, \end{aligned}$$

which shows that $F$ is well-defined and uniformly Lipschitz continuous.
By Theorem 4.2.3, there exists $v \in \operatorname{dom}(\partial_{t,\nu})$ with $\nu > L$ such that

$$\partial_{t,\nu} v = F^\nu(v).$$

We read off from $v = \partial_{t,\nu}^{-1} F^\nu(v)$ that $v = 0$ on $(-\infty, 0]$, and that $v$ is continuous by Theorem 4.1.2. Moreover, we obtain that

$$v(t) = \int_{-\infty}^t \mathbb{1}_{[0,\delta')}(\tau) f(\tau, P(v(\tau)))\,\mathrm{d}\tau = \int_0^{\min\{t,\delta'\}} f(\tau, P(v(\tau)))\,\mathrm{d}\tau,$$

from which we read off that $v$ is continuously differentiable on $(0, \delta')$ since $f$ and $P$ are also continuous. The same equality implies for $0 < t \leqslant \delta := \min\{\frac{\varepsilon}{M}, \delta'\}$, where $M := \sup_{(t,x)\in[0,\delta']\times B[0,\varepsilon]} \|f(t, x)\|$, that

$$\|v(t)\| \leqslant \int_0^t \|f(\tau, P(v(\tau)))\|\,\mathrm{d}\tau \leqslant M\delta \leqslant \varepsilon.$$

Thus, $(t, v(t)) \in [0, \delta'] \times B\,[0, \varepsilon] \subseteq \Omega$ for all $0 \leqslant t \leqslant \delta$ and so $Pv(t) = v(t)$ for $0 \leqslant t \leqslant \delta$. Thus, $u := v|_{[0,\delta]}$ satisfies (4.1).
Finally, concerning uniqueness, let $\widetilde{u}\colon [0, \delta] \to H$ be a continuously differentiable solution of (4.1). Let $\widetilde{v}$ be the extension of $\widetilde{u}$ by 0 to the whole of $\mathbb{R}$. Then we get that

$$\mathbb{1}_{(-\infty,\delta]}\widetilde{v} = \mathbb{1}_{(-\infty,\delta]} \int_0^{\cdot} \mathbb{1}_{[0,\delta')}(\tau) f(\tau, \widetilde{v}(\tau))\,\mathrm{d}\tau$$

$$
\begin{aligned}
&= \mathbb{1}_{(-\infty,\delta]} \int_{-\infty}^{\cdot} \mathbb{1}_{[0,\delta')}(\tau) f(\tau, P(\widetilde{v}(\tau)))\,\mathrm{d}\tau \\
&= \mathbb{1}_{(-\infty,\delta]} \partial_{t,\nu}^{-1} F^{\nu}(\mathbb{1}_{(-\infty,\delta]}\widetilde{v}).
\end{aligned}
$$

Since $\mathbb{1}_{(-\infty,\delta]}v$ is the unique solution of the equation $w = \mathbb{1}_{(-\infty,\delta]}\partial_{t,\nu}^{-1}F^{\nu}(w)$, we obtain that $\mathbb{1}_{(-\infty,\delta]}\widetilde{v} = \mathbb{1}_{(-\infty,\delta]}v$, which yields $u = \widetilde{u}$. □

*Remark* 4.2.7. The reason for the proof of the classical Picard–Lindelöf theorem being seemingly complicated is two-fold. First of all, the Hilbert space solution theory is for $L_2$-functions rather than continuous (or continuously differentiable) functions. The second, maybe more important point is that the Hilbert space Picard–Lindelöf asserts a solution theory, which provides *global* existence in the time variable. The main body of the proof of the classical Picard–Lindelöf theorem presented here is therefore devoted to ‘localisation’ of the abstract theorem. Furthermore, note that the method of proof for obtaining uniqueness and the admittance of the initial value rests on causality. This effect will resurface when we discuss partial differential equations.

## 4.3 Delay Differential Equations

In this section, our study will not be as in depth as done for the local Picard–Lindelöf theorem. Of course, the solution theory afforded would not be a very good one, if it was only applicable to, arguably, the easiest case of ordinary differential equations. We shall see next that the developed theory applies to more elaborate examples.
In what follows, let $H$ be a Hilbert space over $\mathbb{K}$. We start out with a delay differential equation with so-called ‘discrete delay’. For this, we introduce, for $h \in \mathbb{R}$, the *time-shift operator*

$$
\begin{aligned}
\tau_h \colon S_{\mathrm{c}}(\mathbb{R}; H) &\to \bigcap_{\nu\in\mathbb{R}} L_{2,\nu}(\mathbb{R}; H), \\
f &\mapsto f(\cdot + h).
\end{aligned}
$$

**Lemma 4.3.1.** *Let $\mu \in \mathbb{R}$. The mapping $\tau_h \colon S_{\mathrm{c}}(\mathbb{R}; H) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R}; H)$ is uniformly Lipschitz continuous if and only if $h \leqslant 0$. Moreover, for $\nu \in \mathbb{R}$ we have*

$$
\|\tau_h\|_{L_{2,\nu}(\mathbb{R};H)} = \mathrm{e}^{h\nu}.
$$

*Proof.* Let $f \in S_{\mathrm{c}}(\mathbb{R}; H)$. Then for $\nu \in \mathbb{R}$ we compute

$$
\begin{aligned}
\|\tau_h f\|_{L_{2,\nu}(\mathbb{R};H)}^2 &= \int_{\mathbb{R}} \|f(t+h)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}} \|f(t)\|^2 \mathrm{e}^{-2\nu(t-h)}\,\mathrm{d}t \\
&= \|f\|_{L_{2,\nu}(\mathbb{R};H)}^2 \mathrm{e}^{2\nu h}.
\end{aligned}
$$

Since $\sup_{\nu\geqslant\nu} \mathrm{e}^{2\nu h} < \infty$ if and only if $h \leqslant 0$ we obtain the equivalence. Moreover, the above equality also also yields the norm of $\tau_h$ in $L_{2,\nu}(\mathbb{R}; H)$. □

We will reuse $\tau_h$ for the Lipschitz continuous extensions to $L_{2,\nu}(\mathbb{R};H)$. The well-posedness theorem for delay equations with discrete delay is contained in the next theorem. We note here that we only formulate the respective result for right-hand sides that are globally Lipschitz continuous. With a localisation technique, as has already been carried out for the classical Picard–Lindelöf theorem, it is also possible to obtain local results.

**Theorem 4.3.2.** *Let $H$ be a Hilbert space, $\mu \in \mathbb{R}$, $N \in \mathbb{N}$, $h_1, \ldots, h_N \in (-\infty, 0]$, and*

$$G\colon S_{\mathrm{c}}(\mathbb{R};H^N) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R};H)$$

*uniformly Lipschitz. Then there exists an $\eta \in \mathbb{R}$ such that for all $\nu \geqslant \eta$ the equation*

$$\partial_{t,\nu} u = G^\nu\left(\tau_{h_1} u, \ldots, \tau_{h_N} u\right)$$

*admits a solution $u \in \operatorname{dom}(\partial_{t,\nu})$ which is unique in $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R};H)$. Moreover, for all $a \in \mathbb{R}$ the function $u_a \coloneqq \mathbb{1}_{(-\infty,a]} u$ satisfies*

$$u_a = \mathbb{1}_{(-\infty,a]} \partial_{t,\nu}^{-1} G^\nu\left(\tau_{h_1} u_a, \ldots, \tau_{h_N} u_a\right).$$

*Proof.* The assertion follows from Theorem 4.2.3 applied to $F \coloneqq G \circ (\tau_{h_1}, \ldots, \tau_{h_N})$. □

Next, we formulate an initial value problem for a subclass of the latter type of equations.

**Theorem 4.3.3.** *Let $h > 0$, $f\colon \mathbb{R}_{\geq 0} \times H \times H \to H$ continuous, and $f(\cdot, 0, 0) \in L_{2,\mu}(\mathbb{R};H)$ for some $\mu > 0$. Assume that there exists $L \geqslant 0$ with*

$$\|f(t,x,y) - f(t,u,v)\| \leqslant L\|(x,y) - (u,v)\| \quad ((t,x,y),(t,u,v) \in \mathbb{R}_{\geq 0} \times H \times H).$$

*Let $u_0 \in C([-h,0];H)$. Then there exists $\eta \in \mathbb{R}$ such that for all $\nu \geqslant \eta$ the initial value problem*

$$\begin{cases} u'(t) = f(t, u(t), u(t-h)) & (t > 0), \\ u(\tau) = u_0(\tau) & (\tau \in [-h, 0]) \end{cases} \tag{4.2}$$

*admits a unique continuous solution $u\colon [-h,\infty) \to H$, continuously differentiable on $(0,\infty)$.*

*Proof.* For $t < 0$ let $f(t,\cdot,\cdot) \coloneqq 0$. We define $F\colon S_{\mathrm{c}}(\mathbb{R};H) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R};H)$ by

$$\begin{aligned} &F(\phi)(t) \\ &\coloneqq f\big(t, \phi(t) + \mathbb{1}_{[0,\infty)}(t) u_0(0), \phi(t-h) + \mathbb{1}_{[0,\infty)}(t-h) u_0(0) + \mathbb{1}_{[0,h)}(t) u_0(t-h)\big) \end{aligned}$$

for all $t \in \mathbb{R}$. It is easy to see that $F$ is uniformly Lipschitz continuous. Thus, by Theorem 4.2.3, we find $\eta \geqslant \mu$ such that for all $\nu \geqslant \eta$ the equation

$$\partial_{t,\nu} v = F^\nu(v)$$

admits a solution $v \in \bigcap_{\nu \geqslant \eta} \operatorname{dom}(\partial_{t,\nu})$ which is unique in $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R};H)$. Note that $\operatorname{spt} F^\nu(v) \subseteq [0,\infty)$. Hence, $v = 0$ on $(-\infty, 0]$. By Theorem 4.1.2, we obtain that $v(0) = 0$.

We claim that $u \coloneqq v + \mathbb{1}_{[0,\infty)}(\cdot)u_0(0) + \mathbb{1}_{[-h,0)}u_0$ is a solution of (4.2). First of all note that $u$ is continuous on $[-h,\infty)$. Next, for $h > t > 0$ we have that $t-h<0$ and thus $v(t-h)=0$ and so we see that

$$\begin{aligned} F^\nu(v)(t) &= f\big(t, v(t) + \mathbb{1}_{[0,\infty)}(t)u_0(0), v(t-h) + \mathbb{1}_{[0,\infty)}(t-h)u_0(0) + \mathbb{1}_{[0,h)}(t)u_0(t-h)\big) \\ &= f(t, u(t), u_0(t-h)). \end{aligned}$$

Similarily, for $t \geqslant h$ we obtain

$$F^\nu(v)(t) = f(t, u(t), u(t-h))$$

and thus, by continuity of $f$, $u_0$ and $u$, it follows that $v$ is continuously differentiable on $(0,\infty)$ and

$$u'(t) = v'(t) = \partial_{t,\nu} v(t) = f(t, u(t), u(t-h)).$$

It remains to show uniqueness. For this, let $w\colon [-h,\infty) \to H$ be a solution of (4.2). Then

$$w(t) = u_0(0) + \int_0^t f(s, w(s), w(s-h))\,\mathrm{d}s \quad (t \geqslant 0)$$

and $w(t) = u_0(t)$ if $t \in [-h,0]$. We set $\widetilde{v} \coloneqq w - \mathbb{1}_{[0,\infty)}(\cdot)u_0(0) - \mathbb{1}_{[-h,0)}u_0$ and infer

$$\begin{aligned} \widetilde{v}(t) &= \int_0^t f(s, w(s), w(s-h))\,\mathrm{d}s \\ &= \int_{-\infty}^t f\big(s, \widetilde{v}(s) + \mathbb{1}_{[0,\infty)}(s)u_0(0), \\ &\qquad\qquad \widetilde{v}(s-h) + \mathbb{1}_{[0,\infty)}(s-h)u_0(0) + \mathbb{1}_{[0,h)}(s)u_0(s-h)\big)\,\mathrm{d}s \end{aligned}$$

for all $t \in \mathbb{R}$. For $a \in \mathbb{R}$ we set $\widetilde{v}_a \coloneqq \mathbb{1}_{(-\infty,a]}\widetilde{v} \in \bigcap_{\nu\in\mathbb{R}} L_{2,\nu}(\mathbb{R};H)$ and obtain, using the above formula for $\widetilde{v}$,

$$\widetilde{v}_a = \mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1} F^\nu(\widetilde{v}_a).$$

By uniqueness of the solution of

$$\mathbb{1}_{(-\infty,a]}v = \mathbb{1}_{(-\infty,a]}\partial_{t,\nu}^{-1} F^\nu\left(\mathbb{1}_{(-\infty,a]}v\right)$$

it follows that $\widetilde{v}_a = \mathbb{1}_{(-\infty,a]}v$ for all $a \in \mathbb{R}$ and, thus, $u = w$. □

The equation to come involves the whole history of the unknown; that is, the unknown evaluated at $(-\infty,0]$. For a mapping $u\colon \mathbb{R} \to H$ we define the mapping

$$u_{(\cdot)}\colon \mathbb{R} \ni t \mapsto \big(\mathbb{R}_{\leq 0} \ni \theta \mapsto u(t+\theta) \in H\big).$$

**Lemma 4.3.4.** *Let $\mu > 0$. Then*

$$\begin{aligned} \Theta\colon S_{\mathrm{c}}(\mathbb{R};H) &\to \bigcap_{\nu\geqslant\mu} L_{2,\nu}\big(\mathbb{R}; L_2(\mathbb{R}_{\leq 0};H)\big) \\ u &\mapsto u_{(\cdot)} \end{aligned}$$

*is uniformly Lipschitz continuous. More precisely, for all $\nu > 0$ we have*

$$\|\Theta^\nu\| = \frac{1}{\sqrt{2\nu}}.$$

*Proof.* Let $u \in S_{\mathrm{c}}(\mathbb{R}; H)$. Then we compute

$$\begin{aligned}\|\Theta u\|^2_{L_{2,\nu}\left(\mathbb{R};L_2(\mathbb{R}_{\leq 0};H)\right)} &= \int_{\mathbb{R}}\int_{\mathbb{R}_{\leq 0}} \|u(t+\theta)\|^2 \,\mathrm{d}\theta \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}_{\leq 0}} \|u(t)\|^2 \mathrm{e}^{-2\nu(t-\theta)}\,\mathrm{d}\theta\,\mathrm{d}t \\ &= \frac{1}{2\nu}\int_{\mathbb{R}} \|u(t)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t.\end{aligned}$$ □

**Theorem 4.3.5.** *Let $H$ be a Hilbert space, $\mu \in \mathbb{R}$ and let $\Phi\colon S\big(\mathbb{R}; L_2(\mathbb{R}_{\leq 0}; H)\big) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H)$ be uniformly Lipschitz. Then, there exists $\eta > 0$ such that for all $\nu \geqslant \eta$ the equation*

$$\partial_{t,\nu} u = \Phi^{\nu}(u_{(\cdot)})$$

*admits a solution $u \in \bigcap_{\nu \geqslant \eta} \operatorname{dom}(\partial_{t,\nu})$ unique in $\bigcup_{\nu \geqslant \eta} L_{2,\nu}(\mathbb{R}; H)$.*

*Proof.* This is another application of Theorem 4.2.3. □

## 4.4 Comments

In a way, the proof of Theorem 4.2.6 is standard PDE-theory in a nutshell; a solution theory for $L_p$-spaces is used to deduce existence and uniqueness of solutions and a posteriori regularity theory provides more information on the properties of the solution.

Note that – of course – other proofs are available for the Picard–Lindelöf theorem. We chose, however, to present this proof here in order to provide a perspective on classical results. Furthermore, we mention that in order to obtain unique existence for the solution, it suffices to assume that $f$ satisfies a uniform Lipschitz condition with respect to the second variable and that $f$ is measurable. Continuity of $f$ is needed in order to obtain $C^1$-solutions.

A more detailed exposition and more examples of the theory applied to delay differential equations can be found in [Kal+14] and – in a Banach space setting – [PTW14].

There is also a way of dealing with delay differential equations by expanding the state space the problem is formulated in. In this case, it is possible to make use of the rich theory of $C_0$-semigroups. We refer to [BP05] for this.

Causality is one of the main concepts for evolutionary equations. We have provided this notion for mappings defined on $L_{2,\nu}$-type spaces only. The situation becomes different, if one considers merely densely defined mappings. Then it is a priori unclear, whether for a Lipschitz continuous mapping the continuous extension is also causal. For this we refer to Exercise 4.7 below and to [JP00; Wau15], and [Wau16, Chapter 2] as well as to references mentioned there.

## Exercises

**Exercise 4.1.** (a) Let $X$ be a Banach space, $u\colon [a,b] \to X$ continuous. Show that $v\colon (a,b) \to X$ given by

$$v(t) = \int_a^t u(\tau)\,\mathrm{d}\tau$$

is continuously differentiable with $v'(t) = u(t)$.
(b) Let $H$ be a Hilbert space, and $\nu \in \mathbb{R}$. Let $u \in \operatorname{dom}(\partial_{t,\nu})$ with $\partial_{t,\nu}u$ continuous. Show that $u$ is continuously differentiable and $u' = \partial_{t,\nu}u$.

**Exercise 4.2.** Prove Corollary 4.1.3.

**Exercise 4.3.** Let $H$ be a Hilbert space. Show that

$$\operatorname{dom}(\partial_{t,\nu}) \hookrightarrow C_\nu^{1/2}(\mathbb{R}; H) := \left\{ f \in C_\nu(\mathbb{R}; H)\,;\, \mathrm{e}^{-\nu\cdot} f \text{ is } \tfrac{1}{2}\text{-Hölder continuous} \right\},$$

where a function $g\colon \mathbb{R} \to H$ is said to be $\frac{1}{2}$-*Hölder continuous*, if

$$\sup_{\substack{s,t\in\mathbb{R}\\ t\neq s}} \frac{\|g(t) - g(s)\|}{|t-s|^{1/2}} < \infty.$$

**Exercise 4.4.** Let $H$ be a Hilbert space, $C \subseteq H$ closed and convex. Show that the orthogonal projection, $P$, of $H$ onto $C$ defines a Lipschitz continuous mapping with Lipschitz semi-norm bounded by 1.

**Exercise 4.5.** Let $h\colon \mathbb{R} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous satisfying

$$\|h(t,s,x) - h(t,s,y)\| \leqslant L\|x - y\|$$

with $h(\cdot,\cdot,0) = 0$. Let $R > 0$ and $u_0 \in C(\mathbb{R}_{\leq 0}; \mathbb{R}^n)$ have compact support. Show that the initial value problem

$$\begin{cases} u'(t) = \int_{-R}^0 h(t,s,u_{(t)}(s))\,\mathrm{d}s & (t > 0), \\ u(t) = u_0(t) & (t \leqslant 0) \end{cases}$$

admits a unique continuous solution $u\colon \mathbb{R} \to \mathbb{R}^n$, which is continuously differentiable on $\mathbb{R}_{>0}$.
Hint: Modify $\Theta$ from Lemma 4.3.4.

**Exercise 4.6.** Let $H$ be a Hilbert space. Show that for a uniformly Lipschitz continuous $\Phi\colon S\big(\mathbb{R}; L_2(\mathbb{R}_{\leq 0}; H)^2\big) \to \bigcap_{\nu\geqslant\mu} L_{2,\nu}(\mathbb{R}; H)$ the equation

$$\partial_{t,\nu} u = \Phi^\nu\left(u_{(\cdot)}, (\partial_{t,\nu}u)_{(\cdot)}\right)$$

admits a unique solution $u \in \operatorname{dom}(\partial_{t,\nu})$ for $\nu$ large enough.

**Exercise 4.7.** Let $D \subseteq L_2(\mathbb{R})$ be dense and suppose that $F\colon D \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R})$ admits a Lipschitz continuous extension $F^0$.

(a) Show that $F^0$ is causal if and only if for all $\phi \in S_\mathrm{c}(\mathbb{R};\mathbb{R})$ and all $a \in \mathbb{R}$ there exists $L \geqslant 0$ such that

$$\left|\left\langle \mathbb{1}_{(-\infty,a]} \cdot (F(f) - F(g)), \phi\right\rangle_{L_2(\mathbb{R})}\right| \leqslant L \left\|\mathbb{1}_{(-\infty,a]} \cdot (f-g)\right\|_{L_2(\mathbb{R})}$$

for all $f, g \in D$; that is, the mapping

$$\left(D, \left\|\mathbb{1}_{(-\infty,a]} \cdot (\cdot - \cdot)\right\|_{L_2(\mathbb{R})}\right) \ni f \mapsto F(f) \in \left(L_2(\mathbb{R}), \left|\left\langle \mathbb{1}_{(-\infty,a]} \cdot (\cdot - \cdot), \phi\right\rangle_{L_2(\mathbb{R})}\right|\right)$$

is Lipschitz continuous.

(b) For $a \in \mathbb{R}$ let $\operatorname{dom}(F) \cap \operatorname{dom}(F \mathbb{1}_{(-\infty,a]})$ be dense in $L_2(\mathbb{R})$ and if $f, g \in D = \operatorname{dom}(F)$ and $f = g$ on $(-\infty, a]$ then also $F(f) = F(g)$ on $(-\infty, a]$. Show that $F^0$ is causal.

(c) Assume that for all $f, g \in D$ that $f = g$ on $(-\infty, a]$ implies that $F(f) = F(g)$ on $(-\infty, a]$. Show that this is not sufficient for $F^0$ to be causal. Hint: Find a dense subspace $D = \operatorname{dom}(F)$ so that the first condition in (b) is not satisfied.

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# 5 The Fourier–Laplace Transformation and Material Law Operators

In this chapter we introduce the Fourier–Laplace transformation and use it to define operator-valued functions of $\partial_{t,\nu}$; the so-called material law operators. These operators will play a crucial role when we deal with partial differential equations. In the equations of classical mathematical physics, like the heat equation, wave equation or Maxwell's equation, the involved material parameters, such as heat conductivity or permeability of the underlying medium, are incorporated within these operators and hence the name "material law". We start our chapter by defining the Fourier transformation and proving Plancherel's theorem in the Hilbert space-valued case, which states that the Fourier transformation defines a unitary operator on $L_2(\mathbb{R}; H)$.
Throughout, let $H$ be a complex Hilbert space.

## 5.1 The Fourier Transformation

We start by defining the Fourier transformation on $L_1(\mathbb{R}; H)$.

**Definition.** For $f \in L_1(\mathbb{R}; H)$ we define the *Fourier transform* $\widehat{f}$ *of* $f$ by

$$\widehat{f}(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(t) \,\mathrm{d}t \quad (s \in \mathbb{R}).$$

We also introduce

$$C_{\mathrm{b}}(\mathbb{R}; H) := \{f : \mathbb{R} \to H \,;\, f \text{ continuous, bounded}\}$$

endowed with the sup-norm, $\|\cdot\|_\infty$.

**Lemma 5.1.1** (Riemann–Lebesgue). *Let* $f \in L_1(\mathbb{R}; H)$. *Then* $\widehat{f} \in C_{\mathrm{b}}(\mathbb{R}; H)$ *and* $\lim_{|t|\to\infty} \|\widehat{f}(t)\| = 0$. *Moreover,*

$$\|\widehat{f}\|_\infty \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_1.$$

*Proof.* First, note that $\widehat{f}$ is continuous by dominated convergence and bounded with

$$\|\widehat{f}\|_\infty \leqslant \frac{1}{\sqrt{2\pi}} \|f\|_1.$$

This shows that the mapping

$$L_1(\mathbb{R}; H) \to C_{\mathrm{b}}(\mathbb{R}; H), \quad f \mapsto \widehat{f} \tag{5.1}$$

defines a bounded linear operator. Moreover, for $\varphi \in C_{\mathrm{c}}^1(\mathbb{R};H)$ we compute

$$\widehat{\varphi}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st}\varphi(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}s} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st}\varphi'(t)\,\mathrm{d}t$$

for $s \neq 0$ and thus,

$$\limsup_{|s|\to\infty} \|\widehat{\varphi}(s)\| \leqslant \limsup_{|s|\to\infty} \frac{1}{|s|}\frac{1}{\sqrt{2\pi}}\left\|\varphi'\right\|_1 = 0,$$

which shows that $\lim_{|s|\to\infty}\|\widehat{\varphi}(s)\| = 0$. By the facts that $C_{\mathrm{c}}^1(\mathbb{R};H)$ is dense in $L_1(\mathbb{R};H)$ (see Lemma 3.1.9), $\left\{f \in C_{\mathrm{b}}(\mathbb{R};H)\,;\, \lim_{|t|\to\infty}\|f(t)\| = 0\right\}$ is a closed subspace of $C_{\mathrm{b}}(\mathbb{R};H)$ and the operator in (5.1) is bounded, the assertion follows. □

It is our main goal to extend the definition of the Fourier transformation to functions in $L_2(\mathbb{R};H)$. For doing so, we make use of the Schwartz space of rapidly decreasing functions.

**Definition.** We define

$$\mathcal{S}(\mathbb{R};H) := \left\{f \in C^\infty(\mathbb{R};H)\,;\, \forall n,k\in\mathbb{N} : \left(t \mapsto t^k f^{(n)}(t)\right) \in C_{\mathrm{b}}(\mathbb{R};H)\right\}$$

to be the *Schwartz space* of rapidly decreasing functions on $\mathbb{R}$ with values in $H$.

As usual we abbreviate $\mathcal{S}(\mathbb{R}) := \mathcal{S}(\mathbb{R};\mathbb{K})$.

*Remark* 5.1.2. $\mathcal{S}(\mathbb{R};H)$ is a Fréchet space with respect to the seminorms

$$\mathcal{S}(\mathbb{R};H) \ni f \mapsto \sup_{t\in\mathbb{R}} \left\|t^k f^{(n)}(t)\right\| \quad (k,n\in\mathbb{N}).$$

Moreover, $\mathcal{S}(\mathbb{R};H) \subseteq \bigcap_{p\in[1,\infty]} L_p(\mathbb{R};H)$. Indeed, $\mathcal{S}(\mathbb{R};H) \subseteq L_\infty(\mathbb{R};H)$ by definition, and for $1 \leqslant p < \infty$ we have that

$$\begin{aligned}\int_{\mathbb{R}} \|f(t)\|^p\,\mathrm{d}t &= \int_{\mathbb{R}} \frac{1}{(1+|t|)^{2p}}\left\|(1+|t|)^2 f(t)\right\|^p\,\mathrm{d}t\\ &\leqslant \sup_{t\in\mathbb{R}}\left\|(1+|t|)^2 f(t)\right\|^p \int_{\mathbb{R}} \frac{1}{(1+|t|)^{2p}}\,\mathrm{d}t < \infty.\end{aligned}$$

**Proposition 5.1.3.** *For $f \in \mathcal{S}(\mathbb{R};H)$ we have $\widehat{f} \in \mathcal{S}(\mathbb{R};H)$ and the mapping*

$$\mathcal{S}(\mathbb{R};H) \to \mathcal{S}(\mathbb{R};H), \quad f \mapsto \widehat{f}$$

*is bijective. Moreover, for $f, g \in L_1(\mathbb{R};H)$ we have that*

$$\int_{\mathbb{R}} \left\langle \widehat{f}(t), g(t)\right\rangle\,\mathrm{d}t = \int_{\mathbb{R}} \left\langle f(t), \widehat{g}(-t)\right\rangle\,\mathrm{d}t. \tag{5.2}$$

*Additionally, if $f, \widehat{f} \in L_1(\mathbb{R};H)$ then*

$$f(t) = \widehat{\widehat{f}}(-t) \quad (t\in\mathbb{R}). \tag{5.3}$$

*Proof.* Let $f \in \mathcal{S}(\mathbb{R}; H)$. By Exercise 5.1 we have

$$\widehat{f}\,'(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t = -\mathrm{i}\big(\widehat{t \mapsto tf(t)}\big)(s) \quad (s \in \mathbb{R}) \tag{5.4}$$

and

$$s\widehat{f}(s) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{i}s)\,\mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t = -\mathrm{i}\widehat{f'}(s) \quad (s \in \mathbb{R}). \tag{5.5}$$

Using these formulas, one can show that $\widehat{f} \in \mathcal{S}(\mathbb{R}; H)$. Since the bijectivity of the Fourier transformation on $\mathcal{S}(\mathbb{R}; H)$ would follow from (5.3), it suffices to prove the formulas (5.2) and (5.3). Let $f, g \in L_1(\mathbb{R}; H)$. Then we compute using Proposition 3.1.7 and Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}} \Big\langle \widehat{f}(t), g(t) \Big\rangle\,\mathrm{d}t &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \Big\langle \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(s)\,\mathrm{d}s, g(t) \Big\rangle\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}st} \langle f(s), g(t) \rangle\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \Big\langle f(s), \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st} g(t)\,\mathrm{d}t \Big\rangle\,\mathrm{d}s \\
&= \int_{\mathbb{R}} \langle f(s), \widehat{g}(-s) \rangle\,\mathrm{d}s,
\end{aligned}$$

which yields (5.2). For proving formula (5.3), we consider the function $\gamma$ defined by $\gamma(t) := \mathrm{e}^{-\frac{t^2}{2}}$ for $t \in \mathbb{R}$. Clearly, $\gamma \in \mathcal{S}(\mathbb{R})$. We claim that $\widehat{\gamma} = \gamma$. Indeed, we observe that $\gamma$ solves the initial value problem $y' + ty = 0$ subject to $y(0) = 1$; if we can show that $\widehat{\gamma}$ solves the same initial value problem, then their equality would follow from the uniqueness of the solution. First, we observe that $\widehat{\gamma}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t = 1$. Second, we compute using the formulas (5.4) and (5.5) that

$$\widehat{\gamma}'(s) = -\mathrm{i}\big(\widehat{t \mapsto t\gamma(t)}\big)(s) = \mathrm{i}\widehat{\gamma'}(s) = -s\widehat{\gamma}(s) \quad (s \in \mathbb{R}).$$

Altogether, we have shown that $\widehat{\gamma}$ solves the same initial value problem as $\gamma$ and hence, $\widehat{\gamma} = \gamma$. Let now $f \in L_1(\mathbb{R}; H)$ with $\widehat{f} \in L_1(\mathbb{R}; H)$, $a > 0$ and $x \in H$. Then we compute using (5.2)

$$\begin{aligned}
\Big\langle \int_{\mathbb{R}} \widehat{f}(t)\gamma(at)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t, x \Big\rangle &= \int_{\mathbb{R}} \Big\langle \widehat{f}(t), \gamma(at)x\mathrm{e}^{-\mathrm{i}st} \Big\rangle\,\mathrm{d}t = \int_{\mathbb{R}} \Big\langle f(t), \big(\widehat{\gamma(a\cdot)x\mathrm{e}^{-\mathrm{i}s(\cdot)}}\big)(-t) \Big\rangle\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \Big\langle f(t), \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \gamma(ar)x\mathrm{e}^{-\mathrm{i}sr}\mathrm{e}^{\mathrm{i}tr}\,\mathrm{d}r \Big\rangle\,\mathrm{d}t \\
&= \frac{1}{a} \int_{\mathbb{R}} \Big\langle f(t), \widehat{\gamma}\left(\frac{s-t}{a}\right) x \Big\rangle\,\mathrm{d}t = \frac{1}{a} \int_{\mathbb{R}} \Big\langle f(t), \gamma\left(\frac{s-t}{a}\right) x \Big\rangle\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \langle f(s-at), \gamma(t)\,x \rangle\,\mathrm{d}t = \Big\langle \int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t, x \Big\rangle
\end{aligned}$$

for each $s \in \mathbb{R}$. Since this holds for all $x \in H$ we get

$$\int_{\mathbb{R}} \widehat{f}(t)\gamma(at)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = \int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t \quad (s \in \mathbb{R}).$$

Letting $a \to 0$ in the latter equality, we obtain

$$\int_{\mathbb{R}} \widehat{f}(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = \lim_{a\to 0}\int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t \quad (s \in \mathbb{R}), \tag{5.6}$$

where we have used dominated convergence for the term on the left-hand side. In order to compute the limit on the right-hand side, we first observe that

$$\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t \right\| \mathrm{d}s \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \|f(s-at)\|\,\mathrm{d}s\,\gamma(t)\,\mathrm{d}t = \|f\|_1\|\gamma\|_1,$$

and hence, for each $a > 0$ the operator

$$\begin{aligned} S_a \colon L_1(\mathbb{R};H) &\to L_1(\mathbb{R};H),\\ f &\mapsto \left(s \mapsto \int_{\mathbb{R}} f(s-at)\gamma(t)\,\mathrm{d}t\right) \end{aligned}$$

is bounded by $\|\gamma\|_1$. Moreover, since $S_a\psi \to \psi(\cdot)\|\gamma\|_1$ as $a \to 0$ for $\psi \in C_{\mathrm{c}}(\mathbb{R};H)$, we infer that

$$S_a f \to f(\cdot)\|\gamma\|_1 \quad (a \to 0)$$

for each $f \in L_1(\mathbb{R};H)$. Hence, passing to a suitable sequence $(a_n)_n$ in $\mathbb{R}_{>0}$ tending to 0, we get

$$\lim_{n\to\infty} (S_{a_n}f)(s) \to f(s)\|\gamma\|_1 \quad (\text{a.e. } s \in \mathbb{R}).$$

Using this identity for the right-hand side of (5.6), we get

$$\int_{\mathbb{R}} \widehat{f}(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t = f(s)\|\gamma\|_1 \quad (\text{a.e. } s \in \mathbb{R}),$$

and since $\|\gamma\|_1 = \sqrt{2\pi}$, we derive (5.3). □

With these preparations at hand, we are now able to prove the main theorem of this section.

**Theorem 5.1.4** (Plancherel)**.** *The mapping*

$$\mathcal{F}\colon \mathcal{S}(\mathbb{R};H) \subseteq L_2(\mathbb{R};H) \to L_2(\mathbb{R};H),\ f \mapsto \widehat{f}$$

*extends to a unitary operator on $L_2(\mathbb{R};H)$, again denoted by $\mathcal{F}$, the* Fourier transformation. *Moreover, $\mathcal{F}^* = \mathcal{F}^{-1}$ is given by $f \mapsto \widehat{f}(-\cdot)$.*

*Proof.* Using (5.2) and (5.3) we obtain that

$$\left\langle \widehat{f}, \widehat{g} \right\rangle_2 = \int_{\mathbb{R}} \left\langle \widehat{f}(t), \widehat{g}(t) \right\rangle \,\mathrm{d}t = \int_{\mathbb{R}} \left\langle f(t), \widehat{\widehat{g}}(-t) \right\rangle \,\mathrm{d}t = \int_{\mathbb{R}} \langle f(t), g(t) \rangle \,\mathrm{d}t = \langle f, g \rangle_2$$

for all $f, g \in \mathcal{S}(\mathbb{R}; H)$ and thus, in particular,

$$\|f\|_2 = \|\mathcal{F}f\|_2. \tag{5.7}$$

Moreover, $\operatorname{dom}(\mathcal{F}) = \operatorname{ran}(\mathcal{F}) = \mathcal{S}(\mathbb{R}; H)$ is dense in $L_2(\mathbb{R}; H)$ and hence, the first assertion follows by Exercise 5.2. As $\mathcal{F}$ is unitary, we have $\mathcal{F}^* = \mathcal{F}^{-1}$, thus, by (5.2) applied to $f, g \in \mathcal{S}(\mathbb{R}; H)$, we read off (using Proposition 2.3.8) that $\mathcal{F}^{-1} = (f \mapsto \widehat{f}(-\cdot))$, which yields all the claims of the theorem at hand. □

*Remark* 5.1.5. We emphasise that for $f \in L_2(\mathbb{R}; H)$ the Fourier transform $\mathcal{F}f$ is not given by the integral expression for $L_1$-functions, simply because the integral does not need to exist. However, by dominated convergence

$$\mathcal{F}f = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathrm{e}^{-\mathrm{i}t(\cdot)} f(t) \,\mathrm{d}t,$$

where the limit is taken in $L_2(\mathbb{R}; H)$.

## 5.2 The Fourier–Laplace Transformation and its Relation to the Time Derivative

We now use the Fourier transformation to define an analogous transformation on our exponentially weighted $L_2$-type spaces; the so-called Fourier–Laplace transformation. We recall from Corollary 3.2.5 that for $\nu \in \mathbb{R}$ the mapping

$$\exp(-\nu\mathrm{m})\colon L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H),\ f \mapsto \big(t \mapsto \mathrm{e}^{-\nu t} f(t)\big)$$

is unitary. In a similar fashion, we obtain that

$$\exp(-\nu\mathrm{m})\colon L_{1,\nu}(\mathbb{R}; H) \to L_1(\mathbb{R}; H),\ f \mapsto \big(t \mapsto \mathrm{e}^{-\nu t} f(t)\big)$$

defines an isometry.

**Definition.** Let $\nu \in \mathbb{R}$. We define the *Fourier–Laplace transformation* as

$$\mathcal{L}_\nu\colon L_{2,\nu}(\mathbb{R}; H) \to L_2(\mathbb{R}; H),\ f \mapsto \mathcal{F}\exp(-\nu\mathrm{m})f.$$

We can also consider the Fourier–Laplace transformation as a mapping from $L_{1,\nu}(\mathbb{R}; H)$ to $C_\mathrm{b}(\mathbb{R}; H)$; that is,

$$\mathcal{L}_\nu\colon L_{1,\nu}(\mathbb{R}; H) \to C_\mathrm{b}(\mathbb{R}; H),\ f \mapsto \mathcal{F}\exp(-\nu\mathrm{m})f.$$

*Remark* 5.2.1. Note that $\mathcal{L}_\nu = \mathcal{F}\exp(-\nu\mathrm{m})$ is unitary as an operator from $L_{2,\nu}(\mathbb{R};H)$ to $L_2(\mathbb{R};H)$ since it is the composition of two unitary operators. For $\varphi \in C_\mathrm{c}^\infty(\mathbb{R};H)$, we have the expression

$$\left(\mathcal{L}_\nu\varphi\right)(t) = \frac{1}{\sqrt{2\pi}}\int_\mathbb{R} \mathrm{e}^{-(\mathrm{i}t+\nu)s}\varphi(s)\,\mathrm{d}s \quad (t\in\mathbb{R}),$$

which shows that $\mathcal{L}_\nu$ can be interpreted as a shifted variant of the Fourier transformation, where the real part in the exponent equals $\nu$ instead of zero.

Our next goal is to show that the Fourier–Laplace transformation provides a spectral representation of our time derivative, $\partial_{t,\nu}$.

**Definition.** Let $V\colon\mathbb{R}\to\mathbb{K}$ be measurable. We define the *multiplication-by-$V$ operator* as

$$V(\mathrm{m})\colon\ \mathrm{dom}(V(\mathrm{m}))\subseteq L_2(\mathbb{R};H)\to L_2(\mathbb{R};H),\ f\mapsto \big(t\mapsto V(t)f(t)\big)$$

with

$$\mathrm{dom}(V(\mathrm{m})) := \left\{f\in L_2(\mathbb{R};H)\,;\ \big(t\mapsto V(t)f(t)\big)\in L_2(\mathbb{R};H)\right\}.$$

In particular, if $V$ is the identity on $\mathbb{R}$ we will just write m instead of id(m) and call it the *multiplication-by-the-argument operator*.

*Remark* 5.2.2. Note that the multiplication-by-$V$ operator is a vector-valued analogue of the multiplication operator seen in Theorem 2.4.2 and Theorem 2.4.5. The statements in these theorems generalise (easily) to the vector-valued situation at hand. Thus, as in Theorem 2.4.2, one shows that m is selfadjoint. Moreover, when $H\neq\{0\}$, in a similar fashion to the arguments carried out in Theorem 2.4.5 one shows that

$$\sigma(\mathrm{m}) = \mathbb{R}.$$

In order to avoid trivial cases, we shall assume throughout that $H\neq\{0\}$.

**Theorem 5.2.3.** *Let $\nu\in\mathbb{R}$. Then*

$$\partial_{t,\nu} = \mathcal{L}_\nu^*(\mathrm{im}+\nu)\mathcal{L}_\nu.$$

*In particular,*

$$\sigma(\partial_{t,\nu}) = \{\mathrm{i}t+\nu\,;\ t\in\mathbb{R}\}.$$

*Proof.* We first prove the assertion for $\nu\neq 0$ and show that

$$I_\nu = \mathcal{L}_\nu^*\left(\frac{1}{\mathrm{im}+\nu}\right)\mathcal{L}_\nu.$$

The assertion will then follow by Theorem 2.4.2. Note that $\frac{1}{\mathrm{im}+\nu}\in L(L_2(\mathbb{R};H))$ by Theorem 2.4.2, and hence, both operators $I_\nu$ and $\mathcal{L}_\nu^*(\frac{1}{\mathrm{im}+\nu})\mathcal{L}_\nu$ are bounded and defined on the whole of $L_{2,\nu}(\mathbb{R};H)$. Thus, it suffices to prove the equality on a dense subset of

$L_{2,\nu}(\mathbb{R}; H)$, like $C_\mathrm{c}(\mathbb{R}; H)$. We will just do the computation for the case when $\nu > 0$. So, let $\varphi \in C_\mathrm{c}(\mathbb{R}; H)$ and compute

$$\begin{aligned}(\mathcal{L}_\nu I_\nu \varphi)(t) &= \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \mathrm{e}^{-(\mathrm{i}t+\nu)s} \int_{-\infty}^{s} \varphi(r)\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \int_r^\infty \mathrm{e}^{-(\mathrm{i}t+\nu)s}\,\mathrm{d}s\,\varphi(r)\,\mathrm{d}r \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}t+\nu} \int_\mathbb{R} \mathrm{e}^{-(\mathrm{i}t+\nu)r} \varphi(r)\,\mathrm{d}r = \frac{1}{\mathrm{i}t+\nu} (\mathcal{L}_\nu \varphi)(t)\end{aligned}$$

for $t \in \mathbb{R}$. For $\nu < 0$ the computation is analogous. In the case when $\nu = 0$ we observe that

$$\begin{aligned}\partial_{t,0} &= \exp(-\nu \mathrm{m})(\partial_{t,\nu} - \nu)\exp(-\nu\mathrm{m})^{-1} = \exp(-\nu\mathrm{m})\mathcal{L}_\nu^*(\mathrm{im} + \nu - \nu)\mathcal{L}_\nu \exp(-\nu\mathrm{m})^{-1} \\ &= \mathcal{L}_0^*(\mathrm{im})\mathcal{L}_0. \end{aligned} \qquad \square$$

## 5.3 Material Law Operators

Using the multiplication operator representation of $\partial_{t,\nu}$ via the Fourier–Laplace transformation, we can assign a functional calculus to this operator. We will do this in the following and define operator-valued functions of $\partial_{t,\nu}$. The class of functions used for this calculus are the so-called material laws. We begin by defining this function class.

**Definition.** A mapping $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ is called a *material law* if

(a) $\operatorname{dom}(M)$ is open and $M$ is holomorphic (i.e., complex differentiable; see also Exercise 5.3),

(b) there exists some $\nu \in \mathbb{R}$ such that $\mathbb{C}_{\mathrm{Re}>\nu} \subseteq \operatorname{dom}(M)$ and

$$\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} \coloneqq \sup_{z\in\mathbb{C}_{\mathrm{Re}>\nu}} \|M(z)\| < \infty.$$

Moreover, we set

$$\mathrm{s_b}(M) \coloneqq \inf\left\{\nu \in \mathbb{R}\,;\, \mathbb{C}_{\mathrm{Re}>\nu} \subseteq \operatorname{dom}(M) \text{ and } \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} < \infty\right\}$$

to be the *abscissa of boundedness* of $M$.

**Example 5.3.1.** (a) Polynomials in $z^{-1}$: Let $n \in \mathbb{N}_0$, $M_0, \ldots, M_n \in L(H)$. Then

$$M(z) \coloneqq \sum_{k=0}^{n} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\})$$

defines a material law with

$$\mathrm{s_b}(M) = \begin{cases} -\infty, & \text{if } M_1 = \ldots = M_n = 0, \\ 0, & \text{otherwise.}\end{cases}$$

(b) Series in $z^{-1}$: Let $(M_k)_{k\in\mathbb{N}}$ in $L(H)$ such that $\sum_{k=0}^{\infty}\|M_k\|r^{-k}<\infty$ for some $r>0$. Then
$$M(z) \coloneqq \sum_{k=0}^{\infty} z^{-k}M_k \quad (z\in\mathbb{C}\setminus\{0\})$$
defines a material law with $\mathrm{s_b}(M)\leqslant r$.

(c) Exponentials: Let $h\in\mathbb{R}, M_0\in L(H)$ where $M_0\neq 0$ and set
$$M(z)\coloneqq M_0\mathrm{e}^{zh} \quad (z\in\mathbb{C}).$$
Then $M$ is a material law if and only if $h\leqslant 0$. In this case, $\mathrm{s_b}(M)=-\infty$.

(d) Laplace transforms: Let $\nu\in\mathbb{R}$ and $k\in L_{1,\nu}(\mathbb{R})$ with $\operatorname{spt} k\subseteq\mathbb{R}_{\geq 0}$. Then
$$M(z)\coloneqq\sqrt{2\pi}(\mathcal{L}k)(z)\coloneqq\int_0^{\infty}\mathrm{e}^{-zt}k(t)\,\mathrm{d}t \quad (z\in\mathbb{C}_{\mathrm{Re}>\nu})$$
defines a material law with $\mathrm{s_b}(M)\leqslant\nu$.

(e) Fractional powers: Let $M_0\in L(H)$, $M_0\neq 0$, $\alpha\in\mathbb{R}$ and set
$$M(z)\coloneqq M_0z^{-\alpha} \quad (z\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}),$$
where we set
$$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^{-\alpha}\coloneqq r^{-\alpha}\mathrm{e}^{-\mathrm{i}\alpha\theta} \quad (r>0,\theta\in(-\pi,\pi)).$$
Then $M$ is a material law if and only if $\alpha\geqslant 0$ and
$$\mathrm{s_b}(M)=\begin{cases}-\infty & \text{if } \alpha=0,\\ 0 & \text{otherwise.}\end{cases}$$

For material laws $M$ we now define the corresponding material law operators in terms of the functional calculus induced by the spectral representation of $\partial_{t,\nu}$.

**Proposition 5.3.2.** *Let $M\colon \operatorname{dom}(M)\subseteq\mathbb{C}\to L(H)$ be a material law. Then, for $\nu>\mathrm{s_b}(M)$, the operator*
$$M(\mathrm{im}+\nu)\colon L_2(\mathbb{R};H)\to L_2(\mathbb{R};H),\ f\mapsto\big(t\mapsto M(\mathrm{i}t+\nu)f(t)\big)$$
*is bounded. Moreover, we define the* material law operator
$$M(\partial_{t,\nu})\coloneqq\mathcal{L}_\nu^{*}M(\mathrm{im}+\nu)\mathcal{L}_\nu\in L(L_{2,\nu}(\mathbb{R};H))$$
*and obtain*
$$\|M(\partial_{t,\nu})\|\leqslant\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}.$$

*Proof.* The proof is clear. □

*Remark* 5.3.3. The set of material laws is an algebra and the mapping of assigning a material law to its corresponding material law operator is an algebra homomorphism in the following sense. For $j \in \{1,2\}$ let $M_j \colon \operatorname{dom}(M_j) \subseteq \mathbb{C} \to L(H)$ be material laws, $\lambda \in \mathbb{C}$. Then $M_1 + M_2$ (with domain $\operatorname{dom}(M_1) \cap \operatorname{dom}(M_2)$), $\lambda M_1$ and $M_1 \cdot M_2$ (with domain $\operatorname{dom}(M_1) \cap \operatorname{dom}(M_2)$) are material laws as well. Moreover, $\mathrm{s_b}\left(M_1 + M_2\right), \mathrm{s_b}\left(M_1 \cdot M_2\right) \leqslant \max\{\mathrm{s_b}\left(M_1\right), \mathrm{s_b}\left(M_2\right)\}$. Furthermore, if $M_2(z)$ is a multiplication operator for all $z \in \operatorname{dom}(M_2)$, then for $\nu > \max\{\mathrm{s_b}\left(M_1\right), \mathrm{s_b}\left(M_2\right)\}$ we have $(M_1M_2)(\partial_{t,\nu}) = M_1(\partial_{t,\nu})M_2(\partial_{t,\nu}) = M_2(\partial_{t,\nu})M_1(\partial_{t,\nu}) = (M_2M_1)(\partial_{t,\nu})$.

**Example 5.3.4.** We now revisit the material laws presented in Example 5.3.1 and compute their corresponding operators, $M(\partial_{t,\nu})$.

(a) Let $n \in \mathbb{N}_0$, $M_0, \ldots, M_n \in L(H)$ and

$$M(z) \coloneqq \sum_{k=0}^{n} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\}).$$

Then, for $\nu > 0$, one obviously has

$$M(\partial_{t,\nu}) = \sum_{k=0}^{n} \partial_{t,\nu}^{-k} M_k,$$

due to Theorem 5.2.3.

(b) Let $(M_k)_{k\in\mathbb{N}}$ in $L(H)$ such that $\sum_{k=0}^{\infty} \|M_k\| r^{-k} < \infty$ for some $r > 0$ and

$$M(z) \coloneqq \sum_{k=0}^{\infty} z^{-k} M_k \quad (z \in \mathbb{C} \setminus \{0\}).$$

Then, for $\nu > r$, one has

$$M(\partial_{t,\nu}) = \sum_{k=0}^{\infty} \partial_{t,\nu}^{-k} M_k$$

again on account of Theorem 5.2.3.

(c) Let $h \leqslant 0, M_0 \in L(H)$ and

$$M(z) \coloneqq M_0 \mathrm{e}^{zh} \quad (z \in \mathbb{C}).$$

Then, for $\nu \in \mathbb{R}$, we have

$$M(\partial_{t,\nu}) = M_0 \tau_h,$$

where

$$\tau_h \colon L_{2,\nu}(\mathbb{R}; H) \to L_{2,\nu}(\mathbb{R}; H),\ f \mapsto \big(t \mapsto f(t+h)\big).$$

Indeed, for $\varphi \in C_{\mathrm{c}}(\mathbb{R}; H)$ we compute

$$\left(\mathcal{L}_\nu M_0 \tau_h \varphi\right)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s} M_0 \varphi(s+h)\, \mathrm{d}s$$

$$= M_0 \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)(s-h)} \varphi(s)\,\mathrm{d}s = M(\mathrm{i}t+\nu)\left(\mathcal{L}_\nu \varphi\right)(t)$$

for all $t \in \mathbb{R}$, where we have used Proposition 3.1.7 in the second line. Hence,

$$M_0 \tau_h \varphi = \mathcal{L}_\nu^* M(\mathrm{im}+\nu)\mathcal{L}_\nu \varphi = M(\partial_{t,\nu})\varphi$$

and since $C_{\mathrm{c}}(\mathbb{R}; H)$ is dense in $L_{2,\nu}(\mathbb{R}; H)$ the assertion follows.

(d) Let $\nu \in \mathbb{R}$ and $k \in L_{1,\nu}(\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geq 0}$ and

$$M(z) := \sqrt{2\pi}(\mathcal{L}k)(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

Then, by Exercise 5.4,

$$M(\partial_{t,\mu}) = k*$$

for each $\mu > \nu$.

(e) Let $M_0 \in L(H)$, $\alpha > 0$ and

$$M(z) := M_0 z^{-\alpha} \quad (z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

Then for $\nu > 0$ we have

$$\left(M(\partial_{t,\nu})f\right)(t) = M_0 \int_{-\infty}^{t} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad (\text{a.e. } t \in \mathbb{R}) \tag{5.8}$$

for each $f \in L_{2,\nu}(\mathbb{R}; H)$; see Exercise 5.5. This formula gives rise to the definition

$$\left(\partial_{t,\nu}^{-\alpha} f\right)(t) := \int_{-\infty}^{t} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad (t \in \mathbb{R}),$$

which is known as the *(Riemann–Liouville) fractional integral of order* $\alpha$.

Throughout the previous examples, the operator $M(\partial_{t,\nu})$ did not depend on the actual value of $\nu$. Indeed, this is true for all material laws. In order to see this, we need the following lemma.

**Lemma 5.3.5.** *Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$, and set $U := \{z \in \mathbb{C}\,;\, \operatorname{Re} z \in (\mu, \nu)\}$. Moreover, let $g\colon \overline{U} \to H$ be continuous and holomorphic on $U$ such that $g(\mathrm{i}\cdot+\nu), g(\mathrm{i}\cdot+\mu) \in L_2(\mathbb{R}; H)$ and there exists a sequence $(R_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{\geqslant 0}$ such that $R_n \to \infty$ and*

$$\int_{\mu}^{\nu} \|g(\pm \mathrm{i}R_n + \rho)\|\,\mathrm{d}\rho \to 0 \quad (n \to \infty). \tag{5.9}$$

*Then*

$$\mathcal{L}_\mu^* g(\mathrm{i}\cdot+\mu) = \mathcal{L}_\nu^* g(\mathrm{i}\cdot+\nu).$$

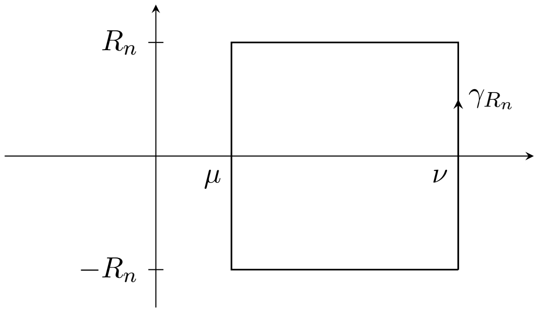


Figure 5.1: Curve $\gamma_{R_n}$.

*Proof.* Let $t \in \mathbb{R}$. By Cauchy's integral theorem, we have that

$$\int_{\gamma_{R_n}} g(z)\mathrm{e}^{zt}\,\mathrm{d}z = 0,$$

where $\gamma_{R_n}$ is the rectangular closed path with corners $\pm \mathrm{i}R_n + \mu, \pm \mathrm{i}R_n + \nu$. Thus, we have that

$$\begin{aligned} &\mathrm{i}\int_{-R_n}^{R_n} g(\mathrm{i}s+\nu)\mathrm{e}^{(\mathrm{i}s+\nu)t}\,\mathrm{d}s - \mathrm{i}\int_{-R_n}^{R_n} g(\mathrm{i}s+\mu)\mathrm{e}^{(\mathrm{i}s+\mu)t}\,\mathrm{d}s \\ &= -\int_{\mu}^{\nu} g(-\mathrm{i}R_n+\rho)\mathrm{e}^{(-\mathrm{i}R_n+\rho)t}\,\mathrm{d}\rho + \int_{\mu}^{\nu} g(\mathrm{i}R_n+\rho)\mathrm{e}^{(\mathrm{i}R_n+\rho)t}\,\mathrm{d}\rho. \end{aligned} \tag{5.10}$$

Note that with the help of the formula for the inverse Fourier-transformation (see Theorem 5.1.4) and $\mathcal{L}_\nu^* = (\mathcal{F}\exp(-\nu\mathrm{m}))^* = \exp(-\nu\mathrm{m})^{-1}\mathcal{F}^*$ the left-hand side of (5.10) is nothing but

$$\sqrt{2\pi}\mathrm{i}\left(\left(\mathcal{L}_\nu^* \mathbb{1}_{[-R_n,R_n]} g(\mathrm{i}\cdot+\nu)\right)(t) - \left(\mathcal{L}_\mu^* \mathbb{1}_{[-R_n,R_n]} g(\mathrm{i}\cdot+\mu)\right)(t)\right),$$

and hence, there is a subsequence of $(R_n)_n$ (which we do not relabel) such that the left-hand side of (5.10) tends to

$$\sqrt{2\pi}\mathrm{i}\left(\left(\mathcal{L}_\nu^* g(\mathrm{i}\cdot+\nu)\right)(t) - \left(\mathcal{L}_\mu^* g(\mathrm{i}\cdot+\mu)\right)(t)\right)$$

for almost every $t \in \mathbb{R}$ as $n \to \infty$. As such, all we need to show is that the right-hand side of (5.10) tends to 0 as $n \to \infty$, which obviously follows by (5.9). □

**Theorem 5.3.6.** *Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then, for $\mu, \nu > \mathrm{s_b}(M)$ and $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$, we have*

$$M(\partial_{t,\nu})f = M(\partial_{t,\mu})f.$$

*Moreover, $M(\partial_{t,\nu})$ is causal for all $\nu > \mathrm{s_b}(M)$.*

*Proof.* Let $\mu < \nu$. We prove the assertion for $f = \mathbb{1}_{[a,b]} \cdot x$ with $a < b$ and $x \in H$ first. For $\rho \in \mathbb{R}$ we compute

$$\left(\mathcal{L}_\rho f\right)(t) = \frac{1}{\sqrt{2\pi}} \int_a^b x \mathrm{e}^{-(\mathrm{i}t+\rho)s} \, \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \frac{1}{\mathrm{i}t+\rho} \left(\mathrm{e}^{-(\mathrm{i}t+\rho)a} - \mathrm{e}^{-(\mathrm{i}t+\rho)b}\right) x \quad (t \in \mathbb{R} \setminus \{0\}).$$

Moreover, we define

$$g(z) := \frac{1}{\sqrt{2\pi}} M(z) x \frac{1}{z} \left(\mathrm{e}^{-za} - \mathrm{e}^{-zb}\right) \quad (z \in \mathbb{C}_{\mathrm{Re} \geqslant \mu} \setminus \{0\})$$

and prove that $g$ satisfies the assumptions of Lemma 5.3.5. First, we note that $g$ is bounded on $\{z \in \mathbb{C}\,;\, \mu \leqslant \operatorname{Re} z \leqslant \nu\} \setminus \{0\}$. Indeed, we only need to prove that it is bounded near 0 provided that $\mu \leqslant 0$. To that end, we observe

$$\frac{1}{z}(\mathrm{e}^{-za} - \mathrm{e}^{-zb}) = \mathrm{e}^{-za} \frac{1 - \mathrm{e}^{-z(b-a)}}{z} \to b - a \quad (z \to 0).$$

Thus, $g$ is bounded near 0. In particular, $z = 0$ is a removable singularity and, hence, $g$ can be extended holomorphically to $\mathbb{C}_{\mathrm{Re} \geqslant \mu}$. Moreover, for $\rho \geqslant \mu$ we have that

$$\int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|^2 \, \mathrm{d}t = \int_{-1}^{1} \|g(\mathrm{i}t+\rho)\|^2 \, \mathrm{d}t + \int_{|t|>1} \|g(\mathrm{i}t+\rho)\|^2 \, \mathrm{d}t.$$

The first term on the right-hand side is finite since $g$ is bounded, while the second term can be estimated by

$$\int_{|t|>1} \|g(\mathrm{i}t+\rho)\|^2 \, \mathrm{d}t \leqslant \|M\|^2_{\infty, \mathbb{C}_{\mathrm{Re}>\mu}} \|x\|^2 \frac{(\mathrm{e}^{-\rho a} + \mathrm{e}^{-\rho b})^2}{2\pi} \int_{|t|>1} \frac{1}{t^2+\rho^2} \, \mathrm{d}t < \infty.$$

This proves that $g(\mathrm{i} \cdot + \rho) \in L_2(\mathbb{R}; H)$ for each $\rho \geqslant \mu$ and hence, particularly for $\rho = \mu$ and $\rho = \nu$. Finally, for $\rho \geqslant \mu$ we have that

$$\|g(\mathrm{i}t+\rho)\| \leqslant \frac{1}{\sqrt{2\pi}} \|M\|_{\infty, \mathbb{C}_{\mathrm{Re}>\mu}} \|x\| \frac{1}{\sqrt{t^2+\rho^2}} \left(\mathrm{e}^{-\rho a} + \mathrm{e}^{-\rho b}\right) \to 0 \quad (|t| \to \infty),$$

which together with the boundedness of $g$ yields (5.9) by dominated convergence. This shows that $g$ satisfies the assumptions of Lemma 5.3.5 and thus

$$M(\partial_{t,\nu}) f = \mathcal{L}_\nu^* g(\mathrm{i} \cdot + \nu) = \mathcal{L}_\mu^* g(\mathrm{i} \cdot + \mu) = M(\partial_{t,\mu}) f.$$

By linearity, this equality extends to $S_{\mathrm{c}}(\mathbb{R}; H)$ and so,

$$F \colon S_{\mathrm{c}}(\mathbb{R}; H) \to \bigcap_{\nu \geqslant \mu} L_{2,\nu}(\mathbb{R}; H), \; f \mapsto M(\partial_{t,\nu}) f$$

is well-defined. Moreover, $F$ is uniformly Lipschitz continuous (with $\sup_{\nu \geqslant \mu} \|F^\nu\| = \|M\|_{\infty, \mathbb{C}_{\mathrm{Re}>\mu}}$) and hence, the assertions follow from Lemma 4.2.5. $\square$

## 5.4 Comments

The Fourier and the Fourier–Laplace transformation introduced in this chapter are used to define an operator-valued functional calculus for the time derivative, $\partial_{t,\nu}$. This functional calculus can be defined since the Fourier–Laplace transformation provides the unitary transformation yielding the spectral representation of the time derivative as multiplication operator. This fact was already noticed in [Pic89], which eventually led to evolutionary equations in [Pic09].
We emphasise that we have used the fundamental property that both $\mathcal{F}$ and $\mathcal{L}_\nu$ are unitary. It is noteworthy that the Fourier transformation is an isometric isomorphism on $L_2(\mathbb{R}; X)$ if and only if $X$ is a Hilbert space, see [Kwa72]. In the Banach space-valued case one has to further restrict the class of functions used to define a functional calculus. For the topic of functional calculus we refer to the 21st ISem [Haa18] by Markus Haase and to his monograph, [Haa06].

## Exercises

**Exercise 5.1.** Let $(\Omega, \Sigma, \mu)$ be a measure space, $X$ a Banach space and $I \subseteq \mathbb{R}$ an open interval. Let $g\colon I \times \Omega \to X$ such that $g(t, \cdot) \in L_1(\mu; X)$ for each $t \in I$, and define

$$h\colon I \to X,\ t \mapsto \int_\Omega g(t,\omega)\,\mathrm{d}\mu(\omega).$$

(a) Assume that $g(\cdot, \omega)$ is continuous for $\mu$-almost every $\omega \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|g(t,\omega)\| \leqslant f(\omega) \quad (t \in I, \omega \in \Omega).$$

Prove that $h$ is continuous.

(b) Assume that $g(\cdot, \omega)$ is differentiable for $\mu$-almost every $\omega \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|\partial_t g(t,\omega)\| \leqslant f(\omega) \quad (t \in I, \omega \in \Omega).$$

Prove that $h$ is differentiable with

$$h'(t) = \int_\Omega \partial_t g(t,\omega)\,\mathrm{d}\mu(\omega).$$

**Exercise 5.2.** Let $H_0, H_1$ be two Hilbert spaces and $U\colon \operatorname{dom}(U) \subseteq H_0 \to H_1$ linear such that

- $\operatorname{dom}(U)$ is dense in $H_0$ and $\operatorname{ran}(U)$ is dense in $H_1$.
- $\forall x \in \operatorname{dom}(U) : \ \|Ux\|_{H_1} = \|x\|_{H_0}$.

Show that $U$ can be uniquely extended to a unitary operator between $H_0$ and $H_1$.

**Exercise 5.3.** Let $\Omega \subseteq \mathbb{C}$ be open, $X$ a complex Banach space and $f\colon \Omega \to X$. Prove that the following statements are equivalent:

(i) $f$ is holomorphic.

(ii) For all $x' \in X'$ the mapping $x' \circ f\colon \Omega \to \mathbb{C}$ is holomorphic.

(iii) $f$ is locally bounded and $x' \circ f\colon \Omega \to \mathbb{C}$ is holomorphic for all $x' \in D$, where $D \subseteq X'$ is a norming set[1] for $X$.

(iv) $f$ is analytic, i.e. for each $z_0 \in \Omega$ there is $r > 0$ and $(a_n)_n$ in $X$ with $B(z_0, r) \subseteq \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (z \in B(z_0, r)).$$

Assume now that $X = L(X_1, X_2)$ for two complex Banach spaces $X_1, X_2$, let $D_1 \subseteq X_1$ be dense and $D_2 \subseteq X_2'$ norming for $X_2$. Prove that the statements (a) to (d) are equivalent to

(v) $f$ is locally bounded and $\Omega \ni z \mapsto x_2'(f(z)(x_1)) \in \mathbb{C}$ is holomorphic for all $x_1 \in D_1$ and $x_2' \in D_2$.

**Exercise 5.4.** Let $\nu \in \mathbb{R}$ and $k \in L_{1,\nu}(\mathbb{R})$. Prove that

$$\mathcal{L}_\nu (k * f) = \sqrt{2\pi}\, (\mathcal{L}_\nu k) \cdot (\mathcal{L}_\nu f)$$

for $f \in L_{2,\nu}(\mathbb{R}; H)$.

**Exercise 5.5.** Let $\alpha > 0$ and define $g_\alpha(t) \coloneqq \mathbb{1}_{[0,\infty)}(t) t^{\alpha-1}$ for $t \in \mathbb{R}$. Show that $g_\alpha \in L_{1,\nu}(\mathbb{R})$ for each $\nu > 0$ and that

$$(\mathcal{L}_\nu g_\alpha)(t) = \frac{1}{\sqrt{2\pi}} \Gamma(\alpha)(\mathrm{i}t + \nu)^{-\alpha}.$$

Use this formula and Exercise 5.4 to derive (5.8).
Hint: To compute the Fourier–Laplace transform of $g_\alpha$, derive that $\mathcal{L}_\nu g_\alpha$ solves a first order ordinary differential equation and use separation of variables to solve this equation.

**Exercise 5.6.** Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ and $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$. Moreover, set $U \coloneqq \{z \in \mathbb{C}\,;\, \mu < \operatorname{Re} z < \nu\}$. Show that $f \in \bigcap_{\mu<\rho<\nu} L_{2,\rho}(\mathbb{R}; H) \cap L_{1,\rho}(\mathbb{R}; H)$ and that

$$U \ni z \mapsto (\mathcal{L}_{\operatorname{Re} z} f)(\operatorname{Im} z)$$

is holomorphic.

**Exercise 5.7.** Let $H_0, H_1$ be Hilbert spaces and $T\colon L_{2,\nu}(\mathbb{R}; H_0) \to L_{2,\nu}(\mathbb{R}; H_1)$ linear and bounded. We call $T$ *autonomous* if $T\tau_h = \tau_h T$ for each $h \in \mathbb{R}$ ($\tau_h$ denotes the translation operator defined in Example 5.3.4). Prove that for autonomous $T$, the following statements are equivalent:

[1] $D \subseteq X'$ is called a norming set for $X$, if $\|x\| = \sup_{x' \in D\setminus\{0\}} \frac{1}{\|x'\|} |x'(x)|$ for each $x \in X$. Note that $X'$ is norming for $X$ by the Hahn–Banach theorem.

(i) $T$ is causal.

(ii) For all $f \in L_{2,\nu}(\mathbb{R}; H_0)$ with $\operatorname{spt} f \subseteq [0, \infty)$ one has $\operatorname{spt} Tf \subseteq [0, \infty)$.

Moreover, prove that for a material law $M$, the operator $M(\partial_{t,\nu})$ is autonomous for each $\nu > \mathrm{s_b}(M)$.

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# 6 Solution Theory for Evolutionary Equations

In this chapter, we shall discuss and present the first major result of this year's internet seminar: Picard's theorem on the solution theory for evolutionary equations which is the main result of [Pic09]. In order to stress the applicability of this theorem, we shall deal with applications first and provide a proof of the actual result afterwards. With an initial interest in applications in mind, we start off with the introduction of some vector-analytic operators.

## 6.1 First Order Sobolev Spaces

Throughout this section let $\Omega \subseteq \mathbb{R}^d$ be an open set.

**Definition.** We define

$$\begin{aligned} \operatorname{grad}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^d \\ \phi &\mapsto (\partial_j \phi)_{j\in\{1,\ldots,d\}}, \end{aligned}$$

$$\begin{aligned} \operatorname{div}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega) \\ (\phi_j)_{j\in\{1,\ldots,d\}} &\mapsto \sum_{j\in\{1,\ldots,d\}} \partial_j \phi_j, \end{aligned}$$

and if $d = 3$,

$$\begin{aligned} \operatorname{curl}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^3 \subseteq L_2(\Omega)^3 &\to L_2(\Omega)^3 \\ (\phi_j)_{j\in\{1,2,3\}} &\mapsto \begin{pmatrix} \partial_2\phi_3 - \partial_3\phi_2 \\ \partial_3\phi_1 - \partial_1\phi_3 \\ \partial_1\phi_2 - \partial_2\phi_1 \end{pmatrix}. \end{aligned}$$

Furthermore, we put

$$\operatorname{div} := -\operatorname{grad}_{\mathrm{c}}^*, \quad \operatorname{grad} := -\operatorname{div}_{\mathrm{c}}^*, \quad \operatorname{curl} := \operatorname{curl}_{\mathrm{c}}^*$$

and

$$\operatorname{div}_0 := -\operatorname{grad}^*, \quad \operatorname{grad}_0 := -\operatorname{div}^*, \quad \operatorname{curl}_0 := \operatorname{curl}^*.$$

**Proposition 6.1.1.** *The relations* $\operatorname{div}, \operatorname{div}_0, \operatorname{grad}, \operatorname{grad}_0, \operatorname{curl}$ *and* $\operatorname{curl}_0$ *are all densely defined, closed linear operators.*

*Proof.* The operators $\operatorname{grad}_c, \operatorname{div}_c$ and $\operatorname{curl}_c$ are densely defined by Exercise 6.3. Thus, $\operatorname{div}, \operatorname{grad}$ and curl are closed linear operators by Lemma 2.2.7. Moreover, it follows from integration by parts that $\operatorname{grad}_c \subseteq \operatorname{grad}$, $\operatorname{div}_c \subseteq \operatorname{div}$ and $\operatorname{curl}_c \subseteq \operatorname{curl}$. Thus, $\operatorname{div}, \operatorname{grad}$ and curl are also densely defined. This, in turn, implies that $\operatorname{grad}_c, \operatorname{div}_c$ and $\operatorname{curl}_c$ are closable by Lemma 2.2.7 with respective closures $\operatorname{grad}_0, \operatorname{div}_0$ and $\operatorname{curl}_0$ by Lemma 2.2.4. □

We shall describe the domains of these operators in more detail in the next theorem.

**Theorem 6.1.2.** *If $f \in L_2(\Omega)$ and $g = (g_j)_{j\in\{1,\ldots,d\}} \in L_2(\Omega)^d$ then the following statements hold:*

(a) $f \in \operatorname{dom}(\operatorname{grad})$ *and* $g = \operatorname{grad} f$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega), j \in \{1,\ldots,d\}\colon\ -\int_\Omega f\partial_j\phi = \int_\Omega g_j\phi.$$

(b) $f \in \operatorname{dom}(\operatorname{grad}_0)$ *and* $g = \operatorname{grad}_0 f$ *if and only if there exists $(f_k)_k$ in $C_c^\infty(\Omega)$ such that $f_k \to f$ in $L_2(\Omega)$ and* $\operatorname{grad} f_k \to g$ *in $L_2(\Omega)^d$ as $k \to \infty$.*

(c) $g \in \operatorname{dom}(\operatorname{div})$ *and* $f = \operatorname{div} g$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega)\colon\ -\int_\Omega g\cdot \operatorname{grad}\phi = \int_\Omega f\phi.$$

(d) $g \in \operatorname{dom}(\operatorname{div}_0)$ *and* $f = \operatorname{div}_0 g$ *if and only if there exists $(g_k)_k$ in $C_c^\infty(\Omega)^d$ such that $g_k \to g$ in $L_2(\Omega)^d$ and* $\operatorname{div} g_k \to f$ *in $L_2(\Omega)$ as $k \to \infty$.*

*If $d = 3$ and $f, g \in L_2(\Omega)^3$ then the following statements hold:*

(e) $f \in \operatorname{dom}(\operatorname{curl})$ *and* $g = \operatorname{curl} f$ *if and only if*

$$\forall \phi \in C_c^\infty(\Omega)^3\colon\ \int_\Omega f\cdot \operatorname{curl}\phi = \int_\Omega g\cdot\phi.$$

(f) $f \in \operatorname{dom}(\operatorname{curl}_0)$ *and* $g = \operatorname{curl}_0 f$ *if and only if there exists $(f_k)_k$ in $C_c^\infty(\Omega)^3$ such that $f_k \to f$ in $L_2(\Omega)^3$ and* $\operatorname{curl} f_k \to g$ *in $L_2(\Omega)^3$ as $k \to \infty$.*

All the statements in Theorem 6.1.2 are elementary consequences of the integration by parts formula and the definitions of the adjoint. We ask the reader to prove these statements in Exercise 6.4.

*Remark* 6.1.3. We remark here that, classically, the following notation has been introduced:

$$\begin{aligned} H^1(\Omega) &\coloneqq \operatorname{dom}(\operatorname{grad}),\\ H_0^1(\Omega) &\coloneqq \operatorname{dom}(\operatorname{grad}_0),\\ H(\operatorname{div},\Omega) &\coloneqq \operatorname{dom}(\operatorname{div}),\\ H(\operatorname{curl},\Omega) &\coloneqq \operatorname{dom}(\operatorname{curl}). \end{aligned}$$

Following the rationale of appending zero as an index for $H_0^1(\Omega)$, we shall also use

$$H_0(\mathrm{div},\Omega) \coloneqq \mathrm{dom}(\mathrm{div}_0),$$
$$H_0(\mathrm{curl},\Omega) \coloneqq \mathrm{dom}(\mathrm{curl}_0).$$

We do, however, caution the reader that other authors also use $H_0(\mathrm{div},\Omega)$ and $H_0(\mathrm{curl},\Omega)$ to denote the kernel of div and curl. All the spaces just defined are so-called Sobolev spaces. We note that for $d = 3$ we clearly have $H^1(\Omega)^3 \subseteq H(\mathrm{div},\Omega) \cap H(\mathrm{curl},\Omega)$. On the other hand, note that $H(\mathrm{div},\Omega)$ is neither a sub- nor a superset of $H(\mathrm{curl},\Omega)$.

*Remark* 6.1.4. We emphasise that $H_0^1(\Omega) \subseteq H^1(\Omega)$ is a proper inclusion for many open $\Omega$. The '0' in the index is a reminder of '0'-boundary conditions. In fact, the only difference between these two spaces lies in the behaviour of their elements at the boundary of $\Omega$. The space $H_0^1$ signifies all $H^1$-functions vanishing at $\partial\Omega$ in a generalised sense. The corresponding statements are true for the inclusions $H_0(\mathrm{div},\Omega) \subseteq H(\mathrm{div},\Omega)$ and $H_0(\mathrm{curl},\Omega) \subseteq H(\mathrm{curl},\Omega)$. The space $H_0(\mathrm{div},\Omega)$ describes $H(\mathrm{div},\Omega)$-vector fields with vanishing normal component and to lie in $H_0(\mathrm{curl},\Omega)$ provides a handy generalisation of vanishing tangential component. We will anticipate these abstractions, when we apply the solution theory of evolutionary equations for particular cases. In a later chapter we will come back to this issue when we discuss inhomogeneous boundary value problems.

For later use, we record the following relationships between the vector-analytical operators introduced above.

**Proposition 6.1.5.** *Let $d = 3$. We have the following inclusions:*

$$\overline{\mathrm{ran}}(\mathrm{curl}_0) \subseteq \ker(\mathrm{div}_0),$$
$$\overline{\mathrm{ran}}(\mathrm{grad}_0) \subseteq \ker(\mathrm{curl}_0),$$
$$\overline{\mathrm{ran}}(\mathrm{curl}) \subseteq \ker(\mathrm{div}),$$
$$\overline{\mathrm{ran}}(\mathrm{grad}) \subseteq \ker(\mathrm{curl}).$$

*Proof.* It is elementary to show that for given $\psi \in C_c^\infty(\Omega)^3$ and $\phi \in C_c^\infty(\Omega)$ we have $\mathrm{div}_0 \,\mathrm{curl}_0 \,\psi = 0$ as well as $\mathrm{curl}_0 \,\mathrm{grad}_0 \,\phi = 0$. Thus, we obtain $\mathrm{ran}(\mathrm{curl}_c) \subseteq \ker(\mathrm{div}_0)$ and $\mathrm{ran}(\mathrm{grad}_c) \subseteq \ker(\mathrm{curl}_0)$. Since $\ker(\mathrm{div}_0)$ and $\ker(\mathrm{curl}_0)$ are closed, and $C_c^\infty(\Omega)^3$ and $C_c^\infty(\Omega)$ are cores for $\mathrm{curl}_0$ and $\mathrm{grad}_0$ respectively, we obtain the first two inclusions. The last two inclusions follow from the first two by taking into account the orthogonal decompositions

$$L_2(\Omega)^3 = \overline{\mathrm{ran}}(\mathrm{grad}) \oplus \ker(\mathrm{div}_0) = \ker(\mathrm{curl}) \oplus \overline{\mathrm{ran}}(\mathrm{curl}_0)$$

and

$$L_2(\Omega)^3 = \overline{\mathrm{ran}}(\mathrm{grad}_0) \oplus \ker(\mathrm{div}) = \ker(\mathrm{curl}_0) \oplus \overline{\mathrm{ran}}(\mathrm{curl})$$

which follow from Corollary 2.2.6. □

## 6.2 Well-Posedness of Evolutionary Equations and Applications

The solution theory of evolutionary equations is contained in the next result, Picard's theorem. This result is central for all the derivations to come. In fact, with the notation of Theorem 6.2.1, we shall prove that for all (well-behaved) $F$ there is a unique solution of

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)\, U = F.$$

The solution $U$ depends continuously and causally on the choice of $F$.
In order to formulate the result, for a Hilbert space $H$, $\nu \in \mathbb{R}$ and a given operator $A\colon \operatorname{dom}(A) \subseteq H \to H$ we define its extended operator in $L_{2,\nu}(\mathbb{R}; H)$, again denoted by $A$, by

$$\begin{aligned} L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A)) \subseteq L_{2,\nu}(\mathbb{R}; H) &\to L_{2,\nu}(\mathbb{R}; H) \\ f &\mapsto \big(t \mapsto Af(t)\big). \end{aligned}$$

We have collected some properties of extended operators in Exercise 6.1 and Exercise 6.2.

**Theorem 6.2.1** (Picard). *Let $\nu_0 \in \mathbb{R}$ and $H$ be a Hilbert space. Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law with $\mathrm{s_b}(M) < \nu_0$ and let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. Assume that*

$$\operatorname{Re} \langle \phi, zM(z)\phi \rangle_H \geqslant c\|\phi\|_H^2 \quad (\phi \in H, z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0})$$

*for some $c > 0$. Then for all $\nu \geqslant \nu_0$ the operator $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable and*

$$S_\nu := \big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H)).$$

*Furthermore, $S_\nu$ is causal and satisfies $\|S_\nu\|_{L(L_{2,\nu})} \leqslant 1/c$, and for all $F \in \operatorname{dom}(\partial_{t,\nu})$ we have*

$$S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A).$$

*Furthermore, for $\eta, \nu \geqslant \nu_0$ and $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ we have that $S_\nu F = S_\eta F$.*

The property that $S_\nu F = S_\eta F$ for all $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ where $\eta, \nu \geqslant \nu_0$, for some $\nu_0 \in \mathbb{R}$, will be referred to as $S_\nu$ being *eventually independent of* $\nu$ in what follows.

*Remark* 6.2.2. Recall that $F \in \operatorname{dom}(\partial_{t,\nu})$ implies $U := S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A)$ in Theorem 6.2.1. Since $M(\partial_{t,\nu})$ leaves the space $\operatorname{dom}(\partial_{t,\nu})$ invariant, this gives that $M(\partial_{t,\nu})U \in \operatorname{dom}(\partial_{t,\nu})$ and thus, $U$ solves the evolutionary equation literally; that is,

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)U = F,$$

while for $F \in L_{2,\nu}(\mathbb{R}; H)$, in general, we just have

$$\big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)U = F.$$

**Definition.** Let $H$ be a Hilbert space and $T \in L(H)$. If $T$ is selfadjoint, we write $T \geqslant c$ for some $c \in \mathbb{R}$ if

$$\forall x \in H : \langle x, Tx \rangle_H \geqslant c\|x\|_H^2.$$

Moreover, we define the *real part of* $T$ by $\operatorname{Re} T := \frac{1}{2}(T + T^*)$.

Note that if $H$ is a Hilbert space and $T \in L(H)$ then $\operatorname{Re} T$ is selfadjoint. Moreover,

$$\langle x, (\operatorname{Re} T)x \rangle_H = \operatorname{Re} \langle x, Tx \rangle_H \quad (x \in H).$$

Hence, in Theorem 6.2.1 the assumption on the material law can be rephrased as

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0}).$$

The following operators will be prototypical examples needed for the applications of the previous theorem.

**Proposition 6.2.3.** *Let $H_0, H_1$ be Hilbert spaces.*

(a) *Let $B\colon \operatorname{dom}(B) \subseteq H_0 \to H_1$, $C\colon \operatorname{dom}(C) \subseteq H_1 \to H_0$ be densely defined linear operators. Then*

$$\begin{aligned} \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} : \operatorname{dom}(B) \times \operatorname{dom}(C) \subseteq H_0 \times H_1 &\to H_0 \times H_1 \\ (\phi, \psi) &\mapsto (C\psi, B\phi) \end{aligned}$$

*is densely defined, and we have*

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix}.$$

(b) *Let $a \in L(H_0)$, and $c > 0$. Assume $\operatorname{Re} a \geqslant c$. Then $a^{-1} \in L(H_0)$ with $\left\|a^{-1}\right\| \leqslant \frac{1}{c}$ and $\operatorname{Re} a^{-1} \geqslant c\|a\|^{-2}$.*

*Proof.* The proof of the first statement can be done in two steps. First, notice that the inclusion $\begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*$ follows immediately. If, on the other hand, $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \operatorname{dom}\left(\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^*\right)$ with $\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$ we get for all $x \in \operatorname{dom}(B)$ that

$$\begin{aligned} \langle Bx, \psi \rangle_{H_1} &= \left\langle \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_{H_0 \times H_1} \\ &= \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_{H_0 \times H_1} = \langle x, \xi \rangle_{H_0}. \end{aligned}$$

Hence, $\psi \in \operatorname{dom}(B^*)$ and $B^*\psi = \xi$. Similarly, we obtain $\phi \in \operatorname{dom}(C^*)$ and $C^*\phi = \zeta$.

For the second statement, we compute for all $\phi \in H_0$ using the Cauchy–Schwarz inequality

$$\|\phi\|\|a\phi\|_{H_0} \geqslant |\langle \phi, a\phi\rangle|_{H_0} \geqslant \operatorname{Re}\langle \phi, a\phi\rangle_{H_0} \geqslant c\,\langle \phi, \phi\rangle_{H_0} = c\|\phi\|^2_{H_0}.$$

Thus, $a$ is one-to-one. Since $\operatorname{Re} a = \operatorname{Re} a^*$ it follows that $a^*$ is one-to-one, as well. Thus, we get that $a$ has dense range by Theorem 2.2.5. The inequality

$$\|a\phi\|_{H_0} \geqslant c\|\phi\|_{H_0}$$

implies that $a^{-1}$ is bounded with $\left\|a^{-1}\right\| \leqslant \frac{1}{c}$. Hence, as $a^{-1}$ is closed, $\operatorname{dom}(a^{-1}) = \operatorname{ran}(a)$ is closed by Lemma 2.1.3 and hence, $\operatorname{dom}(a^{-1}) = H_0$; that is, $a^{-1} \in L(H_0)$. To conclude, let $\psi \in H_0$ and put $\phi \coloneqq a^{-1}\psi$. Then $\|\psi\|_{H_0} = \left\|aa^{-1}\psi\right\|_{H_0} \leqslant \|a\|\left\|a^{-1}\psi\right\|_{H_0}$ and so

$$\begin{aligned}
\operatorname{Re}\left\langle \psi, a^{-1}\psi\right\rangle_{H_0} &= \operatorname{Re}\langle a\phi, \phi\rangle_{H_0} = \operatorname{Re}\langle \phi, a\phi\rangle_{H_0} \geqslant c\,\langle \phi, \phi\rangle_{H_0} = c\left\langle a^{-1}\psi, a^{-1}\psi\right\rangle_{H_0} \\
&\geqslant c\frac{1}{\|a\|^2}\|\psi\|^2_{H_0}.
\end{aligned}$$

□

## The Heat Equation

The first example we will consider is the heat equation in an open subset $\Omega \subseteq \mathbb{R}^d$. Under a heat source, $Q\colon \mathbb{R}\times\Omega \to \mathbb{R}$, the heat distribution, $\theta\colon \mathbb{R}\times\Omega\to\mathbb{R}$, satisfies the so-called heat-flux-balance

$$\partial_t\theta + \operatorname{div} q = Q.$$

Here, $q\colon \mathbb{R}\times\Omega \to \mathbb{R}^d$ is the heat flux which is connected to $\theta$ via Fourier's law

$$q = -a\operatorname{grad}\theta,$$

where $a\colon \Omega \to \mathbb{R}^{d\times d}$ is the heat conductivity, which is measurable, bounded and uniformly strictly positive in the sense that

$$\operatorname{Re} a(x) \geqslant c$$

for all $x \in \Omega$ and some $c > 0$ in the sense of positive definiteness. Moreover, we assume that $\Omega$ is thermally isolated, which is modelled by requiring that the normal component of $q$ vanishes at $\partial\Omega$; that is, $q \in \operatorname{dom}(\operatorname{div}_0)$. Written as a block matrix and incorporating the boundary condition, we obtain

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}\right)\begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

**Theorem 6.2.4.** *For all $\nu > 0$, the operator*

$$\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}$$

*is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega)\times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of $\nu$.*

*Proof.* The assertion follows from Theorem 6.2.1 applied to

$$M(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}.$$

Note that $M$ is a material law with $\mathrm{s_b}(M) = 0$ by Example 5.3.1. Moreover, for $(x, y) \in L_2(\Omega) \times L_2(\Omega)^d$ and $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu}$ with $\nu > 0$ we estimate

$$\begin{aligned} \mathrm{Re}\,\langle (x,y), zM(z)(x,y)\rangle_{L_2(\Omega)\times L_2(\Omega)^d} &\geqslant \mathrm{Re}\, z\|x\|^2_{L_2(\Omega)} + c\|a\|^{-2}\|y\|^2_{L_2(\Omega)^d} \\ &\geqslant \min\{\nu, c\|a\|^{-2}\}\|(x,y)\|^2_{L_2(\Omega)\times L_2(\Omega)^d}, \end{aligned}$$

where we have used Proposition 6.2.3(b) in the first inequality. Moreover, $A$ is skew-selfadjoint by Proposition 6.2.3(a). □

*Remark* 6.2.5. Assume that $Q \in \operatorname{dom}(\partial_{t,\nu})$. It then follows from Theorem 6.2.1 that

$$\begin{aligned} \begin{pmatrix} \theta \\ q \end{pmatrix} &:= \overline{\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}\right)}^{-1} \begin{pmatrix} Q \\ 0 \end{pmatrix} \\ &\in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}\right). \end{aligned} \tag{6.1}$$

Then, as in Remark 6.2.2, it follows that $\theta$ and $q$ satisfy the heat-flux-balance and Fourier's law in the sense that $\theta \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{grad})$ and $q \in \operatorname{dom}(\operatorname{div}_0)$ and

$$\begin{aligned} \partial_t \theta + \operatorname{div}_0 q &= Q, \\ q &= -a \operatorname{grad} \theta. \end{aligned}$$

This regularity result is true even for $Q \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$; see [PTW17].

## The Scalar Wave Equation

The classical scalar wave equation in a medium $\Omega \subseteq \mathbb{R}^d$ (think, for instance, of a vibrating string ($d = 1$) or membrane ($d = 2$)) consists of the equation of the balance of momentum where the acceleration of the (vertical) displacement, $u\colon \mathbb{R} \times \Omega \to \mathbb{R}$, is balanced by external forces, $f\colon \mathbb{R} \times \Omega \to \mathbb{R}$, and the divergence of the stress, $\sigma\colon \mathbb{R} \times \Omega \to \mathbb{R}^d$, in such a way that

$$\partial_t^2 u - \operatorname{div} \sigma = f.$$

The stress is related to $u$ via the following so-called stress-strain relation (here Hooke's law)

$$\sigma = T \operatorname{grad} u,$$

where the so-called elasticity tensor, $T\colon \Omega \to \mathbb{R}^{d\times d}$, is bounded, measurable, and satisfies

$$T(x) = T(x)^* \geqslant c$$

for some $c > 0$ uniformly in $x \in \Omega$. The quantity $\operatorname{grad} u$ is referred to as the strain. We think of $u$ as being fixed at $\partial\Omega$ ("clamped boundary condition"). This is modelled by $u \in \operatorname{dom}(\operatorname{grad}_0)$.
Using $v \coloneqq \partial_t u$ as an unknown, we can rewrite the balance of momentum and Hooke's law as $2 \times 2$-block-operator matrix equation

$$\left(\partial_t \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The solution theory of evolutionary equations for the wave equation now reads as follows:

**Theorem 6.2.6.** *Let $\Omega \subseteq \mathbb{R}^d$ be open, and $T$ as indicated above. Then, for all $\nu > 0$,*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$$

*is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of $\nu$.*

*Proof.* We apply Theorem 6.2.1 to $A = -\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$, which is skew-selfadjoint by Proposition 6.2.3(a), and $M(z) = \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix}$, which defines a material law with $\mathrm{s_b}(M) = -\infty$. The positive definiteness constraint needed in Theorem 6.2.1 is satisfied by Proposition 6.2.3(b) on account of the selfadjointness of $T$, which implies the same for $T^{-1}$. Indeed, for $\nu_0 > 0$ and $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}$ we estimate

$$\begin{aligned} \operatorname{Re} \langle (x, y), zM(z)(x, y) \rangle_{L_2(\Omega) \times L_2(\Omega)^d} &= \operatorname{Re} \langle x, zx \rangle_{L_2(\Omega)} + \operatorname{Re} \left\langle y, zT^{-1} y \right\rangle_{L_2(\Omega)^d} \\ &\geqslant \nu_0 \|x\|^2_{L_2(\Omega)} + \nu_0 \frac{c}{\|T\|^2} \|y\|^2_{L_2(\Omega)^d} \\ &\geqslant \nu_0 \min\{1, c/\|T\|^2\} \|(x, y)\|^2_{L_2(\Omega) \times L_2(\Omega)^d} \end{aligned}$$

for each $(x, y) \in L_2(\Omega) \times L_2(\Omega)^d$, where we used the selfadjointness of $T^{-1}$ in the second line. $\square$

*Remark* 6.2.7. Let $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$, $\nu > 0$, and define

$$\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} = \left(\overline{\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} \partial_{t,\nu}^{-1} f \\ 0 \end{pmatrix}.$$

By Theorem 6.2.1, we obtain $\begin{pmatrix} u \\ \widetilde{\sigma} \end{pmatrix} \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right)$. Hence, we have

$$\begin{aligned} \partial_{t,\nu} u - \operatorname{div} \widetilde{\sigma} &= \partial_{t,\nu}^{-1} f \\ \partial_{t,\nu} T^{-1} \widetilde{\sigma} &= \operatorname{grad}_0 u \end{aligned}$$

or

$$\begin{aligned}\partial_{t,\nu}u - \operatorname{div}\widetilde{\sigma} &= \partial_{t,\nu}^{-1} f\\ \widetilde{\sigma} &= T\partial_{t,\nu}^{-1}\operatorname{grad}_0 u.\end{aligned}$$

Thus, formally, after another time-differentiation and the setting of $\sigma = \partial_{t,\nu}\widetilde{\sigma}$ we obtain a solution of the wave equation, $(u, \sigma)$. Notice, however, that differentiating $\operatorname{div}\widetilde{\sigma}$ cannot be done without any additional knowledge of the regularity of $\widetilde{\sigma}$. In fact, in order to arrive at the balance of momentum equation, one would need to have $\operatorname{div}\widetilde{\sigma} \in \operatorname{dom}(\partial_{t,\nu})$. However, one only has $\widetilde{\sigma} \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{div})$. It is an elementary argument, see [SW17, Lemma 4.6], that we in fact have $\operatorname{div}\partial_{t,\nu}^{-1} = \overline{\partial_{t,\nu}^{-1}\operatorname{div}}$, which suggests that, in general, $\operatorname{div}\widetilde{\sigma} \notin \operatorname{dom}(\partial_{t,\nu})$, see Exercise 6.6.

## Maxwell's Equations

The final example in this lecture forms the archetypical evolutionary equation – Maxwell's equations in a medium $\Omega \subseteq \mathbb{R}^3$. In order to see this (and to finally conclude the $2\times 2$-block matrix formulation historically due to the work of [Min10; Sch68; Lei68]), we start out with Faraday's law of induction, which relates the unknown electric field, $E\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$, to the magnetic induction, $B\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$, via

$$\partial_t B + \operatorname{curl} E = 0.$$

We assume that the medium is contained in a perfect conductor, which is reflected in the so-called electric boundary condition which asks for the vanishing of the tangential component of $E$ at the boundary. This is modelled by $E \in \operatorname{dom}(\operatorname{curl}_0)$. The next constituent of Maxwell's equations is Ampere's law

$$\partial_t D + J_c - \operatorname{curl} H = J_0,$$

which relates the unknown electric displacement, $D\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$, charge, $J_c\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$, and magnetic field, $H\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$, to the (given) external currents, $J_0\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$. Maxwell's equations are completed by constitutive relations specific to each material at hand. Indeed, the (bounded, measurable) dielectricity, $\varepsilon\colon \Omega \to \mathbb{R}^{3\times 3}$, and the (bounded, measurable) magnetic permeability, $\mu\colon \Omega \to \mathbb{R}^{3\times 3}$, are symmetric matrix-valued functions which couple the electric displacement to the electric field and the magnetic field to the magnetic induction via

$$D = \varepsilon E, \text{ and } B = \mu H.$$

Finally, Ohm's law relates the charge to the electric field via the (bounded, measurable) electric conductivity, $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}$, as

$$J_c = \sigma E.$$

All in all, in terms of $(E, H)$, Maxwell's equations read

$$\left(\partial_t \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} J_0 \\ 0 \end{pmatrix}.$$

For the time being, we shall assume that there exists $c > 0$ and $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ we have

$$\nu\varepsilon(x) + \operatorname{Re}\sigma(x) \geqslant c, \quad \mu(x) \geqslant c \quad (x \in \Omega)$$

in the sense of positive definiteness. Note that the latter condition allows particularly for $\varepsilon = 0$ on certain regions, if $\operatorname{Re}\sigma$ compensates for this. This situation is referred to as the eddy current approximation in these regions. With the above preparations at hand, we may now formulate the well-posedness result concerning Maxwell's equations.

**Theorem 6.2.8.** *Let $\Omega \subseteq \mathbb{R}^3$ be open and $\nu \geqslant \nu_0$. Then*

$$\partial_{t,\nu} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}$$

*is densely defined and closable in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega)^3 \times L_2(\Omega)^3\right)$. The respective closure is continuously invertible with causal inverse being eventually independent of $\nu$.*

*Proof.* The assertion follows from Theorem 6.2.1 applied to the material law

$$M(z) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + z^{-1} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and the skew-selfadjoint operator

$$A = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}. \qquad \square$$

*Remark* 6.2.9. In the physics literature (see e.g. [FLS64, Chapter 18]), Maxwell's equations are usually complemented by Gauss' law,

$$\operatorname{div}_0 B = 0,$$

as well as the introduction of the charge density, $\rho = \operatorname{div} \varepsilon E$, and the current, $J = J_0 - J_c$, by the continuity equation

$$\partial_t \rho = \operatorname{div} J.$$

We shall argue in the following that these equations are *automatically* satisfied if $(E, H)$ is a solution to Maxwell's equation. Indeed, assuming $J_0 \in \operatorname{dom}(\partial_{t,\nu})$, then, as a consequence of Theorem 6.2.1, we have that

$$\begin{pmatrix} E \\ H \end{pmatrix} = \left(\overline{\partial_{t,\nu} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} J_0 \\ 0 \end{pmatrix}$$

$$\in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}\left(\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix}\right).$$

Reformulating the latter equation yields

$$\begin{aligned} B = \mu H &= -\partial_{t,\nu}^{-1} \operatorname{curl}_0 E, \\ \varepsilon E &= \partial_{t,\nu}^{-1} \left(-\sigma E + J_0 + \operatorname{curl} H\right) = \partial_{t,\nu}^{-1} J + \partial_{t,\nu}^{-1} \operatorname{curl} H. \end{aligned}$$

Since $\operatorname{curl}_0 E \in \operatorname{ran}(\operatorname{curl}_0)$, we have by Proposition 3.1.7 that $\partial_{t,\nu}^{-1} \operatorname{curl}_0 E \in \overline{\operatorname{ran}}(\operatorname{curl}_0)$. Thus, by Proposition 6.1.5, we obtain

$$\operatorname{div}_0 B = \operatorname{div}_0 \left(-\partial_{t,\nu}^{-1} \operatorname{curl}_0 E\right) = 0.$$

Similarly, we deduce that

$$\rho = \operatorname{div} \varepsilon E = \operatorname{div} \partial_{t,\nu}^{-1} J.$$

If, in addition, we have that $J \in \operatorname{dom}(\operatorname{div})$, we recover the continuity equation. In general, the continuity equation is satisfied in the integrated sense just derived.

We shall keep the list of examples to that for now. In the course of this internet seminar, we will see more (involved) examples. Furthermore, we will study the boundary conditions more deeply and shall relate the conditions introduced abstractly here to more classical formulations involving trace spaces.

## 6.3 Proof of Picard's Theorem

In this section we shall prove the well-posedness theorem. For this, we recall an elementary result from functional analysis. It is remindful of the Lax–Milgram lemma.

**Proposition 6.3.1.** *Let $H$ be a Hilbert space and $B\colon \operatorname{dom}(B) \subseteq H \to H$ densely defined and closed with $\operatorname{dom}(B) \supseteq \operatorname{dom}(B^*)$. Assume there exists $c > 0$ such that*

$$\operatorname{Re}\langle \phi, B\phi\rangle_H \geqslant c\|\phi\|_H^2 \quad (\phi \in \operatorname{dom}(B)).$$

*Then $B^{-1} \in L(H)$ and $\left\|B^{-1}\right\| \leqslant 1/c$.*

*Proof.* The proof is a refinement of the argument in Proposition 6.2.3. In fact, the assumed inequality implies closedness of the range of $B$ as well as continuous invertibility with $B^{-1}\colon \operatorname{ran}(B) \to H$. The fact that $\operatorname{ran}(B)$ is dense in $H$ follows from the fact that $\operatorname{Re}\langle \phi, B^*\phi\rangle_H \geqslant c\|\phi\|_H^2$ for all $\phi \in \operatorname{dom}(B^*) \subseteq \operatorname{dom}(B)$ which, in turn, also follows from the assumed inequality. □

*Proof of Theorem 6.2.1.* Let $\nu \geqslant \nu_0$ and $z \in \mathbb{C}_{\operatorname{Re}\geqslant\nu}$. Define $B(z) \coloneqq zM(z) + A$. Since $M(z) \in L(H)$ it follows from Theorem 2.3.2 that $B(z)^* = (zM(z))^* - A$ and $\operatorname{dom}(B(z)) = \operatorname{dom}(B(z)^*) = \operatorname{dom}(A)$. Moreover, for all $\phi \in \operatorname{dom}(A)$ we have

$$\operatorname{Re}\langle \phi, B(z)\phi\rangle_H = \operatorname{Re}\langle \phi, (zM(z) + A)\,\phi\rangle_H = \operatorname{Re}\langle \phi, zM(z)\phi\rangle_H \geqslant c\|\phi\|_H^2,$$

due to the skew-selfadjointness of $A$. Thus, by Proposition 6.3.1 applied to $B(z)$ instead of $B$, we deduce that

$$S\colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \ni z \mapsto B(z)^{-1}$$

is bounded and assumes values in $L(H)$ with norm bounded by $1/c$. By Exercise 6.5, we have that $S$ is holomorphic. Thus, $S$ is a material law and $\|S(\partial_{t,\nu})\| \leqslant 1/c$ by Proposition 5.3.2. Moreover, Theorem 5.3.6 implies that $S(\partial_{t,\nu})$ is independent of $\nu$ and causal.
Next, if $f \in \operatorname{dom}(\partial_{t,\nu})$, it follows that $(\mathrm{im}+\nu)\,\mathcal{L}_\nu f \in L_2(\mathbb{R};H)$. Hence, for all $t\in\mathbb{R}$ we obtain

$$\begin{aligned} AS(\mathrm{i}t+\nu)\mathcal{L}_\nu f(t) &= A\big((\mathrm{i}t+\nu)\,M(\mathrm{i}t+\nu)+A\big)^{-1}\mathcal{L}_\nu f(t)\\ &= \mathcal{L}_\nu f(t) - (\mathrm{i}t+\nu)\,M(\mathrm{i}t+\nu)S(\mathrm{i}t+\nu)\mathcal{L}_\nu f(t). \end{aligned}$$

Thus, by the boundedness of $M$ and $S$, we deduce $S(\mathrm{i}\cdot+\nu)\mathcal{L}_\nu f \in L_2(\mathbb{R};\operatorname{dom}(A))$. This implies $S(\partial_{t,\nu})f \in L_{2,\nu}(\mathbb{R};\operatorname{dom}(A))$ by Exercise 6.2. Similarly, but more easily, it follows that $(\mathrm{i}\cdot+\nu)\,S(\mathrm{i}\cdot+\nu)\mathcal{L}_\nu f \in L_2(\mathbb{R};H)$ also, which shows $S(\partial_{t,\nu})f \in \operatorname{dom}(\partial_{t,\nu})$.
We now define the operator $B(\mathrm{im}+\nu)$ by

$$\begin{aligned} \operatorname{dom}(B(\mathrm{im}+\nu)) \coloneqq \big\{f \in L_2(\mathbb{R};H)\,;\, &f(t)\in\operatorname{dom}(A) \text{ for a.e. } t\in\mathbb{R},\\ &(t\mapsto B(\mathrm{i}t+\nu)f(t)) \in L_2(\mathbb{R};H)\big\} \end{aligned}$$

and

$$B(\mathrm{im}+\nu)f \coloneqq (t\mapsto B(\mathrm{i}t+\nu)f(t)) \quad (f\in\operatorname{dom}(B(\mathrm{im}+\nu))).$$

Then one easily sees that $B(\mathrm{im}+\nu) = S(\mathrm{im}+\nu)^{-1}$ and since $S(\mathrm{im}+\nu)$ is closed, it follows that $B(\mathrm{im}+\nu)$ is closed as well. Moreover

$$(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A \subseteq B(\mathrm{im}+\nu)$$

and hence, the operator $(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A$ is closable, which also yields the closability of $\partial_{t,\nu}M(\partial_{t,\nu})+A$ by unitary equivalence. To complete the proof, we have to show that

$$\overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A} = B(\mathrm{im}+\nu),$$

as this equality would imply $S(\partial_{t,\nu}) = \big(\overline{\partial_{t,\nu}M(\partial_{t,\nu})+A}\big)^{-1}$ by unitary equivalence. For showing the asserted equality, let $f\in\operatorname{dom}(B(\mathrm{im}+\nu))$. For $n\in\mathbb{N}$ we define $f_n \coloneqq \mathbb{1}_{[-n,n]}f$. Then $f_n \in \operatorname{dom}(\mathrm{im}+\nu)\cap\operatorname{dom}(A) \subseteq \operatorname{dom}\big((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\big)$ for each $n\in\mathbb{N}$ and by dominated convergence, we have that $f_n\to f$ as $n\to\infty$ as well as

$$\big((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\big)f_n = B(\mathrm{im}+\nu)f_n = \mathbb{1}_{[-n,n]}B(\mathrm{im}+\nu)f \to B(\mathrm{im}+\nu)f \quad (n\to\infty).$$

This shows that $f\in\operatorname{dom}\big(\overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A}\big)$ and hence, the assertion follows. □

*Remark* 6.3.2. Note that Theorem 6.2.1 can partly be generalised in the following way (with the same proof). Let $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0}\to L(H)$ be holomorphic and $A$ a closed, densely defined operator in $H$ such that $zM(z)+A$ is boundedly invertible for all $z\in\mathbb{C}_{\mathrm{Re}>\nu_0}$ and that $\sup_{z\in\mathbb{C}_{\mathrm{Re}>\nu_0}}\big\|(zM(z)+A)^{-1}\big\|_{L(H)} < \infty$. Then $S_\nu \in L(L_{2,\nu}(\mathbb{R};H))$ is causal and eventually independent of $\nu$.

*Remark* 6.3.3. As the proof of Theorem 6.2.1 shows, for $\nu \geqslant \nu_0$ we have that $S\colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \ni z \mapsto (zM(z) + A)^{-1} \in L(H)$ is a material law and $S_\nu = S(\partial_{t,\nu})$. Thus, the solution operator is a material law operator, and by Remark 5.3.3 applied to $S$ and $z \mapsto \frac{1}{z}1_H$ we obtain

$$S_\nu \partial_{t,\nu} \subseteq \partial_{t,\nu} S_\nu.$$

## 6.4 Comments

The proof of Theorem 6.2.1 here is rather close to the strategy originally employed in [Pic09], at least where existence and uniqueness are concerned. The causality part is a consequence of some observations detailed in [Kal+14; Wau15]. The original process of proving causality used the Theorem of Paley and Wiener, which we shall discuss later on.

The eddy current approximation has enjoyed great interest in the mathematical and physical community, in particular for the case when $\varepsilon = 0$ everywhere. The reason being that then Maxwell's equations are merely of parabolic type. We shall refer to [Pau+18] and the references therein for an extensive discussion.

Both Proposition 6.3.1 and the Lax–Milgram lemma have been put into a general perspective in [PTW15].

## Exercises

**Exercise 6.1.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $H_0, H_1$ be Hilbert spaces. Let $A\colon \operatorname{dom}(A) \subseteq H_0 \to H_1$ be densely defined and closed. Show that the operator

$$\begin{aligned} A_\mu\colon L_2(\mu; \operatorname{dom}(A)) \subseteq L_2(\mu; H_0) &\to L_2(\mu; H_1) \\ f &\mapsto \big(\omega \mapsto Af(\omega)\big) \end{aligned}$$

is densely defined and closed. Moreover, show that $(A_\mu)^* = (A^*)_\mu$.

**Exercise 6.2.** In the situation of Exercise 6.1, if $(\Omega_1, \Sigma_1, \mu_1)$ is another $\sigma$-finite measure space and $\mathcal{F}\colon L_2(\mu) \to L_2(\mu_1)$ is unitary, show that for $j \in \{0, 1\}$ there exists a unique unitary operator $\mathcal{F}_{H_j}\colon L_2(\mu; H_j) \to L_2(\mu_1; H_j)$ such that

$$\mathcal{F}_{H_j}(\phi x) = (\mathcal{F}\phi)x \quad (\phi \in L_2(\mu), x \in H_j).$$

Furthermore, prove that

$$\mathcal{F}_{H_1} A_\mu \mathcal{F}_{H_0}^* = A_{\mu_1}.$$

**Exercise 6.3.** Show that for $\Omega \subseteq \mathbb{R}^d$ open, the set $C_c^\infty(\Omega) \subseteq L_2(\Omega)$ is dense.

**Exercise 6.4.** Prove Theorem 6.1.2.

**Exercise 6.5.** Let $H$ be a Hilbert space, $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint, and $c > 0$. Moreover, let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be holomorphic with

$$\operatorname{Re} M(z) \geqslant c \quad (z \in \operatorname{dom}(M)).$$

Show that $\operatorname{dom}(M) \ni z \mapsto (M(z) + A)^{-1}$ is holomorphic.

**Exercise 6.6.** Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be a densely defined and closed linear operator acting in Hilbert spaces $H_0$ and $H_1$. For $\nu > 0$ show that

$$\overline{\partial_{t,\nu}^{-1} C} = C \partial_{t,\nu}^{-1}.$$

Hint: Apply Exercise 6.2 and show $\overline{(\mathrm{im} + \nu)^{-1} C} = C(\mathrm{im} + \nu)^{-1}$ with a suitable approximation argument.

**Exercise 6.7.** (a) Compute $H_0^1(\Omega)^{\perp}$ where the orthogonal complement is computed in $H^1(\Omega)$.

(b) Assume that

$$D := \left\{\phi \in H^1(\Omega)\,;\, \operatorname{grad} \phi \in \operatorname{dom}(\operatorname{div}), \phi = \operatorname{div} \operatorname{grad} \phi\right\} \subseteq C^{\infty}(\Omega).$$

and show that $C^{\infty}(\Omega) \cap H^1(\Omega) \subseteq H^1(\Omega)$ is dense.

Remark: The regularity assumption in (b) always holds and is known as Weyl's Lemma, see e.g. [GT83, Corollary 8.11], where the more general situation of an elliptic operator with smooth coefficients is treated. See also [Don69, p.127], where the regularity is shown for harmonic distributions.

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# 7 Examples of Evolutionary Equations

This chapter is devoted to a small tour through a variety of evolutionary equations. More precisely, we shall look into the equations of poro-elastic media, (time-)fractional elasticity, thermodynamic media with delay as well as visco-elastic media. The discussion of these examples will be similar to that of the examples in the previous chapter in the sense that we shall present the equations first, reformulate them suitably and then apply the solution theory to them. The study of visco-elastic media within the framework of partial integro-differential equations will be carried out in the exercises section.

## 7.1 Poro-Elastic Deformations

In this section we will discuss the equations of poro-elasticity, which form a coupled system of equations. More precisely, the equations of (linearised) elasticity are coupled with the diffusion equation. Before properly writing these equations we introduce the following notation and differential operators.

**Definition.** Let $\mathbb{K}^{d\times d}_{\mathrm{sym}} \coloneqq \left\{A \in \mathbb{K}^{d\times d}\,;\, A = A^\top\right\} \subseteq \mathbb{K}^{d\times d}$ the (closed) subspace of symmetric $d \times d$ matrices. Let $\Omega \subseteq \mathbb{R}^d$ be open. Then define

$$\begin{aligned} L_2(\Omega)^{d\times d}_{\mathrm{sym}} &\coloneqq L_2(\Omega;\mathbb{K}^{d\times d}_{\mathrm{sym}}) \\ &= \left\{(\Phi_{jk})_{j,k\in\{1,\ldots,d\}} \in L_2(\Omega)^{d\times d}\,;\, \forall j,k \in \{1,\ldots,d\} \colon \Phi_{jk} = \Phi_{kj}\right\}. \end{aligned}$$

Analogously, we set $C_{\mathrm{c}}^\infty(\Omega)^{d\times d}_{\mathrm{sym}} \coloneqq C_{\mathrm{c}}^\infty(\Omega;\mathbb{K}^{d\times d}_{\mathrm{sym}})$.

Note that the symmetry of a $d \times d$ matrix here means that the matrix elements are symmetric with respect to the main diagonal. For $\mathbb{K} = \mathbb{C}$, this does not correspond to the symmetry of the associated linear operator (which would rather be selfadjointness).

**Definition.** Let $\Omega \subseteq \mathbb{R}^d$ be open. Then we define

$$\begin{aligned} \mathrm{Grad}_{\mathrm{c}} \colon C_{\mathrm{c}}^\infty(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega)^{d\times d}_{\mathrm{sym}} \\ (\phi_j)_{j\in\{1,\ldots,d\}} &\mapsto \frac{1}{2}\left(\partial_k\phi_j + \partial_j\phi_k\right)_{j,k\in\{1,\ldots,d\}}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Div}_{\mathrm{c}} \colon C_{\mathrm{c}}^\infty(\Omega)^{d\times d}_{\mathrm{sym}} \subseteq L_2(\Omega)^{d\times d}_{\mathrm{sym}} &\to L_2(\Omega)^d \\ (\Phi_{jk})_{j,k\in\{1,\ldots,d\}} &\mapsto \left(\sum_{k=1}^d \partial_k\Phi_{jk}\right)_{j\in\{1,\ldots,d\}}. \end{aligned}$$

Similarly to the definitions in the previous chapter, we put $\operatorname{Grad} \coloneqq -\operatorname{Div}_c^*$, $\operatorname{Div} \coloneqq -\operatorname{Grad}_c^*$ and $\operatorname{Grad}_0 \coloneqq -\operatorname{Div}^*$, $\operatorname{Div}_0 \coloneqq -\operatorname{Grad}^*$, where (analogously to the scalar-valued case) we observe that $\operatorname{Grad}_c \subseteq -\operatorname{Div}_c^*$ motivating the notation Grad and $\operatorname{Grad}_0$.

*Remark* 7.1.1. Note that in the literature $\operatorname{Grad} u$ is also denoted by $\varepsilon(u)$ and is called the *strain tensor*. Due to the (obvious) similarity to the scalar case, we refrain from using $\varepsilon$ in this context and prefer Grad instead. Again, the index 0 in the operators refers to generalised Dirichlet (for $\operatorname{Grad}_0$) or Neumann (for $\operatorname{Div}_0$) boundary conditions.

We are now properly equipped to formulate the equations of poro-elasticity; see also [MC96] and below for further details. In an elastic body $\Omega \subseteq \mathbb{R}^d$, the displacement field, $u \colon \mathbb{R} \times \Omega \to \mathbb{R}^d$, and the pressure field, $p \colon \mathbb{R} \times \Omega \to \mathbb{R}$, of a fluid diffusing through $\Omega$ satisfy the following two energy balance equations

$$\begin{aligned} \partial_t \rho \partial_t u - \operatorname{grad} \partial_t \lambda \operatorname{div} u - \operatorname{Div} C \operatorname{Grad} u + \operatorname{grad} \alpha^* p &= f, \\ \partial_t (c_0 p + \alpha \operatorname{div} u) - \operatorname{div} k \operatorname{grad} p &= g. \end{aligned}$$

The right-hand sides $f \colon \mathbb{R} \times \Omega \to \mathbb{R}^d$ and $g \colon \mathbb{R} \times \Omega \to \mathbb{R}$ describe some given external forcing. We assume homogeneous Neumann boundary conditions for the diffusing fluid as well as homogeneous Dirichlet (i.e. clamped) boundary conditions for the elastic body. The operator $\rho \in L(L_2(\Omega)^d)$ describes the density of the medium $\Omega$ (usually realised as a multiplication operator by a bounded, measurable, scalar function). The bounded linear operators $C \in L(L_2(\Omega)^{d\times d}_{\mathrm{sym}})$ and $k \in L(L_2(\Omega)^d)$ are the elasticity tensor and the hydraulic conductivity of the medium, whereas $c_0, \lambda \in L(L_2(\Omega))$ are the porosity of the medium and the compressibility of the fluid, respectively. The operator $\alpha \in L(L_2(\Omega))$ is the so-called Biot–Willis constant. Note that in many applications $\rho, c_0, \lambda$ and $\alpha$ are just positive real numbers, and $C$ and $k$ are strictly positive definite tensors or matrices. The reformulation of the equations for poro-elasticity involves several 'tricks'. One of these is to introduce the matrix trace as the operator

$$\begin{aligned} \operatorname{trace} \colon L_2(\Omega)^{d\times d}_{\mathrm{sym}} &\to L_2(\Omega) \\ (\Phi_{jk})_{j,k\in\{1,\ldots,d\}} &\mapsto \sum_{j=1}^{d} \Phi_{jj}. \end{aligned}$$

Note that the adjoint is given by $\operatorname{trace}^* f = \operatorname{diag}(f, f, \ldots, f) \in L_2(\Omega)^{d\times d}_{\mathrm{sym}}$. It is then elementary to obtain $\operatorname{trace} \operatorname{Grad} \subseteq \operatorname{div}$ as well as $\operatorname{grad} = \operatorname{Div} \operatorname{trace}^*$. Hence, we get formally

$$\begin{aligned} \partial_t \rho \partial_t u - \operatorname{Div}\big((\partial_t \operatorname{trace}^* \lambda \operatorname{trace} + C) \operatorname{Grad} u - \operatorname{trace}^* \alpha^* p\big) &= f, \\ \partial_t \left(c_0 p + \alpha \operatorname{trace} \operatorname{Grad} u\right) - \operatorname{div} k \operatorname{grad} p &= g. \end{aligned}$$

Next, we introduce a new set of unknowns

$$\begin{aligned} v &\coloneqq \partial_t u, \\ T &\coloneqq C \operatorname{Grad} u, \end{aligned}$$

$$\begin{aligned}\omega &:= \lambda \operatorname{trace} \operatorname{Grad} v - \alpha^* p,\\ q &:= -k \operatorname{grad} p.\end{aligned}$$

Here, $v$ is the velocity, $T$ is the stress tensor and $q$ is the heat flux. The quantity $\omega$ is an additional variable, which helps to rewrite the system into the form of evolutionary equations.
In order to finalise the reformulation we shall assume some additional properties on the coefficients involved. Throughout the rest of this section, we assume that

$$\begin{aligned}\rho = \rho^* &\geqslant c,\\ c_0 = c_0^* &\geqslant c,\\ \operatorname{Re} \lambda &\geqslant c,\\ \operatorname{Re} k &\geqslant c, \text{ and}\\ C = C^* &\geqslant c\end{aligned}$$

for some $c > 0$, where all inequalities are thought of in the sense of positive definiteness (compare Chapter 6). As a consequence, we obtain

$$\operatorname{trace} \operatorname{Grad} v = \lambda^{-1}\omega + \lambda^{-1}\alpha^* p.$$

Rewriting the defining equations for $T$, $\omega$, and $q$ together with the two equations we started out with, we obtain the system

$$\begin{aligned}\partial_t \rho v - \operatorname{Div}(T + \operatorname{trace}^* \omega) &= f,\\ \partial_t c_0 p + \alpha\lambda^{-1}\omega + \alpha\lambda^{-1}\alpha^* p + \operatorname{div} q &= g,\\ \lambda^{-1}\omega + \lambda^{-1}\alpha^* p - \operatorname{trace} \operatorname{Grad} v &= 0,\\ \partial_t C^{-1} T - \operatorname{Grad} v &= 0,\\ k^{-1} q + \operatorname{grad} p &= 0.\end{aligned}$$

Note that at this stage of modelling we did assume that we can freely interchange the order of differenation, so that $\operatorname{Grad} \partial_t u = \partial_t \operatorname{Grad} u$. Introducing

$$M_0 := \begin{pmatrix} \rho & 0 & 0 & 0 & 0\\ 0 & c_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & C^{-1} & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & \alpha\lambda^{-1}\alpha^* & \alpha\lambda^{-1} & 0 & 0\\ 0 & \lambda^{-1}\alpha^* & \lambda^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & k^{-1} \end{pmatrix}, \tag{7.1}$$

$$V := \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & \operatorname{trace} & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 0 & 0 & -\operatorname{Div} & 0\\ 0 & 0 & 0 & 0 & \operatorname{div}_0\\ 0 & 0 & 0 & 0 & 0\\ -\operatorname{Grad}_0 & 0 & 0 & 0 & 0\\ 0 & \operatorname{grad} & 0 & 0 & 0 \end{pmatrix}, \tag{7.2}$$

we obtain

$$(\partial_t M_0 + M_1 + VAV^*)\begin{pmatrix} v \\ p \\ \omega \\ T \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This perspective enables us to prove well-posedness for the equations of poro-elasticity by applying Theorem 6.2.1.

**Theorem 7.1.2.** *Put $H := L_2(\Omega)^d \times L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)^{d\times d}_{\mathrm{sym}} \times L_2(\Omega)^d$ and let $M_0, M_1, V \in L(H)$ and $A$ be given as in (7.1) and (7.2). Then there exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the operator $\overline{\partial_{t,\nu} M_0 + M_1 + VAV^*}$ is continuously invertible on $L_{2,\nu}(\mathbb{R}; H)$. The inverse $S_\nu$ of this operator is causal and eventually independent of $\nu$. Moreover, $\sup_{\nu\geqslant\nu_0} \|S_\nu\| < \infty$ and $F \in \operatorname{dom}(\partial_{t,\nu})$ implies $S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(VAV^*)$.*

We will provide two prerequisites for the proof. We ask for the details of the proof in Exercise 7.1.

**Proposition 7.1.3.** *Let $H_0$, $H_1$ be Hilbert spaces, $B\colon \operatorname{dom}(B) \subseteq H_0 \to H_0$ skew-selfadjoint, $V \in L(H_0, H_1)$ bijective. Then $(VBV^*)^* = -VBV^*$.*

The proof of Proposition 7.1.3 is left as (part of) Exercise 7.1.

**Proposition 7.1.4.** *Let $H$ be a Hilbert space, $N_0, N_1 \in L(H)$ with $N_0 = N_0^*$. Assume there exist $c_0, c_1 > 0$ such that $\langle x, N_0 x\rangle \geqslant c_0\|x\|^2$ for all $x \in \operatorname{ran}(N_0)$ and $\operatorname{Re}\langle y, N_1 y\rangle \geqslant c_1\|y\|^2$ for all $y \in \ker(N_0)$. Then for all $0 < c_1' < c_1$ there exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ we have that*

$$\nu N_0 + \operatorname{Re} N_1 \geqslant c_1'.$$

*Proof.* Note that by the selfadjointness of $N_0$ we can decompose $H = \overline{\operatorname{ran}}(N_0) \oplus \ker(N_0)$, see Corollary 2.2.6. Let $z \in H$, and $x \in \overline{\operatorname{ran}}(N_0)$, $y \in \ker(N_0)$ such that $z = x + y$. For $\varepsilon, \nu > 0$ we estimate

$$\begin{aligned}
&\nu\langle x+y, N_0(x+y)\rangle + \operatorname{Re}\langle x+y, N_1(x+y)\rangle \\
&= \nu\langle x, N_0 x\rangle + \operatorname{Re}\langle y, N_1 y\rangle + \operatorname{Re}\langle x, N_1 x\rangle + \operatorname{Re}\langle x, N_1 y\rangle + \operatorname{Re}\langle y, N_1 x\rangle \\
&\geqslant \nu c_0\|x\|^2 + c_1\|y\|^2 - \|N_1\|\|x\|^2 - 2\|N_1\|\|x\|\|y\| \\
&\geqslant \nu c_0\|x\|^2 + c_1\|y\|^2 - \|N_1\|\|x\|^2 - \frac{1}{\varepsilon}\|N_1\|^2\|x\|^2 - \varepsilon\|y\|^2 \\
&= \left(\nu c_0 - \frac{1}{\varepsilon}\|N_1\|^2 - \|N_1\|\right)\|x\|^2 + (c_1 - \varepsilon)\|y\|^2,
\end{aligned}$$

where we have used the Peter–Paul inequality (i.e., Young's inequality for products of non-negative numbers). For $0 < c_1' < c_1$ we find $\varepsilon > 0$ such that $c_1 - \varepsilon > c_1'$. Then we choose $\nu > \frac{1}{c_0}\left(c_1' + \frac{1}{\varepsilon}\|N_1\|^2 + \|N_1\|\right)$. With this choice of $\nu_0$ we deduce for all $\nu \geqslant \nu_0$ that

$$\nu\langle z, N_0 z\rangle + \operatorname{Re}\langle z, N_1 z\rangle \geqslant c_1'\left(\|x\|^2 + \|y\|^2\right) = c_1'\|z\|^2,$$

which yields the assertion. □

## 7.2 Fractional Elasticity

Let $\Omega \subseteq \mathbb{R}^d$ be open. In order to better fit to the experimental data of visco-elastic solids (i.e., to incorporate solids that 'memorise' previous force applied to them) the equations of linearised elasticity need to be extended in some way. The balance law for the momentum, however, is still satisfied; that is, for the displacement $u\colon \mathbb{R} \times \Omega \to \mathbb{R}^d$ we still have that

$$\partial_t \rho \partial_t u - \operatorname{Div} T = f,$$

where $\rho \in L(L_2(\Omega)^d)$ models the density and $f\colon \mathbb{R} \times \Omega \to \mathbb{R}^d$ is a given external forcing term. The stress tensor, $T\colon \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, does **not** follow the classical Hooke's law, which, if it did, would look like

$$T = C \operatorname{Grad} u$$

for $C \in L(L_2(\Omega)^{d\times d}_{\mathrm{sym}})$. Instead it is amended by another material dependent coefficient $D \in L(L_2(\Omega)^{d\times d}_{\mathrm{sym}})$ and a fractional time derivative; that is,

$$T = C \operatorname{Grad} u + D\partial_t^\alpha \operatorname{Grad} u,$$

for some $\alpha \in [0, 1]$, where $\partial_t^\alpha := \partial_t \partial_t^{\alpha-1}$, see Example 5.3.1(e). We shall simplify the present consideration slightly and refer to Exercise 7.2 instead for a more involved example. Throughout this section, we shall assume that

$$C = 0, \ D = D^* \geqslant c, \ \text{and } \rho = \rho^* \geqslant c$$

for some $c > 0$. Thus, putting $v := \partial_t u$ and assuming the clamped boundary conditions again, we study well-posedness of

$$\partial_t \rho v - \operatorname{Div} T = f, \tag{7.3}$$

$$T = D\partial_t^\alpha \operatorname{Grad}_0 u. \tag{7.4}$$

In order to do that, we first rewrite the second equation. We will make use of the following proposition which will serve us to show bounded invertibility of $\partial_t^\alpha$ (in the space $L_{2,\nu}$), and which will also be employed to obtain well-posedness.

**Proposition 7.2.1.** *Let $\nu > 0$, $z \in \mathbb{C}_{\mathrm{Re}\geqslant \nu}$, $\alpha \in [0, 1]$. Then*

$$\operatorname{Re} z^\alpha \geqslant (\operatorname{Re} z)^\alpha \geqslant \nu^\alpha.$$

*Proof.* Let us prove the first inequality. Note that without loss of generality, we may assume that $\operatorname{Re} z = 1$. Let $\varphi := \arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $\ln \circ \cos$ is concave on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (as $(\ln \circ \cos)' = -\tan$ is decreasing) and $(\ln \circ \cos)(0) = 0$, we obtain

$$\ln\cos(\alpha\varphi) = \ln\cos(\alpha\varphi + (1-\alpha)0) \geqslant \alpha \ln\cos(\varphi) + (1-\alpha)\ln\cos(0) = \ln\bigl(\cos(\varphi)^\alpha\bigr),$$

and therefore $\cos(\alpha\varphi) \geqslant \cos(\varphi)^\alpha$. Since $\operatorname{Re} z = 1$ implies $|z| = \frac{1}{\cos(\varphi)}$, we obtain

$$\operatorname{Re} z^\alpha = \frac{\cos(\alpha\varphi)}{(\cos\varphi)^\alpha} \geqslant 1 = (\operatorname{Re} z)^\alpha.$$

The second inequality follows from monotonicity of $x \mapsto x^\alpha$. $\square$

Applying Proposition 7.2.1 and noting that $D$ is boundedly invertible we can reformulate (7.4) as

$$\partial_{t,\nu}^{-\alpha} D^{-1} T - \operatorname{Grad}_0 u = 0,$$

so that (7.4) and (7.3) read

$$\left(\partial_{t,\nu}\begin{pmatrix}\rho & 0\\ 0 & \partial_{t,\nu}^{-\alpha}D^{-1}\end{pmatrix} - \begin{pmatrix}0 & \operatorname{Div}\\ \operatorname{Grad}_0 & 0\end{pmatrix}\right)\begin{pmatrix}v\\ T\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

A solution theory for the latter equation, thus, reads as follows, where again $v \coloneqq \partial_{t,\nu} u$.

**Theorem 7.2.2.** *Put $H \coloneqq L_2(\Omega)^d \times L_2(\Omega)^{d\times d}_{\mathrm{sym}}$. Then for all $\nu > 0$ the operator*

$$\partial_{t,\nu}\begin{pmatrix}\rho & 0\\ 0 & \partial_{t,\nu}^{-\alpha}D^{-1}\end{pmatrix} - \begin{pmatrix}0 & \operatorname{Div}\\ \operatorname{Grad}_0 & 0\end{pmatrix}$$

*is densely defined and closable in $L_{2,\nu}(\mathbb{R}; H)$. The inverse of the closure is continuous, causal and eventually independent of $\nu$.*

*Proof.* The proof rests on Theorem 6.2.1. Since $\begin{pmatrix}0 & \operatorname{Div}\\ \operatorname{Grad}_0 & 0\end{pmatrix}$ is skew-selfadjoint by Proposition 6.2.3(a), it suffices to confirm the positive definiteness condition for the material law. For this let $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu}$ and compute for $x \in L_2(\Omega)^{d\times d}_{\mathrm{sym}}$, using Proposition 7.2.1 and Proposition 6.2.3(b),

$$\operatorname{Re}\left\langle x, z z^{-\alpha} D^{-1} x\right\rangle = \operatorname{Re}\left\langle x, z^{1-\alpha} D^{-1} x\right\rangle \geqslant \nu^{1-\alpha}\left\langle x, D^{-1}x\right\rangle \geqslant \nu^{1-\alpha}\frac{c}{\|D\|^2}\|x\|^2.$$

This yields the assertion. □

## 7.3 The Heat Equation with Delay

Let $\Omega \subseteq \mathbb{R}^d$ be open. In this section we concentrate on a generalisation of the heat equation discussed in the previous chapter. Although we keep the heat-flux-balance in the sense that

$$\partial_t\theta + \operatorname{div} q = Q,$$

with $q\colon \mathbb{R}\times\Omega \to \mathbb{R}^d$ being the heat flux and $\theta\colon \mathbb{R}\times\Omega\to\mathbb{R}$ being the heat, we shall now modify Fourier's law to the extent that

$$q = -a \operatorname{grad}\theta - b\tau_{-h}\operatorname{grad}\theta$$

for some $a, b \in L(L_2(\Omega)^d)$ with $\operatorname{Re} a \geqslant c$ for some $c > 0$, and $h > 0$. We shall again assume homogeneous Neumann boundary conditions for $q$. Written in the now standard block operator matrix form, this modified heat equation reads

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & (a+b\tau_{-h})^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_0\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix}Q\\ 0\end{pmatrix}.$$

In order to actually justify the existence of the operator $(a+b\tau_{-h})^{-1}$ as a bounded linear operator, we provide the following lemma.

**Lemma 7.3.1.** (a) *There exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the operator $a + b\tau_{-h}$ is continuously invertible in $L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d)$.*

(b) *For all $0 < c' < c/\|a\|^2$ there is $\nu_1 \geqslant \nu_0$ such that for all $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu_1}$ we have $\mathrm{Re}\left(a + b\mathrm{e}^{-zh}\right)^{-1} \geqslant c'$.*

*Proof.* Note that $a$ is invertible with $\left\|a^{-1}\right\| \leqslant \frac{1}{c}$ and $\mathrm{Re}\, a^{-1} \geqslant \frac{c}{\|a\|^2}$ by Proposition 6.2.3(b).

(a) By Example 5.3.4(c), for all $\nu > 0$ we obtain

$$\|b\tau_{-h}\|_{L(L_{2,\nu})} \leqslant \|b\|_{L(L_2(\Omega)^d)} \sup_{t\in\mathbb{R}} \left|\mathrm{e}^{-(it+\nu)h}\right| = \|b\|_{L(L_2(\Omega)^d)} \mathrm{e}^{-h\nu}.$$

Thus, we find $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ we obtain $\left\|b\tau_{-h}a^{-1}\right\|_{L(L_{2,\nu})} \leqslant \frac{1}{c}\|b\tau_{-h}\|_{L(L_{2,\nu})} < 1$. Thus,

$$a + b\tau_{-h} = \left(1 + b\tau_{-h}a^{-1}\right) a$$

is continuously invertible by a Neumann series argument.

(b) Let $0 < c' < c/\|a\|^2$. We choose $\nu_1 \geqslant \nu_0$ such that $\left\|b\mathrm{e}^{-zh}a^{-1}\right\|_{L(L_2(\Omega)^d)} \leqslant \min\{\frac{1}{2}, \varepsilon\}$ for all $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu_1}$, where $0 < \varepsilon \leqslant \frac{1}{2}c\left(\frac{c}{\|a\|^2} - c'\right)$. For $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu_1}$ we compute

$$\begin{aligned}
\mathrm{Re}\left(a + b\mathrm{e}^{-zh}\right)^{-1} &= \mathrm{Re}\, a^{-1}\left(1 + b\mathrm{e}^{-zh}a^{-1}\right)^{-1} = \mathrm{Re}\left(a^{-1}\sum_{k=0}^{\infty}\left(-b\mathrm{e}^{-zh}a^{-1}\right)^k\right) \\
&= \mathrm{Re}\left(a^{-1} + \sum_{k=1}^{\infty} a^{-1}\left(-b\mathrm{e}^{-zh}a^{-1}\right)^k\right) \\
&\geqslant \frac{c}{\|a\|^2} - \left\|\sum_{k=1}^{\infty} a^{-1}\left(-b\mathrm{e}^{-zh}a^{-1}\right)^k\right\| \\
&\geqslant \frac{c}{\|a\|^2} - \sum_{k=1}^{\infty}\left\|a^{-1}\left(-b\mathrm{e}^{-zh}a^{-1}\right)^k\right\| \\
&\geqslant \frac{c}{\|a\|^2} - \frac{1}{c}\sum_{k=1}^{\infty}\left\|\left(-b\mathrm{e}^{-zh}a^{-1}\right)\right\|^k \\
&= \frac{c}{\|a\|^2} - \frac{1}{c}\frac{\left\|\left(-b\mathrm{e}^{-zh}a^{-1}\right)\right\|}{1 - \left\|\left(-b\mathrm{e}^{-zh}a^{-1}\right)\right\|} \geqslant \frac{c}{\|a\|^2} - \frac{1}{c}2\varepsilon \geqslant c'.
\end{aligned}$$

□

With this lemma we are in the position to provide the well-posedness for the modified heat equation, as well.

**Theorem 7.3.2.** *Let $H = L_2(\Omega) \times L_2(\Omega)^d$. There exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the operator*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (a + b\tau_{-h})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}$$

*is densely defined and closable with continuously invertible closure in $L_{2,\nu}(\mathbb{R}; H)$. The inverse of the closure is causal and eventually independent of $\nu$.*

*Proof.* The proof rests on Theorem 6.2.1 and Lemma 7.3.1. □

## 7.4 Dual Phase Lag Heat Conduction

The last example is concerned with a different modification of Fourier's law. The heat-flux balance

$$\partial_t \theta + \operatorname{div} q = Q \tag{7.5}$$

is accompanied by the modified Fourier's law

$$\big(1 + s_q \partial_t + \frac{1}{2} s_q^2 \partial_t^2\big) q = -(1 + s_\theta \partial_t) \operatorname{grad} \theta, \tag{7.6}$$

where $s_q \in \mathbb{R}, s_\theta > 0$ are given numbers, which are called 'phases'.

*Remark* 7.4.1. The modified Fourier's law in (7.6) is an attempt to resolve the problem of inifinite propagation speed which stems from a truncated Taylor series expansion of a model given by

$$\tau_{s_q} q = -\tau_{s_\theta} \operatorname{grad} \theta.$$

Note that it can be shown that such a model would even be ill-posed, see [DQR09].

Let us turn back to the system (7.5) and (7.6). Notice, since $s_\theta > 0$, and due to a strictly positive real part of the derivative in our functional analytic setting, we deduce that $(1 + s_\theta \partial_{t,\nu})$ is continuously invertible for $\nu \geqslant 0$. Thus, we obtain

$$\partial_{t,\nu} \big(\partial_{t,\nu}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,\nu}\big)(1 + s_\theta \partial_{t,\nu})^{-1} q = -\operatorname{grad} \theta$$

The block operator matrix formulation of the dual phase lag heat conduction model is thus

$$\left( \partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & (\partial_{t,\nu}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,\nu})(1 + s_\theta \partial_{t,\nu})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

**Theorem 7.4.2.** *Let $H = L_2(\Omega) \times L_2(\Omega)^d$. Assume $s_q \in \mathbb{R} \setminus \{0\}$, $s_\theta > 0$. Then there exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the operator*

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & (\partial_{t,\nu}^{-1} + s_q + \frac{1}{2} s_q^2 \partial_{t,\nu})(1 + s_\theta \partial_{t,\nu})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}$$

*is densely defined and closable with continuously invertible closure in $L_{2,\nu}(\mathbb{R}; H)$. The inverse of the closure is causal and eventually independent of $\nu$.*

The proof of Theorem 7.4.2 is again based on Theorem 6.2.1. Thus, we shall only record the decisive observation in the next result. For this, we define

$$M(z) := \frac{z^{-1} + s_q + \frac{1}{2}s_q^2 z}{1 + s_\theta z} \in \mathbb{C} \quad (z \in \mathbb{C} \setminus \{0, -\tfrac{1}{s_\theta}\}).$$

**Lemma 7.4.3.** *Let $s_q \in \mathbb{R} \setminus \{0\}, s_\theta > 0$. Then there exists $\nu_0 \in \mathbb{R}$ and $c > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}$ we have*

$$\mathrm{Re}\, zM(z) \geqslant c.$$

*Proof.* We put $\sigma := \frac{s_q}{s_\theta}$. Let $z \in \mathbb{C} \setminus \{0, -\frac{1}{s_\theta}\}$. We compute

$$zM(z) = \frac{1 + s_q z + \frac{1}{2}s_q^2 z^2}{1 + s_\theta z} = \frac{1}{2}s_q z\sigma + \sigma\left(1 - \frac{1}{2}\sigma\right) + \frac{1 - \sigma\left(1 - \frac{1}{2}\sigma\right)}{1 + s_\theta z}$$

and therefore

$$\mathrm{Re}\, zM(z) = \frac{1}{2}s_q\sigma\, \mathrm{Re}\, z + \sigma\left(1 - \frac{1}{2}\sigma\right) + \frac{\left(1 - \sigma\left(1 - \frac{1}{2}\sigma\right)\right)\left(1 + s_\theta\, \mathrm{Re}\, z\right)}{\left|1 + s_\theta z\right|^2}.$$

By assumption

$$0 < \frac{s_q^2}{s_\theta} = s_q\sigma,$$

and since

$$\frac{\left(1 - \sigma\left(1 - \frac{1}{2}\sigma\right)\right)\left(1 + s_\theta\, \mathrm{Re}\, z\right)}{\left|1 + s_\theta z\right|^2} \to 0$$

as $\mathrm{Re}\, z \to \infty$, we obtain

$$\mathrm{Re}\, zM(z) \geqslant \frac{1}{2}s_q\sigma\, \mathrm{Re}\, z - \delta$$

for some $\delta > 0$ and all $z \in \mathbb{C}$ with $\mathrm{Re}\, z$ large enough. □

## 7.5 Comments

The equations of poro-elasticity have been proposed in [MC96] and were mathematically studied in [Sho00; MP10].

Equations of fractional elasticity are discussed in [PTW15; Wau14; CW18; NKS03]. The well-posedness conditions stated here and in Exercise 7.2 can be generalised as it is outlined in [PTW15] to the case where both $C$ and $D$ are non-negative, selfadjoint operators so that $C$ and $D$ satisfy the conditions imposed on $N_1$ and $N_0$ in Proposition 7.1.4. We refrained from presenting this argument here, as it seemed too technical for the time being. Note however that the proof is neither fundamentally different nor considerably less elementary.

The heat equation with delay has also been studied in [KPR15] with an entirely different strategy; the dual phase lag models have been dealt with in [Muk+16; Tzo95].

Other ideas to rectify infinite propagation speed of the heat equation can be found in [And+06], where nonlinear models for heat conduction are being discussed.

The visco-elastic equations discussed in Exercise 7.6 are studied with convolution operators more general than below in [Tro15]; see also [CS03; Tro18; Daf70; Prü09].

## Exercises

**Exercise 7.1.** (a) Prove Proposition 7.1.3.

(b) Prove Theorem 7.1.2.

(c) Let $\Omega \subseteq \mathbb{R}^d$ be open, $\nu > 0$, $f \in H^1_\nu(\mathbb{R}; L_2(\Omega)^d)$ and $g \in H^1_\nu(\mathbb{R}; L_2(\Omega))$. With the help of Theorem 7.1.2 show that for large enough $\nu > 0$ there exist a unique $u \in \operatorname{dom}\left(\partial^2_{t,\nu}\right) \cap \operatorname{dom}\left(\operatorname{grad} \lambda \operatorname{div} \partial_{t,\nu}\right) \cap \operatorname{dom}\left(\operatorname{Div} C \operatorname{Grad}_0\right)$ and $p \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\operatorname{grad} \alpha^*) \cap \operatorname{dom}(\operatorname{div}_0 k \operatorname{grad})$ such that

$$\begin{aligned}\partial_{t,\nu}\rho\partial_{t,\nu}u - \operatorname{grad}\lambda\operatorname{div}\partial_{t,\nu}u - \operatorname{Div}C\operatorname{Grad}_0 u + \operatorname{grad}\alpha^* p &= f\\ \partial_{t,\nu}c_0 p + \alpha\operatorname{div}\partial_{t,\nu}u - \operatorname{div}_0 k \operatorname{grad} p &= g.\end{aligned}$$

**Exercise 7.2.** Let $\Omega \subseteq \mathbb{R}^d$ be open, $C, D \in L(L_2(\Omega)^{d\times d}_{\mathrm{sym}})$, $D = D^* \geqslant c$ for some $c > 0$ and $\alpha \in [\frac{1}{2}, 1]$. Show that there exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the system

$$\begin{aligned}\partial_{t,\nu}\rho v - \operatorname{Div} T &= f,\\ T &= \left(C + D\partial^\alpha_{t,\nu}\right)\operatorname{Grad}_0 u,\end{aligned}$$

where $v = \partial_{t,\nu}u$, admits a unique solution $(v, T) \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d \times L_2(\Omega)^{d\times d}_{\mathrm{sym}})$ for all $f \in H^1_\nu(\mathbb{R}; L_2(\Omega)^d)$.

The following exercises are devoted to showing the well-posedness of certain equations in visco-elasticity, where the 'viscous part' is modelled by convolution with certain integral kernels. The proof of the required positive definiteness property requires some preliminary results. We assume the reader to be equipped with the basics from the theory of functions of one complex variable.
For $U \subseteq \mathbb{C}$ open, $u\colon U \to \mathbb{C}$ holomorphic, define $f_{\operatorname{Re} u}\colon \widetilde{U} \to \mathbb{R}$ by $f_{\operatorname{Re} u}(x, y) \coloneqq \operatorname{Re} u(x + \mathrm{i}y)$ for $(x, y) \in \widetilde{U} \coloneqq \left\{(x, y) \in \mathbb{R}^2\,;\, x + \mathrm{i}y \in U\right\}$. We put

$$H_{\mathrm{Re}}(U) \coloneqq \left\{f_{\operatorname{Re} u}\,;\, u\colon U \to \mathbb{C} \text{ holomorphic}\right\}.$$

**Exercise 7.3.** Let $U \subseteq \mathbb{C}$ be open.

(a) Let $f \in H_{\mathrm{Re}}(U)$. Show that $f$ satisfies the *mean value property*; that is, for all $(x, y) \in \widetilde{U}$ and $r > 0$ with $\overline{B\left((x, y), r\right)} \subseteq \widetilde{U}$ we have

$$f(x, y) = \frac{1}{2\pi}\int_0^{2\pi} f(x + r\cos\theta, y + r\sin\theta)\,\mathrm{d}\theta.$$

(b) Let $U \coloneqq \mathbb{C}_{\mathrm{Im}>0}$ and $f \in H_{\mathrm{Re}}(U) \cap C(\mathbb{R} \times \mathbb{R}_{\geqslant 0})$. Moreover, assume that $f(x, 0) = 0$ for each $x \in \mathbb{R}$ and $f(x, y) \to 0$ as $|(x, y)| \to \infty$. Show that $f = 0$ on $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$.

**Exercise 7.4.** In this exercise we show a version of *Poisson's formula*. Let $U \coloneqq \mathbb{C}_{\mathrm{Im}>0}$ and $f \in H_{\mathrm{Re}}(U) \cap C(\mathbb{R} \times \mathbb{R}_{\geqslant 0})$.

(a) Assume that $f(\cdot,0) \in L_p(\mathbb{R})$ for some $1 \leqslant p < \infty$. Show that $\mathbb{C}_{\mathrm{Im}>0} \ni z \mapsto \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathrm{Im}\, z' + \mathrm{i}(\mathrm{Re}\, z - x')}{(\mathrm{Re}\, z - x')^2 + (\mathrm{Im}\, z)^2} f(x',0)\,\mathrm{d}x'$ is holomorphic.

(b) Assume that $f(\cdot,0) \in L_\infty(\mathbb{R})$. Show that $\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-x')^2+y^2)} f(x',0)\,\mathrm{d}x' \to f(x_0,0)$ as $x \to x_0$ and $y \to 0+$.

(c) (Poisson's formula) Assume that $f(\cdot,0) \in L_p(\mathbb{R})$ for some $1 \leqslant p < \infty$ and $f(x,y) \to 0$ as $|(x,y)| \to \infty$ in $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$. Show that

$$f(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-x')^2+y^2} f(x',0)\,\mathrm{d}x' \quad ((x,y) \in \mathbb{R} \times \mathbb{R}_{>0}).$$

Hint: Apply Exercise 7.3(b).

**Exercise 7.5.** Let $\nu_0 \in \mathbb{R}$ and $k \in L_{1,\nu_0}(\mathbb{R};\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geqslant 0}$.

(a) Show that for all $(x,\nu) \in \mathbb{R} \times \mathbb{R}_{>\nu_0}$ we have

$$\mathrm{Im}(\mathcal{L}k)(\mathrm{i}x+\nu) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu-\nu_0}{(x-x')^2+(\nu-\nu_0)^2}\,\mathrm{Im}(\mathcal{L}k)(\mathrm{i}x'+\nu_0)\,\mathrm{d}x'.$$

Hint: Approximate $k$ by functions in $C_{\mathrm{c}}^\infty(\mathbb{R}_{\geqslant 0};\mathbb{R})$ and use Poisson's formula (see Exercise 7.4).

(b) Assume there exists $d \geqslant 0$ such that for all $x \in \mathbb{R}$

$$x\,\mathrm{Im}(\mathcal{L}k)(\mathrm{i}x+\nu_0) \leqslant d.$$

Show that for all $\nu \geqslant \nu_0$ and $x \in \mathbb{R}$ we have

$$x\,\mathrm{Im}(\mathcal{L}k)(\mathrm{i}x+\nu) \leqslant 4d.$$

Hint: Use the formula in (a) and split the integral into positive and negative part of $\mathbb{R}$; use symmetry of $(\mathcal{L}k)$ under conjugation due to the realness of $k$.

**Exercise 7.6.** Let $\Omega \subseteq \mathbb{R}^d$ be open, $\nu_0 \in \mathbb{R}$ and $k \in L_{1,\nu_0}(\mathbb{R};\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geqslant 0}$. Assume there exists $d \geqslant 0$ such that

$$x\,\mathrm{Im}(\mathcal{L}k)(\mathrm{i}x+\nu_0) \leqslant d \quad (x \in \mathbb{R}).$$

Show that there exists $\nu_1 \geqslant \nu_0$ such that for all $\nu \geqslant \nu_1$ the operator

$$\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & (1-k*)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{Div} \\ \operatorname{Grad}_0 & 0 \end{pmatrix}$$

is well-defined, densely defined and closable in $L_{2,\nu}(\mathbb{R};H)$ with $H = L_2(\Omega)^d \times L_2(\Omega)^{d\times d}_{\mathrm{sym}}$. Further, show that its closure is continuously invertible, and that the corresponding inverse is causal and eventually independent of $\nu$.

**Exercise 7.7.** Let $\nu_0 \in \mathbb{R}$ and $k \in L_{1,\nu_0}(\mathbb{R};\mathbb{R})$ with $\operatorname{spt} k \subseteq \mathbb{R}_{\geqslant 0}$.

(a) Assume that $k$ is absolutely continuous with $k' \in L_{1,\nu_0}(\mathbb{R};\mathbb{R})$. Show that there exist $\nu_1 \geqslant \nu_0$ and $d \geqslant 0$ with

$$x \operatorname{Im}(\mathcal{L}k)(\mathrm{i}x + \nu_1) \leqslant d \quad (x \in \mathbb{R}).$$

(b) Assume that $k(t) \geqslant 0$ for all $t \in \mathbb{R}$ and that $k(t) \leqslant k(s)$, whenever $s \leqslant t$. Show that there exist $\nu_1 \geqslant \nu_0$

$$x \operatorname{Im}(\mathcal{L}k)(\mathrm{i}x + \nu_1) \leqslant 0 \quad (x \in \mathbb{R}).$$

Hint: For part (b) use the explicit formula for $\operatorname{Im}(\mathcal{L}k)$ as an integral and the periodicity of sin.

NB: The condition in (a) is a standard assumption for convolution kernels in the framework of visco-elastic equations; the condition in (b) is from [Prü09].

## References

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# 8 Causality and a Theorem of Paley and Wiener

In this chapter we turn our focus back to causal operators. In Chapter 5 we found out that material laws provide a class of causal and autonomous bounded operators. In this chapter we will present another proof of this fact, which rests on a result which characterises functions in $L_2(\mathbb{R}; H)$ with support contained in the non-negative reals; the celebrated Theorem of Paley and Wiener. With the help of this theorem, which is interesting in its own right, the proof of causality for material laws becomes very easy. At a first glance it seems that holomorphy of a material law is a rather strong assumption. In the second part of this chapter, however, we shall see that in designing autonomous and causal solution operators, there is no way of circumventing holomorphy.
In the following, let $H$ be a Hilbert space, and we consider $L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ as the subspace of functions in $L_{2,\nu}(\mathbb{R}; H)$ vanishing on $(-\infty, 0)$.

## 8.1 A Theorem of Paley and Wiener

We start with the following lemma, for which we need the notion of locally integrable functions. We define

$$\begin{aligned} L_{1,\mathrm{loc}}(\mathbb{R}; H) &:= \{f \,;\, \forall\, K \subseteq \mathbb{R} \text{ compact} : \mathbb{1}_K f \in L_1(\mathbb{R}; H)\} \\ &= \{f \,;\, \forall\, \varphi \in C_\mathrm{c}^\infty(\mathbb{R}) : \varphi f \in L_1(\mathbb{R}; H)\}. \end{aligned}$$

**Lemma 8.1.1.** *Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}; H)$. Then we have $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ if and only if $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R}; H)$ with $\sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. In the latter case we have that*

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \lim_{\nu\to 0+} \|f\|_{L_{2,\nu}(\mathbb{R};H)} = \sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)}.$$

*Proof.* Let $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$ and $\nu > 0$. Then we estimate

$$\int_{\mathbb{R}} \|f(t)\|_H^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t = \int_{\mathbb{R}_{\geqslant 0}} \|f(t)\|_H^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant \int_{\mathbb{R}_{\geqslant 0}} \|f(t)\|_H^2\,\mathrm{d}t = \|f\|^2_{L_2(\mathbb{R}_{\geqslant 0};H)},$$

which proves that $f \in L_{2,\nu}(\mathbb{R}; H)$ with $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ for each $\nu > 0$. Moreover, $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \to \|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)}$ as $\nu \to 0$ by monotone convergence and since clearly $\|f\|_{L_{2,\nu}(\mathbb{R};H)} \leqslant \|f\|_{L_{2,\mu}(\mathbb{R};H)}$ for $0 < \mu \leqslant \nu$ we obtain

$$\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \lim_{\nu\to 0+} \|f\|_{L_{2,\nu}(\mathbb{R};H)} = \sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)}.$$

Assume now that $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R};H)$ with $C \coloneqq \sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. This inequality yields

$$\sup_{\nu\in[0,\infty)} \int_{(-\infty,0)} \|f(t)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant C.$$

Hence, the monotone convergence theorem yields that $g(t) \coloneqq \lim_{\nu\to\infty} \|f(t)\|^2 \mathrm{e}^{-2\nu t}$ for $t \in (-\infty,0)$ defines a function $g \in L_1(-\infty,0)$. Thus, $[g=\infty]$ is a set of measure zero and thus $[f=0] = (-\infty,0) \setminus [g=\infty]$ has full measure in $(-\infty,0)$ implying that $\operatorname{spt} f \subseteq \mathbb{R}_{\geqslant 0}$.
Finally, from

$$\sup_{\nu\in[0,\infty)} \int_{(0,\infty)} \|f(t)\|^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t \leqslant C.$$

we infer again by the monotone convergence theorem that $t \mapsto \lim_{\nu\to 0} \|f(t)\|^2 \mathrm{e}^{-2\nu t} = \|f(t)\|^2$ defines a function in $L_1(0,\infty)$, showing the remaining assertion. $\square$

For the proof of the Paley–Wiener theorem we need a suitable space of holomorphic functions on the right half-plane, the so-called *Hardy space* $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)$, which we introduce in the following.

**Definition.** For $\nu \in \mathbb{R}$ we define the Hardy space

$$\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H) \coloneqq \left\{ g\colon \mathbb{C}_{\mathrm{Re}>\nu} \to H\,;\, g \text{ holomorphic, } \sup_{\rho>\nu} \int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|_H^2\,\mathrm{d}t < \infty \right\}$$

and equip it with the norm $\|\cdot\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)}$ defined by

$$\|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)} \coloneqq \sup_{\rho>\nu} \left( \int_{\mathbb{R}} \|g(\mathrm{i}t+\rho)\|_H^2\,\mathrm{d}t \right)^{\frac{1}{2}}.$$

We motivate the Theorem of Paley–Wiener first. For this, let $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0};H)$ and define its *Laplace transform* as

$$\mathbb{C}_{\mathrm{Re}>\nu} \ni z \mapsto \mathcal{L}f(z) \coloneqq \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\mathrm{e}^{-zt}\,\mathrm{d}t. \tag{8.1}$$

Note that $\mathcal{L}f(z) = \mathcal{L}_{\mathrm{Re}\,z} f(\mathrm{Im}\,z)$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$ due to the support constraint on $f$. Moreover, it is not difficult to see that the integral on the right-hand side of (8.1) exists as $\big(t \mapsto \mathrm{e}^{-\rho t} f(t)\big) \in L_1(\mathbb{R}_{\geqslant 0};H) \cap L_2(\mathbb{R}_{\geqslant 0};H)$ for all $\rho > \nu$. Hence, $\mathcal{L}f\colon \mathbb{C}_{\mathrm{Re}>\nu} \to H$ is holomorphic (cf. Exercise 5.6). Moreover, by Lemma 8.1.1

$$\begin{aligned}\sup_{\rho>\nu} \|\mathcal{L}f(\mathrm{i}\cdot+\rho)\|_{L_2(\mathbb{R};H)} &= \sup_{\rho>\nu} \|\mathcal{L}_\rho f\|_{L_2(\mathbb{R};H)} = \sup_{\rho>\nu} \|f\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>0} \left\|\mathrm{e}^{-\nu\cdot} f\right\|_{L_{2,\rho}(\mathbb{R};H)} \\ &= \left\|\mathrm{e}^{-\nu\cdot} f\right\|_{L_2(\mathbb{R};H)} = \|f\|_{L_{2,\nu}(\mathbb{R};H)},\end{aligned}$$

which proves that

$$\begin{aligned}\mathcal{L}\colon L_{2,\nu}(\mathbb{R}_{\geqslant 0};H) &\to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu};H)\\ f &\mapsto \big(z \mapsto (\mathcal{L}_{\mathrm{Re}\,z} f)\,(\mathrm{Im}\,z)\big)\end{aligned}$$

is well-defined and isometric. It turns out that $\mathcal{L}$ is actually surjective, see Corollary 8.1.3 below. The difficult surjectivity statement is contained in the following Theorem of Paley–Wiener, [PW34]. We mainly follow the proof given in [Rud87, 19.2 Theorem].

**Theorem 8.1.2** (Paley–Wiener)**.** *Let $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)$. Then there exists an $f \in L_2(\mathbb{R}_{\geqslant 0};H)$ such that*

$$\mathcal{L}_\nu f = g(\mathrm{i}\cdot + \nu) \quad (\nu > 0).$$

*Proof.* For $\nu > 0$ we set $g_\nu := g(\mathrm{i}\cdot + \nu) \in L_2(\mathbb{R};H)$ and $f_\nu := \mathcal{F}^* g_\nu \in L_2(\mathbb{R};H)$. Moreover, we set $f := \mathrm{e}^{(\cdot)} f_1$. We first prove that $f \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R};H)$ with $\sup_{\nu>0} \|f\|_{L_{2,\nu}(\mathbb{R};H)} < \infty$. For doing so, let $a > 0$, $\rho > 0$ and $x \in \mathbb{R}$. Applying Cauchy's integral theorem to the function $z \mapsto \mathrm{e}^{zx} g(z)$ and the curve $\gamma$, as indicated in Figure 8.1, we obtain

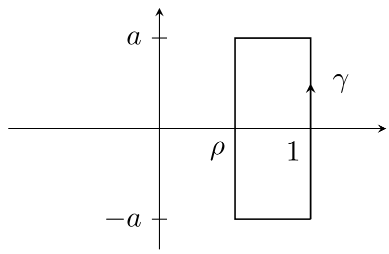


Figure 8.1: Curve $\gamma$.

$$\begin{aligned}0 = \mathrm{i}\int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{\rho}^{1} \mathrm{e}^{(\mathrm{i}a+\kappa)x} g(\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \\ - \mathrm{i}\int_{-a}^{a} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t + \int_{\rho}^{1} \mathrm{e}^{(-\mathrm{i}a+\kappa)x} g(-\mathrm{i}a+\kappa)\,\mathrm{d}\kappa.\end{aligned} \tag{8.2}$$

Moreover, since

$$\begin{aligned}\int_{\mathbb{R}} \left\| \int_{\rho}^{1} \mathrm{e}^{(\pm\mathrm{i}a+\kappa)x} g(\pm\mathrm{i}a+\kappa)\,\mathrm{d}\kappa \right\|_H^2 \mathrm{d}a &\leqslant \int_{\mathbb{R}} \left| \int_{\rho}^{1} \left|\mathrm{e}^{(\pm\mathrm{i}a+\kappa)x}\right|^2 \mathrm{d}\kappa \int_{\rho}^{1} \|g(\pm\mathrm{i}a+\kappa)\|_H^2\,\mathrm{d}\kappa \right| \mathrm{d}a \\ &\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| \left| \int_{\rho}^{1} \int_{\mathbb{R}} \|g(\pm\mathrm{i}a+\kappa)\|_H^2\,\mathrm{d}a\,\mathrm{d}\kappa \right| \\ &\leqslant \left| \int_{\rho}^{1} \mathrm{e}^{2\kappa x}\,\mathrm{d}\kappa \right| |1-\rho| \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}^2 < \infty,\end{aligned}$$

we infer that $\left(a \mapsto \int_\rho^1 \mathrm{e}^{(\pm \mathrm{i}a+\kappa)x} g(\pm \mathrm{i}a+\kappa)\,\mathrm{d}\kappa\right) \in L_2(\mathbb{R};H)$ and thus, we find a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $a_n \to \infty$ and

$$\int_\rho^1 \mathrm{e}^{(\pm \mathrm{i}a_n+\kappa)x} g(\pm \mathrm{i}a_n+\kappa)\,\mathrm{d}\kappa \to 0$$

as $n\to\infty$. Hence, using (8.2) with $a$ replaced by $a_n$ and letting $n$ tend to infinity, we derive that

$$\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t \to 0 \quad (n\to\infty).$$

Noting that for each $\mu>0$ we have

$$\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\mu)x} g(\mathrm{i}t+\mu)\,\mathrm{d}t = \sqrt{2\pi}\mathrm{e}^{\mu x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]}g_\mu)(x) \quad (x\in\mathbb{R})$$

and that $\mathbb{1}_{[-a_n,a_n]}g_\mu \to g_\mu$ in $L_2(\mathbb{R};H)$ as $n\to\infty$, we may choose a subsequence (again denoted by $(a_n)$) such that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(\int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+1)x} g(\mathrm{i}t+1)\,\mathrm{d}t - \int_{-a_n}^{a_n} \mathrm{e}^{(\mathrm{i}t+\rho)x} g(\mathrm{i}t+\rho)\,\mathrm{d}t\right) \\ &= \lim_{n\to\infty}\left(\sqrt{2\pi}\mathrm{e}^{x}\mathcal{F}^*(\mathbb{1}_{[-a_n,a_n]}g_1)(x) - \sqrt{2\pi}\mathrm{e}^{\rho x}\mathcal{F}^*(1_{[-a_n,a_n]}g_\rho)(x)\right) \\ &=\sqrt{2\pi}\big(\mathrm{e}^x f_1(x) - \mathrm{e}^{\rho x} f_\rho(x)\big) \end{aligned}$$

for almost every $x\in\mathbb{R}$. Hence, $f=\mathrm{e}^{(\cdot)}f_1 = \exp(\rho\mathrm{m})f_\rho$ for each $\rho>0$ and thus,

$$\int_{\mathbb{R}} \|f(t)\|_H^2 \mathrm{e}^{-2\rho t}\,\mathrm{d}t = \int_{\mathbb{R}} \|f_\rho(t)\|_H^2\,\mathrm{d}t < \infty$$

which shows $f\in\bigcap_{\rho>0} L_{2,\rho}(\mathbb{R};H)$ with

$$\sup_{\rho>0}\|f\|_{L_{2,\rho}(\mathbb{R};H)} = \sup_{\rho>0}\|f_\rho\|_{L_2(\mathbb{R};H)} = \sup_{\rho>0}\|g_\rho\|_{L_2(\mathbb{R};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}.$$

Thus, $f\in L_2(\mathbb{R}_{\geqslant 0};H)$ with $\|f\|_{L_2(\mathbb{R}_{\geqslant 0};H)} = \|g\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};H)}$ by Lemma 8.1.1. Moreover,

$$\mathcal{L}_\nu f = \mathcal{F}\exp(-\nu\mathrm{m})f = \mathcal{F}\exp(-\nu m)\exp(\nu\mathrm{m})f_\nu = \mathcal{F}f_\nu = g_\nu = g(\mathrm{i}\cdot+\nu)$$

for each $\nu>0$, which shows the representation formula for $g$. □

Summarising the results of Theorem 8.1.2 and the arguments carried out just before Theorem 8.1.2, we obtain the following statement.

**Corollary 8.1.3.** *Let $\nu \in \mathbb{R}$. Then the mapping*

$$\begin{aligned}\mathcal{L}\colon L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) &\to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)\\ f &\mapsto \big(z \mapsto (\mathcal{L}_{\mathrm{Re}\, z} f)(\mathrm{Im}\, z)\big)\end{aligned}$$

*is an isometric isomorphism. In particular, $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ is a Hilbert space.*

*Proof.* We have argued already that $\mathcal{L}$ is well-defined and isometric. Thus, we show that $\mathcal{L}$ is onto, next. For this, let $g \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ and define $\widetilde{g}(z) \coloneqq g(z+\nu)$ for $z \in \mathbb{C}_{\mathrm{Re}>0}$. Then $\widetilde{g} \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0}; H)$ and thus, Theorem 8.1.2 yields the existence of $\widetilde{f} \in L_2(\mathbb{R}_{\geqslant 0}; H)$ with

$$g(\mathrm{i}\cdot{}+\rho) = \widetilde{g}(\mathrm{i}\cdot{}+\rho-\nu) = \mathcal{L}_{\rho-\nu}\widetilde{f} = \mathcal{L}_\rho\big(\mathrm{e}^{\nu\cdot}\widetilde{f}\,\big) \quad (\rho > \nu).$$

Hence, setting $f \coloneqq \mathrm{e}^{\nu\cdot}\widetilde{f} \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$, we obtain $\mathcal{L}f = g$. □

We can now provide an alternative proof of Theorem 5.3.6 by proving causality with the help of the Theorem of Paley–Wiener.

**Proposition 8.1.4.** *Let $M\colon \mathrm{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then for $\nu > \mathrm{s_b}(M)$ we have $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$ and $M(\partial_{t,\nu})$ is causal and autonomous (see Exercise 5.7).*

*Proof.* Let $\nu > \mathrm{s_b}(M)$. Then $M\colon \mathbb{C}_{\mathrm{Re}\geqslant\nu} \to L(H)$ is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$. Hence, by unitary equivalence, $M(\partial_{t,\nu}) \in L(L_{2,\nu}(\mathbb{R}; H))$. Moreover, $M(\partial_{t,\nu})$ is autonomous by Exercise 5.7. Thus, for causality it suffices to check that $\operatorname{spt} M(\partial_{t,\nu})f \subseteq \mathbb{R}_{\geqslant 0}$ whenever $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. So let $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$. Then $\mathcal{L}f \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)$ by Corollary 8.1.3 and since $M$ is bounded and holomorphic on $\mathbb{C}_{\mathrm{Re}>\nu}$, we infer also that

$$\big(z \mapsto M(z)\,(\mathcal{L}f)\,(z)\big) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H).$$

Again by Corollary 8.1.3 there exists $g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\mathcal{L}g(z) = M(z)\,(\mathcal{L}f)\,(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}).$$

Thus, in particular

$$\mathcal{L}_\rho g = M(\mathrm{im}+\rho)\mathcal{L}_\rho f \quad (\rho > \nu).$$

Since $f, g \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ we infer that $\mathcal{L}_\rho g \to \mathcal{L}_\nu g$ and $\mathcal{L}_\rho f \to \mathcal{L}_\nu f$ in $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ by dominated convergence. Moreover, $M(\mathrm{im}+\rho) \to M(\mathrm{im}+\nu)$ strongly on $L_2(\mathbb{R}; H)$ as $\rho \to \nu$ (cf. Exercise 8.2). Hence, we derive

$$\mathcal{L}_\nu g = M(\mathrm{im}+\nu)\mathcal{L}_\nu f,$$

and thus, $g = M(\partial_{t,\nu})f$ which shows the causality. □

## 8.2 A Representation Result

In this section we argue that our solution theory needs holomorphy as a central property for the material law. There are two key properties for rendering $T \in L(L_{2,\nu_0}(\mathbb{R}; H))$ a material law operator. The first one is causality (i.e., $\mathbb{1}_{(-\infty,a]}(\mathrm{m})T\mathbb{1}_{(-\infty,a]}(\mathrm{m}) = \mathbb{1}_{(-\infty,a]}(\mathrm{m})T$ for all $a \in \mathbb{R}$) and, secondly, $T$ needs to be autonomous (i.e., $\tau_h T = T\tau_h$ for all $h \in \mathbb{R}$ where $\tau_h f = f(\cdot + h)$). The main theorem of this section reads as follows:

**Theorem 8.2.1.** *Let $\nu_0 \in \mathbb{R}$ and let $T \in L(L_{2,\nu_0}(\mathbb{R}; H))$ be causal and autonomous. Then $T|_{L_{2,\nu_0}\cap L_{2,\nu}}$ has a unique extension $T_\nu \in L(L_{2,\nu}(\mathbb{R}; H))$ for each $\nu > \nu_0$ and there exists a unique $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ holomorphic and bounded such that $T_\nu = M(\partial_{t,\nu})$ for each $\nu > \nu_0$.*

We consider the following (shifted) variant of Theorem 8.2.1 first.

**Theorem 8.2.2.** *Let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. Then there exists $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that*

$$\left(\mathcal{L}Tf\right)(z) = M(z)\left(\mathcal{L}f\right)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

*Proof.* For $s > 0$ and $x \in H$ we define $f_{x,s} \coloneqq \mathbb{1}_{(0,s)}x$ and compute

$$\mathcal{L}f_{x,s}(z) = \frac{1}{\sqrt{2\pi}}\int_0^s \mathrm{e}^{-zt}x\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\frac{1-\mathrm{e}^{-zs}}{z}x \quad (z \in \mathbb{C}_{\mathrm{Re}>0}). \tag{8.3}$$

We define $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ via

$$M(z)x \coloneqq \frac{\sqrt{2\pi}z}{1-\mathrm{e}^{-z}}\mathcal{L}Tf_{x,1}(z),$$

which is well-defined since $\operatorname{spt} Tf_{x,1} \subseteq [0,\infty)$ (use causality of $T$); $M(z) \in L(H)$, since $T$ is bounded. Also, $M(\cdot)x$ is evidently holomorphic for every $x \in H$ as a product of two holomorphic mappings and thus by Exercise 5.3, $M$ is holomorphic itself. Next, we show that for all $z \in \mathbb{C}_{\mathrm{Re}>0}$ and $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$, we have

$$\left(\mathcal{L}Tf\right)(z) = M(z)\left(\mathcal{L}f\right)(z). \tag{8.4}$$

By definition of $M$, the equality is true for $f$ replaced by $f_{x,1}$, $x \in H$. Next, observe that $\operatorname{lin}\left\{\mathbb{1}_{(a,a+1/n)}x\,;\, a \geqslant 0, n \in \mathbb{N}, x \in H\right\}$ is dense in $L_2(\mathbb{R}_{\geqslant 0}; H)$. Hence, for (8.4), it suffices to show

$$\left(\mathcal{L}T\mathbb{1}_{(a,a+1/n)}x\right)(z) = M(z)\left(\mathcal{L}\mathbb{1}_{(a,a+1/n)}x\right)(z) \tag{8.5}$$

for all $a \geqslant 0$, $n \in \mathbb{N}$, $x \in H$, and $z \in \mathbb{C}_{\mathrm{Re}>0}$. Next, using that $T$ is autonomous in the situation of (8.5), we see $\left(T\mathbb{1}_{(a,a+1/n)}x\right) = \left(T\tau_{-a}\mathbb{1}_{(0,1/n)}x\right) = \tau_{-a}\left(T\mathbb{1}_{(0,1/n)}x\right)$ and, by a straightforward computation, $(\mathcal{L}\tau_{-a}f)(z) = \mathrm{e}^{-za}\mathcal{L}f(z)$ for all $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$. Thus,

$$\left(\mathcal{L}T\mathbb{1}_{(a,a+1/n)}x\right)(z) = \mathrm{e}^{-za}\left(\mathcal{L}T\mathbb{1}_{(0,1/n)}x\right)(z),$$

which yields that it suffices to show (8.5) for $a = 0$ only, that is, for $f = f_{x,1/n}$. Furthermore, we compute for $n \in \mathbb{N}$ and $z \in \mathbb{C}_{\mathrm{Re}>0}$

$$\begin{aligned}\mathcal{L}Tf_{x,1}(z) &= \sum_{k=0}^{n-1} (\mathcal{L}T\mathbb{1}_{(k/n,(k+1)/n)}x)(z) \\ &= \sum_{k=0}^{n-1} \mathrm{e}^{-zk/n} (\mathcal{L}T\mathbb{1}_{(0,1/n)}x)(z) = \frac{1-\mathrm{e}^{-z}}{1-\mathrm{e}^{-z/n}} (\mathcal{L}Tf_{x,1/n})(z).\end{aligned}$$

Thus, using (8.3) for $s = 1/n$, we deduce from the definition of $M$,

$$\begin{aligned}\mathcal{L}Tf_{x,1/n}(z) &= \frac{1-\mathrm{e}^{-z/n}}{\sqrt{2\pi}z} \frac{\sqrt{2\pi}z}{1-\mathrm{e}^{-z}} \mathcal{L}Tf_{x,1}(z) \\ &= \frac{1-\mathrm{e}^{-z/n}}{\sqrt{2\pi}z} M(z)x \\ &= M(z)\mathcal{L}f_{x,1/n}(z).\end{aligned}$$

Hence, (8.4) holds for all $f \in L_2(\mathbb{R}_{\geqslant 0}; H)$. It remains to show boundedness of $M$. For this, let $z \in \mathbb{C}_{\mathrm{Re}>0}$ and $x \in H$. Set $f := \mathbb{1}_{[0,\infty)}\mathrm{e}^{-z^*}x$ as well as $c := 2\,\mathrm{Re}\, z\sqrt{2\pi}$. Then

$$\mathcal{L}f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-zt-z^*t} x \,\mathrm{d}t = \frac{x}{c}.$$

By virtue of (8.4), we get $\mathcal{L}Tf(z) = M(z)\mathcal{L}f(z)$ and thus $M(z)x = c\mathcal{L}Tf(z)$. This leads to

$$\begin{aligned}\|M(z)x\| &\leqslant \frac{c}{\sqrt{2\pi}} \int_0^\infty \left\|\mathrm{e}^{-zt}Tf(t)\right\| \mathrm{d}t \leqslant \frac{c}{\sqrt{2\pi}} \left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})} \|Tf\|_{L_2(\mathbb{R})} \\ &\leqslant \frac{c}{\sqrt{2\pi}} \left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})}^2 \|T\|_{L(L_2(\mathbb{R};H))} \|x\|_H = \|T\|_{L(L_2(\mathbb{R};H))}\|x\|_H,\end{aligned}$$

where we used that $\|f\|_{L_2(\mathbb{R};H)} = \left\|\mathbb{1}_{[0,\infty)}\mathrm{e}^{-z(\cdot)}\right\|_{L_2(\mathbb{R})} \|x\|_H$. Thus, $\|M(z)\| \leqslant \|T\|$, which yields boundedness of $M$ and the assertion of the theorem. □

We can now prove our main result of this section.

*Proof of Theorem 8.2.1.* We just prove the existence of a function $M$. The proof of its uniqueness is left as Exercise 8.3.

We first prove the assertion for $\nu_0 = 0$. So, let $T \in L(L_2(\mathbb{R}; H))$ be causal and autonomous. According to Theorem 8.2.2 we find $M \colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that

$$(\mathcal{L}Tf)(z) = M(z)(\mathcal{L}f)(z) \quad (f \in L_2(\mathbb{R}_{\geqslant 0}; H), z \in \mathbb{C}_{\mathrm{Re}>0}).$$

Let now $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}; H)$ and set $a := \inf \operatorname{spt} \varphi$. Then $\tau_a\varphi \in L_2(\mathbb{R}_{\geqslant 0}; H)$, and for $\nu > 0$ we compute

$$\mathcal{L}_\nu T\varphi = \mathcal{L}_\nu \tau_{-a} T\tau_a\varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a}\mathcal{L}_\nu T\tau_a\varphi = \mathrm{e}^{-(\mathrm{im}+\nu)a} M(\mathrm{im}+\nu)\mathcal{L}_\nu\tau_a\varphi$$

$$= M(\mathrm{im}+\nu)\mathcal{L}_\nu\varphi. \tag{8.6}$$

The latter implies

$$\|T\varphi\|_{L_{2,\nu}(\mathbb{R};H)} = \|\mathcal{L}_\nu T\varphi\|_{L_2(\mathbb{R};H)} = \|M(\mathrm{im}+\nu)\mathcal{L}_\nu\varphi\|_{L_2(\mathbb{R};H)} \leqslant \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>0}}\|\varphi\|_{L_{2,\nu}(\mathbb{R};H)}$$

and hence, $T|_{C_c^\infty(\mathbb{R};H)}$ has a unique continuous extension $T_\nu \in L(L_{2,\nu}(\mathbb{R};H))$. Using (8.6) we obtain

$$T_\nu = \mathcal{L}_\nu^* M(\mathrm{im}+\nu)\mathcal{L}_\nu = M(\partial_{t,\nu})$$

by approximation.
Let now $\nu_0 \in \mathbb{R}$. Then the operator

$$\widetilde{T} := \mathrm{e}^{-\nu_0\mathrm{m}} T \mathrm{e}^{\nu_0\mathrm{m}} \in L(L_2(\mathbb{R};H))$$

is causal and autonomous as well. Thus, $\widetilde{T}|_{C_c^\infty(\mathbb{R};H)}$ has continuous extensions $\widetilde{T}_\rho \in L(L_{2,\rho}(\mathbb{R};H))$ for each $\rho > 0$ and there is $\widetilde{M}\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ holomorphic and bounded such that $\widetilde{T}_\rho = \widetilde{M}(\partial_{t,\rho})$ for each $\rho > 0$. Using $T|_{C_c^\infty(\mathbb{R};H)} = \mathrm{e}^{\nu_0\mathrm{m}}\widetilde{T}|_{C_c^\infty(\mathbb{R};H)}\mathrm{e}^{-\nu_0\mathrm{m}}$, we derive that $T|_{C_c^\infty(\mathbb{R};H)}$ has the unique continuous extension $T_\nu = \mathrm{e}^{\nu_0\mathrm{m}}\widetilde{T}_{\nu-\nu_0}\mathrm{e}^{-\nu_0\mathrm{m}} \in L(L_{2,\nu}(\mathbb{R};H))$ for each $\nu > \nu_0$ and

$$\begin{aligned}\mathcal{L}_\nu T_\nu = \mathcal{L}_\nu \mathrm{e}^{\nu_0\mathrm{m}}\widetilde{T}_{\nu-\nu_0}\mathrm{e}^{-\nu_0\mathrm{m}} &= \mathcal{L}_{\nu-\nu_0}\widetilde{T}_{\nu-\nu_0}\mathrm{e}^{-\nu_0\mathrm{m}} = \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_{\nu-\nu_0}\mathrm{e}^{-\nu_0\mathrm{m}}\\ &= \widetilde{M}(\mathrm{im}+\nu-\nu_0)\mathcal{L}_\nu.\end{aligned}$$

Hence,

$$T_\nu = M(\partial_{t,\nu})$$

for the holomorphic and bounded function $M$ given by $M(z) := \widetilde{M}(z-\nu_0)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. □

## 8.3 Comments

The stated Theorem of Paley and Wiener is of course not the only theorem characterising properties of the support of $L_2$-functions in terms of their Fourier or Laplace transform. For instance, a similar result holds for functions having compact support, see e.g. [Rud87, 19.3 Theorem] and Exercise 8.7. These theorems provide a nice connection between $L_2$-functions and spaces of holomorphic functions in the form of the so-called Hardy spaces. In this chapter we just introduced the Hardy space $\mathcal{H}_2$ and it is not surprising that there are also the Hardy spaces $\mathcal{H}_p$ for $1 \leqslant p \leqslant \infty$. We refer to [Dur70] for this topic.
The representation result presented in the second part of this chapter was originally proved by Fourès and Segal in 1955, [FS55]. In this article the authors prove an analogous representation result for causal operators on $L_2(\mathbb{R}^d;H)$, where causality is defined with respect to a closed and convex cone on $\mathbb{R}^d$. The quite elementary proof of Theorem 8.2.2 for $d = 1$ presented here was kindly communicated to us by Hendrik Vogt.

## Exercises

**Exercise 8.1.** Let $\Lambda \subseteq \mathbb{R}_{>0}$ be a set with an accumulation point in $\mathbb{R}_{>0}$. Prove that $\{(x \mapsto \mathrm{e}^{-\lambda x})\ ;\ \lambda \in \Lambda\}$ is a total set in $L_1(\mathbb{R}_{\geqslant 0})$.
Hint: Use that the set is total if and only if

$$\forall f \in L_\infty(\mathbb{R}_{\geqslant 0}) : \left( \forall \lambda \in \Lambda : \int\limits_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x = 0 \Rightarrow f = 0 \right).$$

**Exercise 8.2.** Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Moreover, let $\nu > \mathrm{s_b}(M)$. Show that $\lim_{\rho \to \nu+} M(\mathrm{im} + \rho) = M(\mathrm{im} + \nu)$ where the limit is meant in the strong operator topology on $L_2(\mathbb{R}; H)$.

**Exercise 8.3.** Prove the uniqueness statement in Theorem 8.2.1.

**Exercise 8.4.** Give an example of a continuous and bounded function $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H)$ such that the corresponding operator $M(\partial_{t,\nu})$ is not causal for any $\nu > 0$.

**Exercise 8.5.** Prove the following distributional variant of the Paley–Wiener theorem: Let $\nu_0 > 0$, $k \in \mathbb{N}$, $f\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathbb{C}$, and set $h(z) \coloneqq \frac{1}{z^k} f(z)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. We assume that $h \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; \mathbb{C})$. For $\nu > \nu_0$ we define the distribution $u\colon C_\mathrm{c}^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$u(\psi) \coloneqq \left\langle \mathcal{L}_\nu^* h(\mathrm{i}\cdot + \nu), (\partial_{t,\nu}^*)^k \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} \quad (\psi \in C_\mathrm{c}^\infty(\mathbb{R}; \mathbb{C})).$$

Prove that $\operatorname{spt} u \subseteq \mathbb{R}_{\geqslant 0}$, where

$$\operatorname{spt} u \coloneqq \mathbb{R} \setminus \bigcup \{U \subseteq \mathbb{R} \text{ open}\,;\, \forall \psi \in C_\mathrm{c}^\infty(U; \mathbb{C}) : u(\psi) = 0\}.$$

What is $u$ if $f = \mathbb{1}_{\mathbb{C}_{\mathrm{Re}>\nu_0}}$?

**Exercise 8.6.** Let $g \in L_2(\mathbb{R}), a > 0$ such that $\operatorname{spt} g \subseteq [-a, a]$. Show that $f \coloneqq \mathcal{F} g$ extends to a holomorphic function $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ with $\widetilde{f}(\mathrm{i}t) = f(t)$ for each $t \in \mathbb{R}$ such that

$$\exists C \geqslant 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}.$$

**Exercise 8.7.** Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic such that

(a) $\exists C \geqslant 0,\ a > 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|\operatorname{Re} z|}$,

(b) $f(\mathrm{i}\cdot) \in L_2(\mathbb{R})$.

Prove that $g \coloneqq \mathcal{F}^* f(\mathrm{i}\cdot)$ satisfies $\operatorname{spt} g \subseteq [-a, a]$.
Hint: Apply Theorem 8.1.2 to the function $h : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}$ given by

$$h(z) \coloneqq \mathrm{e}^{-za} \frac{f(z)}{z+1} \quad (z \in \mathbb{C}_{\mathrm{Re}>0})$$

to derive that $\operatorname{spt} g \subseteq \mathbb{R}_{\geqslant -a}$.
Note: The assertion even holds true if one replaces condition (a) by

$$\exists C \geqslant 0,\ a > 0\, \forall z \in \mathbb{C} : |f(z)| \leqslant C \mathrm{e}^{a|z|}.$$

## References

[Dur70] P. L. Duren. *Theory of $H^p$ spaces.* New York and London: Academic Press XII, 258 p. (1970). 1970.

[FS55] Y. Fourès and I. Segal. 'Causality and analyticity.' In: *Trans. Am. Math. Soc.* 78 (1955), pp. 385–405.

[PW34] R. E. Paley and N. Wiener. *Fourier transforms in the complex domain.* (Am. Math. Soc. Colloq. Publ. 19) New York: Am. Math. Soc. VIII. 1934.

[Rud87] W. Rudin. *Real and complex analysis.* Mathematics series. McGraw-Hill, 1987.

# 9 Initial Value Problems and Extrapolation Spaces

Up until now we have dealt with evolutionary equations of the form

$$\big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)U = F$$

for some given $F \in L_{2,\nu}(\mathbb{R}; H)$ for some Hilbert space $H$, a skew-selfadjoint operator $A$ in $H$ and a material law $M$ defined on a suitable half space satisfying an appropriate positive definiteness condition with $\nu \in \mathbb{R}$ chosen suitably large. Under these conditions, we established that the solution operator, $S_\nu := \big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H))$, is eventually independent of $\nu$ and causal; that is, if $F = 0$ on $(-\infty, a]$ for some $a \in \mathbb{R}$, then so too is $U$.
Solving for $U \in L_{2,\nu}(\mathbb{R}; H)$ for some non-negative $\nu$ penalises $U$ having support on $\mathbb{R}_{\leq 0}$. This might be interpreted as an implicit *initial condition at* $-\infty$. In this chapter, we shall study how to obtain a solution for initial value problems with an initial condition at 0, based on the solution theory developed in the previous chapters.

## 9.1 What are Initial Values?

This section is devoted to the motivation of the framework to follow in the subsequent section. Let us consider the following, arguably easiest but not entirely trivial, initial value problem: find a 'causal' $u \colon \mathbb{R} \to \mathbb{R}$ such that for $u_0 \in \mathbb{R}$ we have

$$\begin{cases} u'(t) = 0 & (t > 0), \\ u(0) = u_0. \end{cases} \tag{9.1}$$

First of all note that there is no condition for $u$ on $\mathbb{R}_{<0}$. Since, there is no source term or right-hand side supported on $\mathbb{R}_{<0}$, causality would imply that $u = 0$ on $(-\infty, 0)$. Moreover, $u = c$ for some constant $c \in \mathbb{R}$ on $(0, \infty)$. Thus, in order to match with the initial condition,

$$u(t) = u_0 \mathbb{1}_{[0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Notice also that $u$ is not continuous. Hence, by the Sobolev embedding theorem (Theorem 4.1.2), $u \notin \bigcup_{\nu > 0} \operatorname{dom}(\partial_{t,\nu})$.

**Proposition 9.1.1.** *Let $H$ be a Hilbert space, $u_0 \in H$. Define*

$$\begin{aligned} \delta_0 u_0 \colon C_c^\infty(\mathbb{R}; H) &\to \mathbb{K} \\ f &\mapsto \langle u_0, f(0) \rangle_H . \end{aligned}$$

*Then, for all $\nu \in \mathbb{R}_{>0}$, $\delta_0 u_0$ extends to a continuous linear functional on* $\operatorname{dom}(\partial_{t,\nu})$. *Re-using the notation for this extension, for all $f \in \operatorname{dom}(\partial_{t,\nu})$ we have*

$$(\delta_0 u_0)(f) = -\left\langle \mathbb{1}_{[0,\infty)} u_0, (\partial_{t,\nu} - 2\nu) f \right\rangle_{L_{2,\nu}(\mathbb{R};H)}. \tag{9.2}$$

*Proof.* The equality (9.2) is obvious for $f \in C_c^\infty(\mathbb{R}; H)$ as it is a direct consequence of the fundamental theorem of calculus (look at the right-hand side first). The continuity of $\delta_0 u_0$ follows from the Cauchy–Schwarz inequality applied to the right-hand side of (9.2). Note that $\mathbb{1}_{[0,\infty)} u_0 \in L_{2,\nu}(\mathbb{R}; H)$. □

Recall from Corollary 3.2.6 that

$$\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

Hence, if we *formally* apply this formula to (9.2), we obtain

$$\left\langle \partial_{t,\nu} \mathbb{1}_{[0,\infty)} u_0, f \right\rangle = \left\langle \mathbb{1}_{[0,\infty)} u_0, \partial_{t,\nu}^* f \right\rangle = (\delta_0 u_0)(f).$$

Therefore, in order to use the introduced time derivative operator for the above initial value problem, we need to extend the time derivative to a broader class of functions than just $\operatorname{dom}(\partial_{t,\nu})$. To utilise the adjoint operator in this way will be central to the construction to follow. It will turn out that indeed

$$\partial_{t,\nu} \mathbb{1}_{[0,\infty)} u_0 = \delta_0 u_0.$$

Moreover, we shall show below that

$$\partial_{t,\nu} u = \delta_0 u_0$$

considered on the full time-line $\mathbb{R}$ is one possible replacement of the initial value problem (9.1).

## 9.2 Extrapolating Operators

Since we are dealing with functionals, let us recall the definition of the dual space. Throughout this section let $H, H_0, H_1$ be Hilbert spaces.

**Definition.** The space

$$H' \coloneqq \{\varphi \colon H \to \mathbb{K}\,;\, \varphi \text{ linear and bounded}\}$$

is called the *dual space of* $H$. We equip $H'$ with the linear structure

$$(\lambda \odot \varphi + \psi)(x) \coloneqq \lambda^* \varphi(x) + \psi(x) \quad (\lambda \in \mathbb{K}, \varphi, \psi \in H', x \in H).$$

*Remark* 9.2.1. Note that $H'$ is a Hilbert space itself, since by the Riesz representation theorem for each $\varphi \in H'$ we find a unique element $R_H\varphi \in H$ such that

$$\forall x \in H: \ \varphi(x) = \langle R_H\varphi, x\rangle .$$

Due to the linear structure on $H'$, the so induced mapping $R_H\colon H' \to H$ (which is one-to-one and onto) becomes linear and

$$H' \times H' \ni (\varphi, \psi) \mapsto \langle R_H\varphi, R_H\psi\rangle$$

defines an inner product on $H'$, which induces the usual norm on functionals.

From now on we will identify elements $x \in H$ with their representatives in $H'$; that is, we identify $x$ with $R_H^{-1}x$.
Let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be linear, densely defined and closed. We recall that in this case $\operatorname{dom}(C)$ endowed with the graph inner product

$$(u, v) \mapsto \langle u, v\rangle_{H_0} + \langle Cu, Cv\rangle_{H_1}$$

becomes a Hilbert space. Clearly, $\operatorname{dom}(C) \hookrightarrow H_0$ is continuous with dense range. Moreover, we see that $\operatorname{dom}(C) \ni x \mapsto Cx \in H_1$ is continuous. We define

$$\begin{aligned} C^\diamond\colon H_1 &\to \operatorname{dom}(C)' =: H^{-1}(C),\\ (C^\diamond\phi)(x) &:= \langle \phi, Cx\rangle_{H_1} \quad (\phi \in H_1, x \in \operatorname{dom}(C)). \end{aligned}$$

Note that $C^\diamond$ is related to the dual operator $C'$ of $C$ considered as a bounded operator from $\operatorname{dom}(C)$ to $H_1$ by

$$C^\diamond = C' R_{H_1}^{-1}.$$

**Proposition 9.2.2.** *With the notions and definitions from this section, the following statements hold:*

(a) $C^\diamond$ *is continuous and linear.*

(b) $C^* \subseteq C^\diamond$.

(c) $\ker(C^*) = \ker(C^\diamond)$.

(d) $C \subseteq (C^*)^\diamond : H_0 \to \operatorname{dom}(C^*)' = H^{-1}(C^*)$.

(e) $H_0 \cong H_0' \hookrightarrow H^{-1}(C)$ *densely and continuously.*

*Proof.* (a) Let $\phi, \psi \in H_1$, $\lambda \in \mathbb{K}$. Then

$$C^\diamond(\lambda\phi + \psi)(x) = \lambda^*(C^\diamond\phi)(x) + (C^\diamond\psi)(x) = (\lambda \odot C^\diamond\phi + C^\diamond\psi)(x) \quad (x \in \operatorname{dom}(C)).$$

To show continuity, let $\phi \in H_1$ and $x \in \operatorname{dom}(C)$. Then

$$\left|\langle \phi, Cx\rangle_{H_1}\right| \leqslant \|\phi\|_{H_1}\|Cx\|_{H_1} \leqslant \|\phi\|_{H_1}\|x\|_{\operatorname{dom}(C)}.$$

Hence, $\|C^\diamond\| = \sup_{\phi\in H_1,\|\phi\|_{H_1}\leqslant 1} \|C^\diamond\phi\|_{\mathrm{dom}(C)'} \leqslant 1$.
(b) Let $\phi \in \mathrm{dom}(C^*)$. Then we have for all $x \in \mathrm{dom}(C)$

$$(C^\diamond\phi)(x) = \langle \phi, Cx\rangle_{H_1} = \langle C^*\phi, x\rangle_{H_0} = (C^*\phi)(x).$$

We obtain $C^\diamond\phi = C^*\phi$. (Note that a functional on $H_0$ is uniquely determined by its values on $\mathrm{dom}(C)$.)
(c) Using (b), we are left with showing $\ker(C^\diamond) \subseteq \ker(C^*)$. So, let $\phi \in \ker(C^\diamond)$. Then for all $x \in \mathrm{dom}(C)$ we have

$$0 = (C^\diamond\phi)(x) = \langle \phi, Cx\rangle_{H_1},$$

which leads to $\phi \in \mathrm{dom}(C^*)$ and $\phi \in \ker(C^*)$.
(d) is a direct consequence of (b) applied to $C^*$.
(e) Since $\mathrm{dom}(C) \hookrightarrow H_0$ is dense and continuous, we obtain that $H_0' \hookrightarrow \mathrm{dom}(C)'$ is so as well; cf. Exercise 9.2. □

We will also write $C_{-1} := (C^*)^\diamond$ for the so-called *extrapolated operator* of $C$. Then $(C^*)_{-1} = C^\diamond$. We will record the index $-1$ at the beginning, but in order to avoid too much clutter in the notation we will drop this index again, bearing in mind that $C_{-1} \supseteq C$ and $(C^*)_{-1} \supseteq C^*$.

**Example 9.2.3.** We have shown that for all $\nu \in \mathbb{R}$ the operator $\partial_{t,\nu}$ is densely defined and closed. Then for $f \in L_{2,\nu}(\mathbb{R})$ we have for all $\phi \in C_c^\infty(\mathbb{R})$

$$((\partial_{t,\nu})_{-1}f)(\phi) = \left\langle f, \partial_{t,\nu}^*\phi\right\rangle_{L_{2,\nu}} = \langle f, (-\partial_{t,\nu} + 2\nu)\phi\rangle_{L_{2,\nu}} = -\int_{\mathbb{R}} \left\langle f, (\mathrm{e}^{-2\nu\cdot}\phi)'\right\rangle_{\mathbb{C}}.$$

Hence, $(\partial_{t,\nu})_{-1}f$ acts as the 'usual' distributional derivative taking into account the exponential weight in the scalar product.
With this observation we deduce that for $\nu > 0$ we have

$$(\partial_{t,\nu})_{-1}\mathbb{1}_{[0,\infty)} = \partial_{t,\nu}\mathbb{1}_{[0,\infty)} = \delta_0.$$

Hence, the initial value problem from the beginning reads: find $u$ such that

$$(\partial_{t,\nu})_{-1}u = \delta_0 u_0.$$

**Example 9.2.4.** Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider $\mathrm{grad}_0\colon H_0^1(\Omega) \subseteq L_2(\Omega) \to L_2(\Omega)^d$. We compute $\mathrm{div}_{-1}\colon L_2(\Omega)^d \to H^{-1}(\Omega)$ with $H^{-1}(\Omega) := H_0^1(\Omega)'$. For $q \in L_2(\Omega)^d$ we obtain for all $\phi \in H_0^1(\Omega)$

$$(\mathrm{div}_{-1}\, q)(\phi) = \langle q, \mathrm{div}^*\,\phi\rangle_{L_2(\Omega)^d} = -\langle q, \mathrm{grad}_0\,\phi\rangle_{L_2(\Omega)^d}.$$

Also, with similar arguments, we see that

$$\big((\mathrm{grad})_{-1} f\big)(q) = -\langle f, \mathrm{div}_0\, q\rangle_{L_2(\Omega)}$$

for all $f \in L_2(\Omega)$ and $q \in H_0(\mathrm{div}, \Omega)$.

We consider a case of particular interest within the framework of evolutionary equations.

**Proposition 9.2.5.** *Let* $A\colon \operatorname{dom}(C) \times \operatorname{dom}(C^*) \subseteq H_0 \times H_1 \to H_0 \times H_1$ *be given by*

$$A \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} C^*\psi \\ -C\phi \end{pmatrix}.$$

*Then* $A_{-1}\colon H_0 \times H_1 \to H^{-1}(C) \times H^{-1}(C^*)$ *acts as*

$$A_{-1} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & (C^*)_{-1} \\ -C_{-1} & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} (C^*)_{-1}\psi \\ -C_{-1}\phi \end{pmatrix}.$$

Next, we will look at the solution theory when carried over to distributional right-hand sides.
An immediate consequence of the introduction of extrapolated operators, however, is that we are now in the position to omit the closure bar for the operator sum in an evolutionary equation, which we will see in an abstract version in Theorem 9.2.6 and for evolutionary equations in Theorem 9.3.2. The main advantage is that we can calculate an operator sum much easier than the closure of it. The price we have to pay is that we have to work in a larger space $H^{-1}$ of an operator in $L_{2,\nu}(\mathbb{R}; H)$ rather than in the original Hilbert space $L_{2,\nu}(\mathbb{R}; H)$. Put differently, this provides another notion of "solutions" for evolutionary equations. For this, we need to introduce the set

$$\operatorname{Fun}(H) := \{\phi\colon \operatorname{dom}(\phi) \subseteq H \to \mathbb{K}\,;\, \phi \text{ linear}\}$$

of not necessearily everywhere defined linear functionals on $H$. Any $u \in H$ is thus identified with an element in $\operatorname{Fun}(H)$ via $\psi \mapsto \langle u, \psi\rangle_H$. Note that we can add and scalarly multiply elements in $\operatorname{Fun}(H)$ with respect to the same addition and multiplication defined on $H'$ and with their natural domains. As usual, we will use the $\subseteq$-sign for extension/restriction of mappings.

**Theorem 9.2.6.** *Let* $A\colon \operatorname{dom}(A) \subseteq H \to H$, $B\colon \operatorname{dom}(B) \subseteq H \to H$ *be densely defined and closed such that* $A + B$ *is closable, and assume that there exists* $(T_n)_{n\in\mathbb{N}}$ *in* $L(H)$ *such that* $T_n \to 1_H$ *in the strong operator topology with* $\operatorname{ran}(T_n) \subseteq \operatorname{dom}(B)$ *and*

$$T_nA \subseteq AT_n, \quad T_nB \subseteq BT_n \text{ for all } n \in \mathbb{N}.$$

*Then* $T_n^*A^* \subseteq A^*T_n^*$ *and* $T_n^*B^* \subseteq B^*T_n^*$ *for each* $n \in \mathbb{N}$ *and* $\operatorname{ran}(T_n^*) \subseteq \operatorname{dom}(B^*)$. *Moreover, for* $x, f \in H$ *the following conditions are equivalent:*

(i) $x \in \operatorname{dom}(\overline{A + B})$ *and* $(\overline{A + B})x = f$.

(ii) $A_{-1}x + B_{-1}x \subseteq f \in \operatorname{Fun}(H)$.

*Proof.* Let $n \in \mathbb{N}$. Taking adjoints in the inclusion $T_nA \subseteq AT_n$, we derive $(AT_n)^* \subseteq (T_nA)^*$. By Theorem 2.3.4 and Remark 2.3.7 we obtain

$$T_n^*A^* \subseteq \overline{T_n^*A^*} = (AT_n)^* \subseteq (T_nA)^* = A^*T_n^*.$$

The same argument shows the claim for $B^*$. Moreover, since $BT_n$ is a closed linear operator defined on the whole space $H$, it follows that $BT_n \in L(H)$ by the closed graph theorem. Hence, $(BT_n)^*$ is bounded by Lemma 2.2.9 and since $(BT_n)^* \subseteq (T_nB)^* = B^*T_n^*$, we derive that $\operatorname{dom}(B^*T_n^*) = H$, showing that $\operatorname{ran}(T_n^*) \subseteq \operatorname{dom}(B^*)$.
We now prove the asserted equivalence.
(i)⇒(ii): By definition, there exists $(x_n)_n$ in $\operatorname{dom}(A) \cap \operatorname{dom}(B)$ such that $x_n \to x$ in $H$ and $Ax_n + Bx_n \to f$. By continuity, we obtain $A_{-1}x_n \to A_{-1}x$ and $B_{-1}x_n \to B_{-1}x$ in $H^{-1}(A^*)$ and $H^{-1}(B^*)$, respectively. Thus, we have

$$(A_{-1}x + B_{-1}x)(y) = \lim_{n\to\infty}(A_{-1}x_n + B_{-1}x_n)(y) = \lim_{n\to\infty}\langle Ax_n + Bx_n, y\rangle = \langle f, y\rangle\,,$$

for each $y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$, which shows the asserted inclusion.
(ii)⇒(i): For $n \in \mathbb{N}$ we put $x_n \coloneqq T_nx$. Then $x_n \in \operatorname{dom}(B)$ and for all $y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$, we obtain

$$\begin{aligned}\langle T_nf - Bx_n, y\rangle &= \langle T_nf, y\rangle - \langle T_nx, B^*y\rangle = \langle f, T_n^*y\rangle - \langle x, T_n^*B^*y\rangle\\ &= \langle f, T_n^*y\rangle - \langle x, B^*T_n^*y\rangle = f(T_n^*y) - (B_{-1}x)(T_n^*y) = (A_{-1}x)(T_n^*y)\\ &= \langle x, A^*T_n^*y\rangle = \langle x, T_n^*A^*y\rangle = \langle x_n, A^*y\rangle\,,\end{aligned}$$

where we have used that $T_n^*y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$. Let now $y \in \operatorname{dom}(A^*)$. Then $T_k^*y \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*)$ for each $k \in \mathbb{N}$ and thus, by what we have shown above

$$\langle T_k(T_nf - Bx_n), y\rangle = \langle T_nf - Bx_n, T_k^*y\rangle = \langle x_n, A^*T_k^*y\rangle = \langle x_n, T_k^*A^*y\rangle = \langle T_kx_n, A^*y\rangle$$

for each $k \in \mathbb{N}$. Letting $k$ tend to infinity, we derive

$$\langle T_nf - Bx_n, y\rangle = \langle x_n, A^*y\rangle\,.$$

Since this holds for each $y \in \operatorname{dom}(A^*)$, this implies that $x_n \in \operatorname{dom}(A)$ and $Ax_n + Bx_n = T_nf$. Letting $n \to \infty$, we deduce $x_n \to x$ and $Ax_n + Bx_n \to f$; that is, (i). □

**Lemma 9.2.7.** *Let $T\colon \operatorname{dom}(T) \subseteq H \to H$ be densely defined and closed with $0 \in \rho(T)$. Then $T_{-1}\colon H \to H^{-1}(T^*)$ is an isomorphsim. In particular, the norms $\left\|(T_{-1})^{-1}\cdot\right\|_H$ and $\|\cdot\|_{H^{-1}(T^*)}$ are equivalent.*

*Proof.* Note that since $0 \in \rho(T)$ we obtain $\{0\} = \ker(T) = \ker((T^*)^\diamond) = \ker(T_{-1})$, see Proposition 9.2.2(c). Thus, $T_{-1}$ is one-to-one. Next, let $f \in H^{-1}(T^*)$. Since $0 \in \rho(T)$, we obtain $0 \in \rho(T^*)$ by Exercise 2.4, which implies that $\langle T^*\cdot, T^*\cdot\rangle$ defines an equivalent scalar product on $\operatorname{dom}(T^*)$. Thus, by the Riesz representation theorem, we find $\phi \in \operatorname{dom}(T^*)$ such that for all $\psi \in \operatorname{dom}(T^*)$ we have

$$f(\psi) = \langle T^*\phi, T^*\psi\rangle = \left((T^*)^\diamond\,(T^*\phi)\right)(\psi).$$

Hence, $f \in \operatorname{ran}((T^*)^\diamond) = \operatorname{ran}(T_{-1})$, thus proving that $T_{-1}$ is onto. □

The following alternative description of $H^{-1}(T^*)$ is content of Exercise 9.5.

**Proposition 9.2.8.** *Let $T\colon \operatorname{dom}(T) \subseteq H \to H$ be densely defined and closed with $0 \in \rho(T)$. Then*

$$H^{-1}(T^*) \cong \big(\widetilde{H, \|T^{-1}\cdot\|_H}\big),$$

*where $\cong$ means isomorphic as Banach spaces and $\widetilde{(\cdot)}$ denotes the completion.*

**Proposition 9.2.9.** *Let $B \in L(H)$. Assume that $T\colon \operatorname{dom}(T) \subseteq H \to H$ is densely defined and closed with $0 \in \rho(T)$ and $T^{-1}B = BT^{-1}$. Then $B$ admits a unique continuous extension $\overline{B} \in L(H^{-1}(T^*))$.*

*Proof.* By Proposition 9.2.2(e), $\operatorname{dom}(B) = H$ is dense in $H^{-1}(T^*)$. Thus, it suffices to show that $B\colon H \subseteq H^{-1}(T^*) \to H^{-1}(T^*)$ is continuous. For this, let $\phi \in H$ and compute for all $q \in \operatorname{dom}(T^*)$

$$\begin{aligned}|(B\phi)(q)| &= |\langle B\phi, q\rangle| = \left|\left\langle B\phi, (T^*)^{-1}T^*q\right\rangle\right| = \left|\langle T^{-1}B\phi, T^*q\rangle\right| = \left|\langle BT^{-1}\phi, T^*q\rangle\right| \\ &\leqslant \|B\|\left\|T^{-1}\phi\right\|\|q\|_{\operatorname{dom}(T^*)}.\end{aligned}$$

The statement now follows upon invoking Lemma 9.2.7. □

The abstract notions and concepts just developed will be applied to evolutionary equations next.

## 9.3 Evolutionary Equations in Distribution Spaces

In this section, we will specialise the results from the previous section and provide an extension of the solution theory in $L_{2,\nu}(\mathbb{R}; H)$. For this, and throughout this whole section, we let $H$ be a Hilbert space, $\mu \in \mathbb{R}$ and $M\colon \mathbb{C}_{\mathrm{Re}>\mu} \to L(H)$ be a material law. Furthermore, let $\nu > \max\{\mathrm{s_b}(M), 0\}$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. In order to keep track of the Hilbert spaces involved, we shall put

$$\begin{aligned}H^1_\nu(\mathbb{R}; H) &\coloneqq \operatorname{dom}(\partial_{t,\nu}), \\ H^{-1}_\nu(\mathbb{R}; H) &\coloneqq \operatorname{dom}(\partial_{t,\nu})' \cong \operatorname{dom}(\partial_{t,\nu}^*)'.\end{aligned}$$

**Proposition 9.3.1.** *Let $H$ be a Hilbert space. Let $D\colon \operatorname{dom}(D) \subseteq H \to H$ be densely defined and closed and $B \in L(H)$. Assume that $DB$ is densely defined. Then for all $\phi \in H$, $(DB)_{-1}(\phi) = (D_{-1}B)(\phi)$ on $\operatorname{dom}(D^*)$.*

*Proof.* First of all, note that $(DB)^* = \overline{B^*D^*}$, by Theorem 2.3.4. Next, let $\phi \in H$ and $x \in \operatorname{dom}(D^*)$. Then

$$\begin{aligned}((DB)_{-1}\phi)(x) &= \langle \phi, (DB)^*x\rangle = \left\langle \phi, \overline{B^*D^*}x\right\rangle \\ &= \langle \phi, B^*D^*x\rangle = \langle B\phi, D^*x\rangle = (D_{-1}B\phi)(x).\end{aligned}$$ □

The first application of the theory developed in the previous section reads as follows.

**Theorem 9.3.2.** *Let $U, F \in L_{2,\nu}(\mathbb{R}; H)$. Then the following statements are equivalent:*

(i) $U \in \operatorname{dom}(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A})$ *and* $\big(\overline{\partial_{t,\nu}M(\partial_{t,\nu}) + A}\big)U = F$.

(ii) $\partial_{t,\nu}M(\partial_{t,\nu})U + AU \subseteq F$ *where the left-hand side is considered as an element of* $H_\nu^{-1}(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H^{-1}(A)) \subseteq \operatorname{Fun}(L_{2,\nu}(\mathbb{R}; H))$.

Before we come to the proof, we state the following lemma, whose proof is left as Exercise 9.7.

**Lemma 9.3.3.** (a) *Let* $B : \operatorname{dom}(B) \subseteq H \to H$ *and* $C : \operatorname{dom}(C) \subseteq H \to H$ *be densely defined closed linear operators. Moreover, let* $\lambda, \mu \in \rho(C)$ *be in the same connected component of* $\rho(C)$ *and*

$$(\mu - C)^{-1}B \subseteq B(\mu - C)^{-1}.$$

*Then* $(\lambda - C)^{-1}B \subseteq B(\lambda - C)^{-1}$.

(b) *For* $\nu > 0$ *we have* $(1+\varepsilon\partial_{t,\nu})^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ *and* $(1+\varepsilon\partial_{t,\nu}^*)^{-1} \to 1_{L_{2,\nu}(\mathbb{R};H)}$ *strongly as* $\varepsilon \to 0+$.

*Proof of Theorem 9.3.2.* For $n \in \mathbb{N}$ we set $T_n := (1 + \frac{1}{n}\partial_{t,\nu})^{-1}$. By Lemma 9.3.3 we obtain $T_n, T_n^* \to 1_{L_{2,\nu}(\mathbb{R};H)}$ strongly in $L_{2,\nu}(\mathbb{R}; H)$ as $n \to \infty$. Moreover, by Hille's theorem (see Proposition 3.1.7) we have $\partial_{t,\nu}^{-1}A \subseteq A\partial_{t,\nu}^{-1}$ and thus, $T_nA \subseteq AT_n$ for each $n \in \mathbb{N}$ by Lemma 9.3.3, which also yields $T_n^*A \subseteq AT_n^*$ for each $n \in \mathbb{N}$ by Theorem 9.2.6. The latter, together with the strong convergence of $(T_n)$ and $(T_n^*)$, yields that $T_n, T_n^* \to 1_{L_{2,\nu}(\mathbb{R};\operatorname{dom}(A))}$ strongly in $L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ as $n \to \infty$.
Consider the inclusion

$$(\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U + A_{-1}U \subseteq F. \tag{9.3}$$

We show next that (9.3) is equivalent to (ii) by applying Proposition 9.3.1 to the case $D = \partial_{t,\nu}, B = M(\partial_{t,\nu})$. For this assume that (9.3) holds. By Proposition 9.3.1, we deduce that

$$((\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U+A_{-1}U)|_{\operatorname{dom}(\partial_{t,\nu}^*)\cap\operatorname{dom}(A)} = ((\partial_{t,\nu})_{-1}M(\partial_{t,\nu})U+A_{-1}U)|_{\operatorname{dom}(\partial_{t,\nu}^*)\cap\operatorname{dom}(A)}$$

Thus, (9.3) implies (ii).
On the other hand, assume that (ii) holds. Let $\phi \in \operatorname{dom}((\partial_{t,\nu}M(\partial_{t,\nu}))^*) \cap L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$. Then, for $n \in \mathbb{N}$, $\phi_n := T_n^*\phi \to \phi$ as $n \to \infty$ in $L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ and

$$(\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi_n = T_n^*(\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi \to (\partial_{t,\nu}M(\partial_{t,\nu}))^*\phi \quad (n \to \infty)$$

in $L_{2,\nu}(\mathbb{R}; H)$. By (ii) we obtain

$$((\partial_{t,\nu})_{-1}M(\partial_{t,\nu})U + A_{-1}U)(\phi_n) = F(\phi_n).$$

Using Proposition 9.3.1, we infer

$$((\partial_{t,\nu}M(\partial_{t,\nu}))_{-1}U + A_{-1}U)(\phi_n) = F(\phi_n).$$

Letting $n \to \infty$, we deduce (9.3).

We are now in the position to apply Theorem 9.2.6 from above to the case $L_{2,\nu}(\mathbb{R};H)$ being the Hilbert space, $A$ the operator in $L_{2,\nu}(\mathbb{R};H)$, $B = \partial_{t,\nu}M(\partial_{t,\nu})$, and $T_n = \left(1+\frac{1}{n}\partial_{t,\nu}\right)^{-1}$. We need to establish the commutativity properties next. The relation $T_nA \subseteq AT_n$ was already shown above. Next, we infer $\operatorname{ran}(T_n) \subseteq \operatorname{dom}(\partial_{t,\nu}) \subseteq \operatorname{dom}(B)$ and

$$T_nB \subseteq BT_n$$

for all $n \in \mathbb{N}$ by using the Fourier–Laplace transformation, see also Theorem 5.2.3. The closability of $A+B$ is implied by Theorem 6.2.1. $\square$

Assume now that there exists $c>0$ such that

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re}\geqslant\nu}).$$

We recall from Theorem 6.2.1 that the operator $\overline{\partial_{t,\nu}M(\partial_{t,\nu})+A}$ is continuously invertible in $L_{2,\nu}(\mathbb{R};H)$.

**Theorem 9.3.4.** *The operator $S_\nu \coloneqq \left(\overline{\partial_{t,\nu}M(\partial_{t,\nu})+A}\right)^{-1} \in L(L_{2,\nu}(\mathbb{R};H))$ admits a continuous extension to $L(H_\nu^{-1}(\mathbb{R};H))$.*

*Proof.* We apply Proposition 9.2.9 to $L_{2,\nu}(\mathbb{R};H)$ being the Hilbert space, $T=\partial_{t,\nu}$ and $B=S_\nu$. For this, it remains to prove that $T^{-1}S_\nu = S_\nu T^{-1}$. This however follows from the fact that $z \mapsto S(z) \coloneqq (zM(z)+A)^{-1}$ is a material law and $S(\partial_{t,\nu}) = S_\nu$. $\square$

## 9.4 Initial Value Problems for Evolutionary Equations

Let $H$ be a Hilbert space, $\mu \in \mathbb{R}$, $M\colon \mathbb{C}_{\operatorname{Re}>\mu} \to L(H)$ a material law, $\nu > \max\{\mathrm{s_b}(M),0\}$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint. In this section we shall focus on the implementation of initial value problems for evolutionary equations. A priori there is no explicit initial condition implemented in the theory established in $L_{2,\nu}(\mathbb{R};H)$. Indeed, choosing $\nu>0$ we have only an implicit exponential decay condition at $-\infty$. For initial values at 0, we would rather want to solve the following type of equation. In the situation of the previous section, for a given initial value $U_0 \in H$ we seek to solve the initial value problem

$$\begin{cases} (\partial_{t,\nu}M(\partial_{t,\nu})+A)\,U = 0 & \text{on } (0,\infty)\,, \\ U(0+) = U_0. \end{cases} \tag{9.4}$$

In this generality the initial value problem cannot be solved. Indeed, for $U \in L_{2,\nu}(\mathbb{R};H)$ evaluation at 0 is not well-defined. A way to overcome this difficulty is to weaken the attainment of the initial value. For this, we specialise to the case when

$$M(\partial_{t,\nu}) = M_0 + \partial_{t,\nu}^{-1}M_1$$

with $M_0, M_1 \in L(H)$.
We start with the following lemma, which will also be useful in the next chapter.

**Lemma 9.4.1.** *Let* $U_0 \in \operatorname{dom}(A)$, $U \in L_{2,\nu}(\mathbb{R}; H)$ *such that* $M_0 U - \mathbb{1}_{[0,\infty)} M_0 U_0 : \mathbb{R} \to H^{-1}(A)$ *is continuous,* $\operatorname{spt} U \subseteq [0, \infty)$ *and*

$$\begin{cases} \partial_{t,\nu} M_0 U + M_1 U + AU = 0 & \text{on } (0, \infty), \\ M_0 U(0+) = M_0 U_0 & \text{in } H^{-1}(A), \end{cases}$$

*where the first equality is meant in the sense that*

$$\forall \varphi \in H^1_\nu(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A)), \operatorname{spt} \varphi \subseteq [0, \infty) : \big(\partial_{t,\nu} M_0 U + M_1 U + AU\big)(\varphi) = 0.$$

*Then* $U - \mathbb{1}_{[0,\infty)} U_0 \in \operatorname{dom}(\overline{\partial_{t,\nu} M_0 + M_1 + A})$ *and*

$$\big(\overline{\partial_{t,\nu} M_0 + M_1 + A}\big)(U - \mathbb{1}_{[0,\infty)} U_0) = -(M_1 + A) U_0 \mathbb{1}_{[0,\infty)}.$$

*Proof.* We apply Theorem 9.3.2 for showing the claim; that is, we show that

$$\big((\partial_{t,\nu} M_0 + M_1)(U - \mathbb{1}_{[0,\infty)} U_0) + A(U - \mathbb{1}_{[0,\infty)} U_0)\big)(\psi) = (-(M_1 + A) U_0 \mathbb{1}_{[0,\infty)})(\psi)$$

for each $\psi \in H^1_\nu(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$. Note that by continuity, it suffices to show the equality for $\psi \in C_c^\infty(\mathbb{R}; \operatorname{dom}(A))$. So, let $\psi \in C_c^\infty(\mathbb{R}; \operatorname{dom}(A))$ and for $n \in \mathbb{N}$ we define the function $\varphi_n \in H^1_\nu(\mathbb{R})$ by

$$\varphi_n(t) \coloneqq \begin{cases} 0 & \text{if } t \leqslant 0, \\ nt & \text{if } t \in (0, 1/n), \\ 1 & \text{if } t \geqslant 1/n. \end{cases}$$

Note that $\varphi_n \psi \in H^1_\nu(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; \operatorname{dom}(A))$ and $\operatorname{spt}(\varphi_n \psi) \subseteq [0, \infty)$ for each $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned} &\big((\partial_{t,\nu} M_0 + M_1 + A)(U - \mathbb{1}_{[0,\infty)} U_0)\big)(\psi) \\ &= \big((\partial_{t,\nu} M_0 + M_1 + A) U\big)(\psi) - \big((\partial_{t,\nu} M_0 + M_1 + A)(\mathbb{1}_{[0,\infty)} U_0)\big)(\psi) \\ &= \big((\partial_{t,\nu} M_0 + M_1 + A) U\big)(\varphi_n \psi) + \big((\partial_{t,\nu} M_0 + M_1 + A) U\big)((1 - \varphi_n)\psi) - \\ &\quad - \big((\partial_{t,\nu} M_0 + M_1 + A)(\mathbb{1}_{[0,\infty)} U_0)\big)(\psi) \\ &= \big((\partial_{t,\nu} M_0 + M_1 + A) U\big)((1 - \varphi_n)\psi) - (\delta_0 M_0 U_0)(\psi) - \big((M_1 + A)(\mathbb{1}_{[0,\infty)} U_0)\big)(\psi) \end{aligned}$$

for each $n \in \mathbb{N}$. Thus, the claim follows if we can show that

$$\big((\partial_{t,\nu} M_0 + M_1 + A) U\big)((1 - \varphi_n)\psi) - (\delta_0 M_0 U_0)(\psi) \to 0 \quad (n \to \infty).$$

For doing so, we first observe that for all $n \in \mathbb{N}$ we have

$$(\delta_0 M_0 U_0)(\psi) = (\delta_0 M_0 U_0)((1 - \varphi_n)\psi) = (\partial_{t,\nu} M_0 \mathbb{1}_{[0,\infty)} U_0)((1 - \varphi_n)\psi),$$

since $\varphi_n(0) = 0$. Moreover,

$$\big((M_1 + A) U\big)((1 - \varphi_n)\psi) = \langle U, (1 - \varphi_n)(M_1^* + A^*)\psi \rangle_{L_{2,\nu}} \to 0 \quad (n \to \infty),$$

since $1-\varphi_n(\mathrm{m}) \to \mathbb{1}_{(-\infty,0]}(\mathrm{m})$ strongly in $L_{2,\nu}(\mathbb{R};H)$ and $\operatorname{spt} U \subseteq [0,\infty)$. Thus, it remains to show that

$$\big(\partial_{t,\nu} M_0(U - \mathbb{1}_{[0,\infty)}U_0)\big)((1-\varphi_n)\psi) \to 0 \quad (n\to\infty).$$

We compute

$$\begin{aligned}
&\big(\partial_{t,\nu} M_0(U - \mathbb{1}_{[0,\infty)}U_0)\big)((1-\varphi_n)\psi)\\
&= \big\langle M_0(U - \mathbb{1}_{[0,\infty)}U_0), \partial_{t,\nu}^*((1-\varphi_n)\psi)\big\rangle_{L_{2,\nu}}\\
&= \big\langle M_0(U - \mathbb{1}_{[0,\infty)}U_0), n\mathbb{1}_{[0,1/n]}\psi\big\rangle_{L_{2,\nu}} - \big\langle M_0(U - \mathbb{1}_{[0,\infty)}U_0), (1-\varphi_n)\partial_{t,\nu}\psi\big\rangle_{L_{2,\nu}}\\
&+ 2\nu\left\langle M_0(U - \mathbb{1}_{[0,\infty)}U_0), (1-\varphi_n)\psi\right\rangle_{L_{2,\nu}}.
\end{aligned}$$

Note that the last two terms on the right-hand side tend to 0 as $n\to\infty$ since, as above, $1-\varphi_n(\mathrm{m}) \to \mathbb{1}_{(-\infty,0]}(\mathrm{m})$ strongly in $L_{2,\nu}(\mathbb{R};H)$ and $\operatorname{spt} U \subseteq [0,\infty)$. For the first term, we observe that

$$\begin{aligned}
&\left|\big\langle M_0(U - \mathbb{1}_{[0,\infty)}U_0), n\mathbb{1}_{[0,1/n]}\psi\big\rangle_{L_{2,\nu}}\right|\\
&\leqslant n\int_0^{1/n} |\langle M_0(U(t)-U_0), \psi(t)\rangle_H| \mathrm{e}^{-2\nu t}\,\mathrm{d}t\\
&\leqslant n\int_0^{1/n} \|M_0(U(t)-U_0)\|_{H^{-1}(A)}\|\psi(t)\|_{\mathrm{dom}(A^*)}\mathrm{e}^{-2\nu t}\,\mathrm{d}t \to 0 \quad (n\to\infty),
\end{aligned}$$

by the fundamental theorem of calculus, since $(M_0U)(t) \to M_0U_0$ in $H^{-1}(A)$ as $t\to 0+$. □

Assume now additionally that there exists $c>0$ such that

$$zM_0 + M_1 \geqslant c \quad (z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu}).$$

Then we can actually prove a stronger result than in the previous lemma.

**Theorem 9.4.2.** *Let $U_0 \in \operatorname{dom}(A)$, $U \in L_{2,\nu}(\mathbb{R};H)$. Then the following statements are equivalent:*

(i) $M_0U - \mathbb{1}_{[0,\infty)}M_0U_0 \colon \mathbb{R} \to H^{-1}(A)$ *is continuous,* $\operatorname{spt} U \subseteq [0,\infty)$ *and*

$$\begin{cases} \partial_{t,\nu}M_0U + M_1U + AU = 0 & \text{on } (0,\infty),\\ M_0U(0+) = M_0U_0 & \text{in } H^{-1}(A), \end{cases}$$

*where the first equality is meant as in Lemma 9.4.1.*

(ii) $U - \mathbb{1}_{[0,\infty)}U_0 \in \operatorname{dom}(\overline{\partial_{t,\nu}M_0 + M_1 + A})$, *and* $\big(\overline{\partial_{t,\nu}M_0 + M_1 + A}\big)(U - \mathbb{1}_{[0,\infty)}U_0) = -(M_1 + A)U_0\mathbb{1}_{[0,\infty)}$.

(iii) $U = S_\nu\delta_0M_0U_0$, *with* $S_\nu \in L(H_\nu^{-1}(\mathbb{R};H))$ *as in Theorem 9.3.4.*

*Moreover, in either case we have* $M_0U - \mathbb{1}_{[0,\infty)}M_0U_0 \in H^1_\nu(\mathbb{R}; H^{-1}(A))$.

*Proof.* (i)⇒(ii): This was shown in Lemma 9.4.1.
(ii)⇒(iii): We have that

$$U - \mathbb{1}_{[0,\infty)}U_0 = -S_\nu((M_1 + A)\mathbb{1}_{[0,\infty)}U_0).$$

Applying $\partial_{t,\nu}^{-1}$ to both sides of this equality we infer that

$$\begin{aligned}\partial_{t,\nu}^{-1}(U - \mathbb{1}_{[0,\infty)}U_0) &= -S_\nu((M_1 + A)\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0)\\ &= -\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0 + S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0),\end{aligned}$$

which gives

$$\partial_{t,\nu}^{-1}U = S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0) = S_\nu(M_0\mathbb{1}_{[0,\infty)}U_0).$$

Applying $\partial_{t,\nu}$ to both sides and taking into account Theorem 9.3.4, we derive the claim.
(iii)⇒(ii): We do the argument in the proof of (ii)⇒(iii) backwards. First, we apply $\partial_{t,\nu}^{-1}$ to $U = S_\nu(\delta_0M_0U_0)$, which yields

$$\partial_{t,\nu}^{-1}U = \partial_{t,\nu}^{-1}S_\nu(\delta_0M_0U_0) = S_\nu(M_0\mathbb{1}_{[0,\infty)}U_0) = S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0).$$

Thus,

$$\begin{aligned}\partial_{t,\nu}^{-1}(U - \mathbb{1}_{[0,\infty)}U_0) &= S_\nu(\partial_{t,\nu}M_0\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0) - \partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0\\ &= -S_\nu((M_1 + A)\partial_{t,\nu}^{-1}\mathbb{1}_{[0,\infty)}U_0).\end{aligned}$$

An application of $\partial_{t,\nu}$ yields the claim.
(ii),(iii)⇒(i): Since $U = S_\nu(\delta_0M_0U_0)$, we derive that

$$(\partial_{t,\nu}M_0 + M_1 + A)U \subseteq \delta_0M_0U_0,$$

which in particular yields $(\partial_{t,\nu}M_0 + M_1 + A)U = 0$ on $(0,\infty)$. By (ii) we infer

$$U - \mathbb{1}_{[0,\infty)}U_0 = -S_\nu((M_1 + A)\mathbb{1}_{[0,\infty)}U_0),$$

which shows that $\operatorname{spt}(U - \mathbb{1}_{[0,\infty)}U_0) \subseteq [0,\infty)$ due to causality and hence, $\operatorname{spt} U \subseteq [0,\infty)$. It remains to show that $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in H^1_\nu(\mathbb{R}; H^{-1}(A))$, since this would imply the continuity of $M_0(U - \mathbb{1}_{[0,\infty)}U_0)$ with values in $H^{-1}(A)$ by Theorem 4.1.2 and thus,

$$M_0(U - \mathbb{1}_{[0,\infty)}U_0)(0+) = M_0(U - \mathbb{1}_{[0,\infty)}U_0)(0-) = 0 \text{ in } H^{-1}(A)$$

since the function is supported on $[0,\infty)$ only. We compute

$$\begin{aligned}M_0(U - \mathbb{1}_{[0,\infty)}U_0) &= -M_0S_\nu((M_1 + A)\mathbb{1}_{[0,\infty)}U_0)\\ &= \partial_{t,\nu}M_0S_\nu(\partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)}U_0)\\ &= \partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)}U_0 - (M_1 + A)S_\nu(\partial_{t,\nu}^{-1}(M_1 + A)\mathbb{1}_{[0,\infty)}U_0),\end{aligned}$$

and since the right-hand side belongs to $H^1_\nu(\mathbb{R}; H^{-1}(A))$, the assertion follows. □

*Remark* 9.4.3. By Theorem 9.3.4, we always have $U = S_\nu \delta_0 M_0 U_0 \in H_\nu^{-1}(\mathbb{R}; H)$. This then serves as our generalisation for the initial value problem even if $U_0 \notin \operatorname{dom}(A)$.

The upshot of Theorem 9.4.2(ii) is that, provided $U_0 \in \operatorname{dom}(A)$, we can reformulate initial value problems with the help of our theory as evolutionary equations with $L_{2,\nu}$-right-hand sides. Thus, we do not need the detour to extrapolation spaces for being able to solve the initial value problem (9.4) (with an adapted initial condition as in (i)) in this situation.
Also note that it may seem that $U$ does depend on the 'full information' of $U_0$ as it is indicated in (ii). In fact, $U$ only depends on the values of $U_0$ orthogonal to the kernel of $M_0$ as it is seen in (iii). We conclude this chapter with two examples; the first one is the heat equation, the second example considers Maxwell's equations.

**Example 9.4.4** (Initial Value Problems for the Heat Equation)**.** We recall the setting for the heat equation outlined in Theorem 6.2.4. This time, we will use homogeneous Dirichlet boundary conditions for the heat distribution $\theta$. Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $a \in L_\infty(\Omega)^{d\times d}$ with $\operatorname{Re} a(x) \geqslant c > 0$ for a.e. $x \in \Omega$ for some $c > 0$. In this case, we have

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}.$$

For the unknown heat distribution, $\theta$, we ask it to have the initial value $\theta_0 \in \operatorname{dom}(\operatorname{grad}_0)$. Let $\nu > 0$ and $V \in L_{2,\nu}\big(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\big)$ be the unique solution of

$$\big(\overline{\partial_{t,\nu} M_0 + M_1 + A}\big) V = -\left(M_1 + A\right) \mathbb{1}_{[0,\infty)} \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix} = -\mathbb{1}_{[0,\infty)} \begin{pmatrix} 0 \\ \operatorname{grad}_0 \theta_0 \end{pmatrix}.$$

Then $(\theta, q) \coloneqq U \coloneqq V + \mathbb{1}_{[0,\infty)} \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix} \in L_{2,\nu}\big(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d\big)$ satisfies (ii) from Theorem 9.4.2. Hence, on $(0,\infty)$ we have

$$\begin{pmatrix} \partial_{t,\nu}\theta \\ a^{-1} q \end{pmatrix} + \begin{pmatrix} \operatorname{div} q \\ \operatorname{grad}_0 \theta \end{pmatrix} = 0$$

and the initial value is attained in the sense that

$$\left(M_0\left(\theta, q\right)\right)(0+) = \begin{pmatrix} \theta(0+) \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix} \quad \text{in} \quad H^{-1}(A) = H^{-1}(\operatorname{grad}_0) \times H^{-1}(\operatorname{div}),$$

which follows from Proposition 9.2.5 where we computed $H^{-1}(A)$. Let us have a closer look at the attainment of the initial value. As a particular consequence of strong convergence in $H^{-1}(\operatorname{grad}_0)$, we obtain for all $\phi \in \operatorname{dom}(\operatorname{div})$

$$\langle \theta(t), \operatorname{div} \phi \rangle \to \langle \theta_0, \operatorname{div} \phi \rangle$$

as $t \to 0+$. Since $\operatorname{grad}_0$ is one-to-one and has closed range, we see that div has dense and closed range. Hence div is onto. This implies that for all $\psi \in L_2(\Omega)$

$$\langle \theta(t), \psi \rangle \to \langle \theta_0, \psi \rangle \quad (t \to 0+).$$

We deduce that the initial value is attained weakly. This might seem a bit unsatisfactory, however, we shall see stronger assertions for more particular cases in the next chapter. Next, we have a look at Maxwell's equations.

**Example 9.4.5** (Initial Value Problems for Maxwell's equations)**.** We briefly recall the situation of Maxwell's equations from Theorem 6.2.8. Let $\varepsilon, \mu, \sigma \colon \Omega \to \mathbb{R}^{3\times 3}$ satisfy the assumptions in Theorem 6.2.8 and let $(E_0, H_0) \in \operatorname{dom}(\operatorname{curl}_0) \times \operatorname{dom}(\operatorname{curl})$. Let $(\widehat{E}, \widehat{H}) \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^6)$ satisfy

$$\left(\overline{\partial_{t,\nu}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}}\right)\begin{pmatrix}\widehat{E}\\ \widehat{H}\end{pmatrix}$$
$$= -\left(\begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}\right)\mathbb{1}_{[0,\infty)}\begin{pmatrix}E_0\\ H_0\end{pmatrix} = \mathbb{1}_{[0,\infty)}\begin{pmatrix}-\sigma E_0 + \operatorname{curl} H_0\\ -\operatorname{curl}_0 E_0\end{pmatrix}.$$

Then, as we have argued for the heat equation,

$$\begin{pmatrix}E\\ H\end{pmatrix} \coloneqq \begin{pmatrix}\widehat{E}\\ \widehat{H}\end{pmatrix} + \mathbb{1}_{[0,\infty)}\begin{pmatrix}E_0\\ H_0\end{pmatrix}$$

satisfies a corresponding initial value problem. We note here that although often the second component in the right-hand side is set to 0, as there are 'no magnetic monopoles', in the theory of evolutionary equations the second component of the right-hand side does appear as an initial value in disguise.

## 9.5 Comments

There are many ways to define spaces generalising the action of an operator to a bigger class of elements; both in a concrete setting and in abstract situations; see e.g. [CG10; EN00]. People have also taken into account simultaneous extrapolation spaces for operators that commute, see e.g. [Pal65; Pic00].

These spaces are particularly useful for formulating initial value problems as was exemplified above; see also the concluding chapter of [PM11] for more insight. Yet there is more to it as one can in fact generalise the equation under consideration or even force the attainment of the initial value in a stronger sense. These issues, however, imply that either the initial value is attained in a much weaker sense, or that there are other structural assumptions needed to be imposed on the material law $M$ (as well as on the operator $A$).

In fact, quite recently, it was established that a particular proper subclass of evolutionary equations can be put into the framework of $C_0$-semigroups. The conditions required to allow for statements in this direction are, on the other hand, rather hard to check in practice; see [Tro18].

## Exercises

**Exercise 9.1.** Let $H_0$ be a Hilbert space, $T \in L(H_0)$. Compute $H^{-1}(T)$ and $H^{-1}(T^*)$.

**Exercise 9.2.** Let $H_0, H_1$ be Hilbert spaces such that $H_0 \hookrightarrow H_1$ is dense and continuous. Prove that $H_1' \hookrightarrow H_0'$ is dense and continuous as well.

**Exercise 9.3.** Prove the following statement which generalises Proposition 9.2.9 from above: Let $H_0$ be a Hilbert space, $A \in L(H_0)$. Assume that $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ is densely defined and closed with $0 \in \rho(T)$ and $T^{-1}A = AT^{-1} + T^{-1}BT^{-1}$ for some bounded $B \in L(H_0)$. Then $A$ admits a unique continuous extension, $\overline{A} \in L(H^{-1}(T^*))$.

**Exercise 9.4.** Let $H_0$ be a Hilbert space, $N\colon \operatorname{dom}(N) \subseteq H_0 \to H_0$ be a *normal* operator; that is, $N$ is densely defined and closed and $NN^* = N^*N$. Show that $H^{-1}(N) \cong H^{-1}(N^*)$ and deduce $H^{-1}(\partial_{t,\nu}) \cong H^{-1}(\partial_{t,\nu}^*)$.

**Exercise 9.5.** Prove Proposition 9.2.8.

**Exercise 9.6.** Let $H_0$ be a Hilbert space, $n \in \mathbb{N}$ and $T\colon \operatorname{dom}(T) \subseteq H_0 \to H_0$ be a densely defined, closed linear operator with $0 \in \rho(T)$. We define $H^n(T) := \operatorname{dom}(T^n)$ and $H^{-n}(T) := H^{-1}(T^n)$. Show that for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ we have that $H^{k+\ell}(T) \hookrightarrow H^\ell(T)$ continuously and densely. Also show that $\mathcal{D} := \bigcap_{n\in\mathbb{N}} \operatorname{dom}(T^n)$ is dense in $H^\ell(T)$ and dense in $H^{-\ell}(T^*)$ for all $\ell \in \mathbb{N}$ and that $T|_{\mathcal{D}}$ can be continuously extended to a topological isomorphism $H^\ell(T) \to H^{\ell-1}(T)$ and to an isomorphism $H^{-\ell+1}(T^*) \to H^{-\ell}(T^*)$ for each $\ell \in \mathbb{N}$.

**Exercise 9.7.** Prove Lemma 9.3.3.
Hint: Prove a similar equality with $\partial_{t,\nu}^{-1}$ formally replaced by $z \in \partial B\,(r,r) \subseteq \mathbb{C}$ and deduce the assertion with the help of Theorem 5.2.3.

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# 10 Differential Algebraic Equations

Let $H$ be a Hilbert space and $\nu \in \mathbb{R}$. We saw in the previous chapter how initial value problems can be formulated within the framework of evolutionary equations. More precisely, we have studied problems of the form

$$\begin{cases} (\partial_{t,\nu} M_0 + M_1 + A)\, U = 0 \quad \text{on } (0,\infty), \\ M_0 U(0+) = M_0 U_0 \end{cases} \tag{10.1}$$

for $U_0 \in H$, $M_0, M_1 \in L(H)$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint; that is, we have considered material laws of the form

$$M(z) \coloneqq M_0 + z^{-1} M_1 \quad (z \in \mathbb{C} \setminus \{0\}).$$

Here, the initial value is attained in a weak sense as an equality in the extrapolation space $H^{-1}(A)$. The first line is also meant in a weak sense since the left-hand side turned out to be a functional in $H_\nu^{-1}(\mathbb{R}; H) \cap L_{2,\nu}(\mathbb{R}; H^{-1}(A))$. In Theorem 9.4.2 it was shown that the latter problem can be rewritten as

$$(\partial_{t,\nu} M_0 + M_1 + A)\, U = \delta_0 M_0 U_0.$$

In this chapter we aim to inspect initial value problems a little closer but in the particularly simple case when $A = 0$. However, we want to impose the initial condition for $U$ and not just $M_0 U$. Thus, we want to deal with the problem

$$\begin{cases} (\partial_{t,\nu} M_0 + M_1)\, U = 0 \quad \text{on } (0,\infty), \\ U(0+) = U_0 \end{cases} \tag{10.2}$$

for two bounded operators $M_0, M_1$ and an initial value $U_0 \in H$. This class of differential equations is known as *differential algebraic equations* since the operator $M_0$ is allowed to have a non-trivial kernel. Thus, (10.2) is a coupled problem of a differential equation (on $(\ker M_0)^\perp$) and an algebraic equation (on $\ker M_0$). We begin by treating these equations in the finite-dimensional case; that is, $H = \mathbb{C}^n$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$.

## 10.1 The finite-dimensional Case

Throughout this section let $n \in \mathbb{N}$ and $M_0, M_1 \in \mathbb{C}^{n\times n}$.

**Definition.** We define the *spectrum of the matrix pair* $(M_0, M_1)$ by

$$\sigma(M_0, M_1) \coloneqq \{z \in \mathbb{C}\,;\, \det(zM_0 + M_1) = 0\},$$

and the *resolvent set of the matrix pair* $(M_0, M_1)$ by

$$\rho(M_0, M_1) \coloneqq \mathbb{C} \setminus \sigma(M_0, M_1).$$

*Remark* 10.1.1. (a) It is immediate that $\sigma(M_0, M_1)$ is closed since the mapping $z \mapsto \det(zM_0 + M_1)$ is continuous.

(b) Note in particular that the spectrum (the set of eigenvalues) of a matrix $A$ corresponds in this setting to the spectrum of the matrix pair $(1, -A)$.

In contrast to the case of the spectrum of one matrix, it may happen that $\sigma(M_0, M_1) = \mathbb{C}$ (for example we can choose $M_0 = 0$ and $M_1$ singular). More precisely, we have the following result.

**Lemma 10.1.2.** *The set $\sigma(M_0, M_1)$ is either finite or equals the whole complex plane $\mathbb{C}$. If $\sigma(M_0, M_1)$ is finite then* $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$.

*Proof.* The function $z \mapsto \det(zM_0 + M_1)$ is a polynomial of order less than or equal to $n$. If it is constantly zero, then $\sigma(M_0, M_1) = \mathbb{C}$ and otherwise $\operatorname{card}(\sigma(M_0, M_1)) \leqslant n$. $\square$

**Definition.** The matrix pair $(M_0, M_1)$ is called *regular* if $\sigma(M_0, M_1) \neq \mathbb{C}$.

The main problem in solving an initial value problem of the form (10.2) is that one cannot expect a solution for each initial value $U_0 \in \mathbb{C}^n$ as the following simple example shows.

**Example 10.1.3.** Let $M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $U_0 \in \mathbb{C}^2$. We assume that there exists a solution $U \colon \mathbb{R}_{\geqslant 0} \to \mathbb{C}^2$ satisfying (10.2); that is,

$$\begin{aligned} U_1'(t) + U_2'(t) + U_1(t) &= 0 \quad (t > 0), \\ U_2(t) &= 0 \quad (t > 0), \\ U(0+) &= U_0. \end{aligned}$$

The second and third equation yield that the second coordinate of $U_0$ has to be zero. Then, for $U_0 = (x, 0) \in \mathbb{C}^2$ the unique solution of the above problem is given by

$$U(t) = \big(U_1(t), U_2(t)\big) = (x\mathrm{e}^{-t}, 0) \quad (t \geqslant 0).$$

**Definition.** We call an initial value $U_0 \in \mathbb{C}^n$ *consistent* for (10.2), if there exists $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n)$ such that (10.2) holds. We denote the set of all consistent initial values for (10.2) by

$$\mathrm{IV}(M_0, M_1) := \{U_0 \in \mathbb{C}^n \,;\, U_0 \text{ consistent}\}.$$

*Remark* 10.1.4. It is obvious that $\mathrm{IV}(M_0, M_1)$ is a subspace of $\mathbb{C}^n$. In particular, $0 \in \mathrm{IV}(M_0, M_1)$.

It is now our goal to determine the space $\mathrm{IV}(M_0, M_1)$. One possibility for doing so uses the so-called *quasi-Weierstraß normal form*.

**Proposition 10.1.5** (quasi-Weierstraß normal form). *Assume that $(M_0, M_1)$ is regular. Then there exist invertible matrices $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

*where $C \in \mathbb{C}^{k\times k}$ and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ for some $k \in \{0,\dots,n\}$. Moreover, the matrix $N$ is nilpotent; that is, there exists $\ell \in \mathbb{N}$ such that $N^\ell = 0$.*

*Proof.* Since $(M_0, M_1)$ is regular we find $\lambda \in \mathbb{C}$ such that $\lambda M_0 + M_1$ is invertible. We set $P_1 := (\lambda M_0 + M_1)^{-1}$ and obtain

$$\begin{aligned} M_{0,1} &:= P_1 M_0 = (\lambda M_0 + M_1)^{-1} M_0, \\ M_{1,1} &:= P_1 M_1 = (\lambda M_0 + M_1)^{-1} M_1 = 1 - \lambda M_{0,1}. \end{aligned}$$

Let now $P_2 \in \mathbb{C}^{n\times n}$ such that

$$M_{0,2} := P_2 M_{0,1} P_2^{-1} = \begin{pmatrix} J & 0 \\ 0 & \widetilde{N} \end{pmatrix}$$

for some invertible matrix $J \in \mathbb{C}^{k\times k}$ and a nilpotent matrix $\widetilde{N} \in \mathbb{C}^{(n-k)\times(n-k)}$ (use the Jordan normal form of $M_{0,1}$ here). Then

$$M_{1,2} := P_2 M_{1,1} P_2^{-1} = \begin{pmatrix} 1-\lambda J & 0 \\ 0 & 1-\lambda\widetilde{N} \end{pmatrix}.$$

Now, by the nilpotency of $\widetilde{N}$, the matrix $(1-\lambda\widetilde{N})$ is invertible by the Neumann series. We set

$$P_3 := \begin{pmatrix} J^{-1} & 0 \\ 0 & (1-\lambda\widetilde{N})^{-1} \end{pmatrix}$$

and obtain

$$P_3 M_{0,2} = \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda\widetilde{N})^{-1}\widetilde{N} \end{pmatrix}, \text{ and } \quad P_3 M_{1,2} = \begin{pmatrix} J^{-1}-\lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $(1-\lambda\widetilde{N})^{-1}\widetilde{N}$ is nilpotent, since the matrices commute and $\widetilde{N}$ is nilpotent. Thus, the assertion follows with $N := (1-\lambda\widetilde{N})^{-1}\widetilde{N}$, $C := J^{-1} - \lambda$, $P = P_3P_2P_1$, and $Q = P_2^{-1}$. □

It is clear that the matrices $P$, $Q$, $C$ and $N$ in the previous proposition are not uniquely determined by $M_0$ and $M_1$. However, the size of $N$ and $C$ as well as the degree of nilpotency of $N$ are determined by $M_0$ and $M_1$ as the following proposition shows.

**Proposition 10.1.6.** *Let $P, Q \in \mathbb{C}^{n\times n}$ be invertible such that*

$$PM_0Q = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \quad PM_1Q = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix},$$

*where $C \in \mathbb{C}^{k\times k}$, $N \in \mathbb{C}^{(n-k)\times(n-k)}$ for some $k \in \{0,\dots,n\}$, and $N$ is nilpotent. Then $(M_0, M_1)$ is regular and*

(a) *$k$ is the degree of the polynomial* $z \mapsto \det(zM_0 + M_1)$.

(b) $N^\ell = 0$ *if and only if*

$$\sup_{|z|\geqslant r} \left\| z^{-\ell+1}(zM_0 + M_1)^{-1} \right\| < \infty$$

*for one (or equivalently all)* $r > 0$ *such that* $B(0, r) \supseteq \sigma(M_0, M_1)$.

*Proof.* First, note that

$$\det(zM_0+M_1) = \frac{1}{\det P \, \det Q} \det \begin{pmatrix} z+C & 0 \\ 0 & zN+1 \end{pmatrix} = \frac{1}{\det P \, \det Q} \det(z+C) \quad (z \in \mathbb{C}).$$

Hence, $(M_0, M_1)$ is regular and

$$k = \deg \det((\cdot) + C) = \deg \det((\cdot)M_0 + M_1),$$

which shows (a). Moreover, we have $\rho(M_0, M_1) = \rho(-C)$ and

$$(zM_0 + M_1)^{-1} = Q \begin{pmatrix} (z+C)^{-1} & 0 \\ 0 & (zN+1)^{-1} \end{pmatrix} P \quad (z \in \rho(M_0, M_1)),$$

and hence, for $r > 0$ with $B(0, r) \supseteq \sigma(M_0, M_1)$ we have

$$\left\|(zM_0 + M_1)^{-1}\right\| \leqslant K_1 \left\|(zN+1)^{-1}\right\| \quad (|z| \geqslant r)$$

for some $K_1 \geqslant 0$, since $\sup_{|z|\geqslant r} \left\|(z+C)^{-1}\right\| < \infty$. Now let $\ell \in \mathbb{N}$ such that $N^\ell = 0$. Then

$$\left\|(zN+1)^{-1}\right\| = \left\| \sum_{k=0}^{\ell-1} (-1)^k z^k N^k \right\| \leqslant K_2 |z|^{\ell-1} \quad (|z| \geqslant r)$$

for some constant $K_2 \geqslant 0$ and thus,

$$\left\|(zM_0 + M_1)^{-1}\right\| \leqslant K_1 K_2 |z|^{\ell-1} \quad (|z| \geqslant r).$$

Assume on the other hand that

$$\sup_{|z|\geqslant r} \left\| z^{-\ell+1}(zM_0 + M_1)^{-1} \right\| < \infty$$

for some $\ell \in \mathbb{N}$ and $r > 0$ with $\sigma(M_0, M_1) \subseteq B(0, r)$. Then there exist $\widetilde{K}_1, \widetilde{K}_2 \geqslant 0$ such that

$$\left\|(zN+1)^{-1}\right\| \leqslant \left\| \begin{pmatrix} (z+C)^{-1} & 0 \\ 0 & (zN+1)^{-1} \end{pmatrix} \right\| \leqslant \widetilde{K}_1 \left\|(zM_0 + M_1)^{-1}\right\| \leqslant \widetilde{K_2} |z|^{\ell-1}$$

for all $z \in \mathbb{C}$ with $|z| \geqslant r$. Now, let $p \in \mathbb{N}$ be minimal such that $N^p = 0$. We show that $p \leqslant \ell$ by contradiction. Assume $p > \ell$. Then we compute

$$0 = \lim_{n\to\infty} \frac{1}{n^\ell}(nN+1)^{-1} N^{p-\ell-1} = \lim_{n\to\infty} \sum_{k=0}^{p-1} (-1)^k n^{k-\ell} N^{k+p-\ell-1}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{\ell-1} (-1)^k n^{k-\ell} N^{k+p-\ell-1} + (-1)^\ell N^{p-1}$$
$$= (-1)^\ell N^{p-1},$$

which contradicts the minimality of $p$. □

**Theorem 10.1.7.** *Let $(M_0, M_1)$ be regular and $P, Q \in \mathbb{C}^{n\times n}$ be chosen according to Proposition 10.1.5. Let $k = \deg\det((\cdot)M_0 + M_1)$. Then*

$$\mathrm{IV}(M_0, M_1) = \left\{ U_0 \in \mathbb{C}^n \,;\, Q^{-1}U_0 \in \mathbb{C}^k \times \{0\} \right\}.$$

*Moreover, for each $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution $U$ of (10.2) is unique and satisfies $U \in C(\mathbb{R}_{\geqslant 0}; \mathbb{C}^n) \cap C^1(\mathbb{R}_{>0}; \mathbb{C}^n)$ as well as*

$$\begin{aligned} M_0 U'(t) + M_1 U(t) &= 0 \quad (t > 0), \\ U(0+) &= U_0. \end{aligned}$$

*Proof.* Let $C \in \mathbb{C}^{k\times k}$ and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be nilpotent as in Proposition 10.1.5. Obviously $U$ is a solution of (10.2) if and only if $V := Q^{-1}U$ is continuous on $\mathbb{R}_{\geqslant 0}$ and solves

$$\begin{aligned} \left( \partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \right) V &= 0 \quad \text{on } (0,\infty), \\ V(0+) &= Q^{-1}U_0 =: V_0. \end{aligned} \tag{10.3}$$

Clearly, if $Q^{-1}U_0 = (x, 0) \in \mathbb{C}^k \times \{0\}$ then $V$ given by $V(t) := (\mathrm{e}^{-tC}x, 0)$ for $t \geqslant 0$ is a solution of (10.3) for $\nu > 0$ large enough. On the other hand, if $V$ given by $V(t) = (V_1(t), V_2(t)) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ $(t \geqslant 0)$ is a solution of (10.3) then we have

$$\partial_{t,\nu} N V_2 + V_2 = 0 \quad \text{on } (0,\infty).$$

Since $N$ is nilpotent, there exists $\ell \in \mathbb{N}$ with $N^\ell = 0$. Hence,

$$N^{\ell-1}V_2(t) = -N^{\ell-1}\partial_{t,\nu} N V_2(t) = \partial_{t,\nu} N^\ell V_2(t) = 0 \quad (t > 0),$$

which in turn implies $\partial_{t,\nu} N^{\ell-1} V_2 = 0$ on $(0,\infty)$. Using again the differential equation, we infer $N^{\ell-2}V_2(t) = 0$ for $t > 0$. Inductively, we deduce $V_2(t) = 0$ for $t > 0$ and by continuity $V_2(0+) = 0$, which yields $V_0 = Q^{-1}U_0 \in \mathbb{C}^k \times \{0\}$. The uniqueness follows from Proposition 10.2.7 below. □

## 10.2 The infinite-dimensional Case

Let now $M_0, M_1 \in L(H)$. Again, it is our aim to determine the space of consistent initial values for the problem

$$\begin{cases} (\partial_{t,\nu} M_0 + M_1)\, U = 0 & \text{on } (0,\infty), \\ U(0+) = U_0. \end{cases} \tag{10.4}$$

Here, consistent initial values are defined as in the finite-dimensional setting:

**Definition.** We call an initial value $U_0 \in H$ *consistent* for (10.4), if there exist $\nu > 0$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ such that (10.4) holds. We denote the set of all consistent initial values for (10.4) by

$$\mathrm{IV}(M_0, M_1) \coloneqq \{U_0 \in H\,;\, U_0 \text{ consistent}\}\,.$$

Before we try to determine $\mathrm{IV}(M_0, M_1)$ we prove a regularity result for solutions of (10.4).

**Proposition 10.2.1.** *Let $\nu > 0$, $U_0 \in H$ and $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ be a solution of (10.4). Then $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in H^1_\nu(\mathbb{R}; H)$ and*

$$\partial_{t,\nu} M_0 \left(U - \mathbb{1}_{[0,\infty)}U_0\right) + M_1 U = 0.$$

*Proof.* We extend $U$ to $\mathbb{R}$ by 0. First, observe that $M_0(U - \mathbb{1}_{[0,\infty)}U_0)\colon \mathbb{R} \to H$ is continuous, since $U$ is continuous and $U(0+) = U_0$. By Lemma 9.4.1 (with $A = 0$), we obtain $U - \mathbb{1}_{[0,\infty)}U_0 \in \mathrm{dom}\big(\overline{\partial_{t,\nu} M_0 + M_1}\big)$ and $\big(\overline{\partial_{t,\nu} M_0 + M_1}\big)(U - \mathbb{1}_{[0,\infty)}U_0) = -M_1 U_0 \mathbb{1}_{[0,\infty)}$. Since $\partial_{t,\nu}$ is closed and $M_0$ is bounded, $\partial_{t,\nu} M_0$ is closed as well. Since $M_1$ is bounded, therefore also $\partial_{t,\nu} M_0 + M_1$ is closed. Thus, $U - \mathbb{1}_{[0,\infty)}U_0 \in \mathrm{dom}(\partial_{t,\nu} M_0 + M_1) = \mathrm{dom}(\partial_{t,\nu} M_0)$ and therefore $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in \mathrm{dom}(\partial_{t,\nu})$, and

$$\partial_{t,\nu} M_0 (U - \mathbb{1}_{[0,\infty)}U_0) + M_1 U = 0. \qquad \square$$

We now come back to the space $\mathrm{IV}(M_0, M_1)$. Since we are now dealing with an infinite-dimensional setting, we cannot use normal forms to determine $\mathrm{IV}(M_0, M_1)$ without dramatically restricting the class of operators. Thus, we follow a different approach using so-called Wong sequences.

**Definition.** We set

$$\mathrm{IV}_0 \coloneqq H$$

and for $k \in \mathbb{N}_0$ we set

$$\mathrm{IV}_{k+1} \coloneqq M_1^{-1}[M_0[\mathrm{IV}_k]].$$

The sequence $(\mathrm{IV}_k)_{k \in \mathbb{N}_0}$ is called the *Wong sequence* associated with $(M_0, M_1)$.

*Remark* 10.2.2. By induction, we infer $\mathrm{IV}_{k+1} \subseteq \mathrm{IV}_k$ for each $k \in \mathbb{N}_0$.

As in the matrix case, we denote by

$$\rho(M_0, M_1) \coloneqq \big\{z \in \mathbb{C}\,;\, (zM_0 + M_1)^{-1} \in L(H)\big\}$$

the *resolvent set of* $(M_0, M_1)$.

**Lemma 10.2.3.** *Let $k \in \mathbb{N}_0$. Then:*

(a) *$M_1 (zM_0 + M_1)^{-1} M_0 = M_0 (zM_0 + M_1)^{-1} M_1$ for each $z \in \rho(M_0, M_1)$.*

(b) *$(zM_0 + M_1)^{-1} M_0[\mathrm{IV}_k] \subseteq \mathrm{IV}_{k+1}$ for each $z \in \rho(M_0, M_1)$.*

(c) *If $x \in \mathrm{IV}_k$ we find $x_1, \ldots, x_{k+1} \in H$ such that for each $z \in \rho(M_0, M_1) \setminus \{0\}$*

$$(zM_0 + M_1)^{-1} M_0 x = \frac{1}{z} x + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}} x_\ell + \frac{1}{z^{k+1}} (zM_0 + M_1)^{-1} x_{k+1}.$$

(d) *If $\rho(M_0, M_1) \neq \varnothing$ then $M_1^{-1}[M_0[\overline{\mathrm{IV}_k}]] \subseteq \overline{\mathrm{IV}_{k+1}}$.*

*Proof.* The proof of the statements (a) to (c) are left as Exercise 10.6. We now prove (d). If $k = 0$ there is nothing to show. So assume that the statement holds for some $k \in \mathbb{N}_0$ and let $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_{k+1}}\right]\right]$. Since $\overline{\mathrm{IV}_{k+1}} \subseteq \overline{\mathrm{IV}_k}$, we infer $x \in M_1^{-1}\left[M_0\left[\overline{\mathrm{IV}_k}\right]\right] \subseteq \overline{\mathrm{IV}_{k+1}}$ by induction hypothesis. Hence, we find a sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathrm{IV}_{k+1}$ with $w_n \to x$. Let now $z \in \rho(M_0, M_1)$. Then, by (b), we have $(zM_0 + M_1)^{-1} M_0 w_n \subseteq \mathrm{IV}_{k+2}$ for each $n \in \mathbb{N}$ and hence, $(zM_0 + M_1)^{-1} M_0 x \in \overline{\mathrm{IV}_{k+2}}$. Moreover, since $M_1 x \in M_0\left[\overline{\mathrm{IV}_{k+1}}\right]$, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathrm{IV}_{k+1}$ with $M_0 y_n \to M_1 x$. Setting now

$$x_n \coloneqq (zM_0 + M_1)^{-1} zM_0 x + (zM_0 + M_1)^{-1} M_0 y_n \in \overline{\mathrm{IV}_{k+2}}$$

(where, again, we have used (b)) for $n \in \mathbb{N}$, we derive

$$\begin{aligned} x_n &= (zM_0 + M_1)^{-1} zM_0 x + (zM_0 + M_1)^{-1} M_0 y_n = x - (zM_0 + M_1)^{-1}\left(M_1 x - M_0 y_n\right) \\ &\to x \end{aligned}$$

as $n \to \infty$ and thus, $x \in \overline{\mathrm{IV}_{k+2}}$. □

The importance of the Wong sequence becomes apparent if we consider solutions of (10.4).

**Lemma 10.2.4.** *Assume that $\rho(M_0, M_1) \neq \varnothing$. Let $\nu > 0$ and $U \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H) \cap C(\mathbb{R}_{\geqslant 0}; H)$ be a solution of (10.4). Then $U(t) \in \bigcap_{k\in\mathbb{N}_0} \overline{\mathrm{IV}_k}$ for each $t \geqslant 0$.*

*Proof.* We prove the claim, $U(t) \in \overline{\mathrm{IV}_k}$ for all $t \geqslant 0$ and $k \in \mathbb{N}_0$, by induction. For $k = 0$ there is nothing to show. Assume now that $U(t) \in \overline{\mathrm{IV}_k}$ for each $t \geqslant 0$ and some $k \in \mathbb{N}_0$. By Proposition 10.2.1 we know that

$$\partial_{t,\nu} M_0 (U - \mathbb{1}_{[0,\infty)} U_0) + M_1 U = 0$$

and thus, in particular,

$$M_0 U(t) - M_0 U_0 + \int_0^t M_1 U(s) \, \mathrm{d}s = 0 \quad (t \geqslant 0).$$

Let now $t \geqslant 0$ and $h > 0$. Then we infer

$$M_0 U(t+h) - M_0 U(t) + M_1 \int_t^{t+h} U(s) \, \mathrm{d}s = 0$$

and hence,

$$\int_t^{t+h} U(s) \, \mathrm{d}s \in M_1^{-1}\left[M_0[\overline{\mathrm{IV}_k}]\right] \subseteq \overline{\mathrm{IV}_{k+1}}$$

by Lemma 10.2.3(d). Since $U$ is continuous, the fundamental theorem of calculus implies $U(t) \in \overline{\mathrm{IV}_{k+1}}$, which yields the assertion. □

In particular, the space of consistent initial values has to be a subspace of $\bigcap_{k\in\mathbb{N}_0}\overline{\mathrm{IV}_k}$. We now impose an additional constraint on the operator pair $(M_0, M_1)$, which is equivalent to being regular in the finite-dimensional setting (cf. Proposition 10.1.6).

**Definition.** We call the operator pair $(M_0, M_1)$ *regular* if there exists $\nu_0 \geqslant 0$ such that

(a) $\mathbb{C}_{\mathrm{Re}>\nu_0} \subseteq \rho(M_0, M_1)$, and

(b) there exist $C \geqslant 0$ and $\ell \in \mathbb{N}$ such that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have $\left\|(zM_0+M_1)^{-1}\right\| \leqslant C|z|^{\ell-1}$.

Moreover, we call the smallest $\ell \in \mathbb{N}$ satisfying (b) the *index of* $(M_0, M_1)$, which is denoted by $\mathrm{ind}(M_0, M_1)$.

*Remark* 10.2.5. Note that for matrices $M_0$ and $M_1$ the index equals the degree of nilpotency of $N$ in the quasi-Weierstraß normal form by Proposition 10.1.6.

From now on, we will require that $(M_0, M_1)$ is regular. First, we prove an important result on the Wong sequence in this case.

**Proposition 10.2.6.** *Let $(M_0, M_1)$ be regular, $k \in \mathbb{N}_0$, and $k \geqslant \mathrm{ind}(M_0, M_1)$. Then*

$$\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}}.$$

*Proof.* We show that $\overline{\mathrm{IV}_k} = \overline{\mathrm{IV}_{k+1}}$ for each $k \geqslant \mathrm{ind}(M_0, M_1)$. Since the inclusion "$\supseteq$" holds trivially, it suffices to show $\mathrm{IV}_k \subseteq \overline{\mathrm{IV}_{k+1}}$. For doing so, let $k \geqslant \mathrm{ind}(M_0, M_1)$ and $x \in \mathrm{IV}_k$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$(zM_0 + M_1)^{-1}M_0x = \frac{1}{z}x + \sum_{\ell=1}^{k} \frac{1}{z^{\ell+1}}x_\ell + \frac{1}{z^{k+1}}(zM_0 + M_1)^{-1}x_{k+1}$$

for each $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Since $k \geqslant \mathrm{ind}(M_0, M_1)$, we derive

$$z(zM_0 + M_1)^{-1}M_0x \to x \quad (\mathrm{Re}\, z \to \infty),$$

and since the elements on the left-hand side belong to $\mathrm{IV}_{k+1}$, by Lemma 10.2.3(b), the assertion immediately follows. □

We now prove that in case of a regular operator pair $(M_0, M_1)$ the solution of (10.4) for a consistent initial value $U_0$ is uniquely determined.

**Proposition 10.2.7.** *Let $(M_0, M_1)$ be regular, $U_0 \in \mathrm{IV}(M_0, M_1)$, and $\nu > 0$ such that a solution $U \in C(\mathbb{R}_{\geqslant 0}; H) \cap L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ of (10.4) exists. Then this solution is unique. In particular*

$$(\mathcal{L}_\rho U)(t) = \frac{1}{\sqrt{2\pi}}\big((\mathrm{i}t + \rho)M_0 + M_1\big)^{-1}M_0U_0 \quad (a.e.\ t \in \mathbb{R})$$

*for each $\rho > \max\{\nu, \nu_0\}$.*

*Proof.* By Proposition 10.2.1 we have $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in H^1_\nu(\mathbb{R}; H)$ and

$$\partial_{t,\nu} M_0(U - \mathbb{1}_{[0,\infty)}U_0) + M_1 U = 0.$$

Applying the Fourier–Laplace transformation, $\mathcal{L}_\rho$, for $\rho > \max\{\nu, \nu_0\}$ the latter yields

$$(\mathrm{i}t + \rho)M_0\big(\mathcal{L}_\rho U(t) - \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}t+\rho}U_0\big) + M_1\mathcal{L}_\rho U(t) = 0 \quad (\text{a.e. } t \in \mathbb{R})$$

which in turn yields

$$\mathcal{L}_\rho U(t) = \frac{1}{\sqrt{2\pi}}\big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1}M_0U_0 \quad (\text{a.e. } t \in \mathbb{R})$$

and, in particular, proves the uniqueness of the solution. □

*Remark* 10.2.8. The formula in Proposition 10.2.7 shows that $U \in L_{2,\nu_0}(\mathbb{R}; H)$ for all solutions $U$ of (10.4) for a consistent initial value $U_0$ and hence, we also have $M_0(U - \mathbb{1}_{[0,\infty)}U_0) \in H^1_{\nu_0}(\mathbb{R}; H)$.

One interesting consequence of the latter proposition is the following.

**Corollary 10.2.9.** *Let $(M_0, M_1)$ be regular. Then the operator $M_0\colon \mathrm{IV}(M_0, M_1) \to H$ is injective.*

*Proof.* Let $U_0 \in \mathrm{IV}(M_0, M_1)$ with $M_0U_0 = 0$. By Proposition 10.2.7, the solution $U$ of (10.4) with $U(0+) = U_0$ satisfies

$$\mathcal{L}_\rho U(t) = \big((\mathrm{i}t+\rho)M_0 + M_1\big)^{-1}M_0U_0 = 0$$

and hence, $U = 0$, which in turn implies $U_0 = U(0+) = 0$. □

We now want to determine the space $\mathrm{IV}(M_0, M_1)$ in terms of the Wong sequence.

**Proposition 10.2.10.** *Let $(M_0, M_1)$ be regular. Then*

$$\mathrm{IV}_{\mathrm{ind}(M_0,M_1)} \subseteq \mathrm{IV}(M_0, M_1) \subseteq \overline{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}}.$$

*Proof.* The second inclusion follows from Lemma 10.2.4 and Proposition 10.2.6. Let now $U_0 \in \mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ and set

$$V(z) \coloneqq \frac{1}{\sqrt{2\pi}}(zM_0 + M_1)^{-1}M_0U_0 \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

Let $k \coloneqq \mathrm{ind}(M_0, M_1)$. By Lemma 10.2.3(c) we find $x_1, \ldots, x_{k+1} \in H$ such that

$$V(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{z}U_0 + \sum_{\ell=1}^{k}\frac{1}{z^{\ell+1}}x_\ell + \frac{1}{z^{k+1}}(zM_0+M_1)^{-1}x_{k+1}\right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

In particular, we read off that $V \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; H)$ and hence, by the Theorem of Paley–Wiener (more precisely by Corollary 8.1.3) there exists $U \in L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H)$ such that

$$\left(\mathcal{L}_\rho U\right)(t) = V(\mathrm{i}t + \rho) \quad (\text{a.e. } t \in \mathbb{R}, \rho > \nu_0).$$

Moreover,

$$zV(z) - \frac{1}{\sqrt{2\pi}} U_0 = \frac{1}{\sqrt{2\pi}} \left( \sum_{\ell=1}^{k} \frac{1}{z^\ell} x_\ell + \frac{1}{z^k} (zM_0 + M_1)^{-1} x_{k+1} \right) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0})$$

and hence $\left(z \mapsto zV(z) - \frac{1}{\sqrt{2\pi}} U_0\right) \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; H)$ as well. Since

$$\begin{aligned} \left(\mathcal{L}_\rho \partial_{t,\rho}(U - \mathbb{1}_{[0,\infty)} U_0)\right)(t) &= (\mathrm{i}t + \rho)\left(\mathcal{L}_\rho U\right)(t) - \frac{1}{\sqrt{2\pi}} U_0 \\ &= (\mathrm{i}t + \rho) V(\mathrm{i}t + \rho) - \frac{1}{\sqrt{2\pi}} U_0 \quad (\text{a.e. } t \in \mathbb{R}, \rho > \nu_0), \end{aligned}$$

we infer $U - \mathbb{1}_{[0,\infty)} U_0 \in H^1_{\nu_0}(\mathbb{R}; H)$ and, thus, $U - \mathbb{1}_{[0,\infty)} U_0$ is continuous by Theorem 4.1.2. Hence, $U \in C(\mathbb{R}_{\geqslant 0}; H)$ and since $\operatorname{spt} U \subseteq \mathbb{R}_{\geqslant 0}$ we derive $U(0+) = U_0$. Finally, by the definition of $V$,

$$M_0 \left( zV(z) - \frac{1}{\sqrt{2\pi}} U_0 \right) = -\frac{1}{\sqrt{2\pi}} M_1 (zM_0 + M_1)^{-1} M_0 U_0 = -M_1 V(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}).$$

Hence,

$$\partial_{t,\nu_0} M_0 (U - \mathbb{1}_{[0,\infty)} U_0) + M_1 U = 0,$$

from which we see that $U$ solves (10.4). □

Finally, we treat the case when $\mathrm{IV}(M_0, M_1)$ is closed.

**Theorem 10.2.11.** *Let $(M_0, M_1)$ be regular and $\mathrm{IV}(M_0, M_1)$ closed. Then the operator $S\colon \mathrm{IV}(M_0, M_1) \to C(\mathbb{R}_{\geqslant 0}; H)$, which assigns to each initial state, $U_0 \in \mathrm{IV}(M_0, M_1)$, its corresponding solution, $U \in C(\mathbb{R}_{\geqslant 0}; H)$, of (10.4) is bounded in the sense that*

$$S_n\colon \mathrm{IV}(M_0, M_1) \to C([0,n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

*is bounded for each $n \in \mathbb{N}$.*

*Proof.* By Proposition 10.2.10 we infer that $\mathrm{IV}(M_0, M_1) = \overline{\mathrm{IV}_k}$ with $k := \operatorname{ind}(M_0, M_1)$. Let $\nu > \nu_0 \geqslant 0$. By Proposition 10.2.7 and Corollary 8.1.3, there exists $C \geqslant 0$ such that

$$\left\| \partial_{t,\nu}^{-k} S u_0 \right\|_{L_{2,\nu}(\mathbb{R}_{\geq 0}; H)} = \left\| \left( z \mapsto z^{-k} (zM_0 + M_1)^{-1} M_0 U_0 \right) \right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)} \leqslant C \sqrt{\frac{\pi}{\nu}} \|M_0 U_0\|_H$$

for each $U_0 \in \mathrm{IV}(M_0, M_1)$, where we have used the regularity of $(M_0, M_1)$ and

$$\left\| (z \mapsto z^{-1} M_0 U_0) \right\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu}; H)} = \sqrt{\frac{\pi}{\nu}} \|M_0 U_0\|_H.$$

In particular, $S\colon \operatorname{IV}(M_0, M_1) \to H^{-1}(\partial_{t,\nu}^k)$ is bounded. Since $L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H) \hookrightarrow H^{-1}(\partial_{t,\nu}^k)$ continuously, we infer that $S\colon \operatorname{IV}(M_0, M_1) \to L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; H)$ is bounded by the closed graph theorem. Hence, also

$$S_n\colon \operatorname{IV}(M_0, M_1) \to L_2([0, n]; H), \quad U_0 \mapsto SU_0|_{[0,n]}$$

is bounded for each $n \in \mathbb{N}$ and since $C([0, n]; H) \hookrightarrow L_2([0, n]; H)$ continuously, we infer that $S_n$ is bounded with values in $C([0, n]; H)$ again by the closed graph theorem. $\square$

*Remark* 10.2.12. The variant of the closed graph theorem used in the proof above is the following: Let $X, Y$ be Banach spaces and $Z$ a Hausdorff topological vector space (e.g. a Banach space) such that $Y \hookrightarrow Z$ continuously. Let $T\colon X \to Z$ be linear and continuous with $T[X] \subseteq Y$. Then $T \in L(X, Y)$. Indeed, by the closed graph theorem it suffices to show that $T : X \to Y$ is closed. For doing so, let $(x_n)_n$ be a sequence in $X$ with $x_n \to x$ and $Tx_n \to y$ for some $x \in X, y \in Y$. Then $Tx_n \to Tx$ in $Z$ by the continuity of $T$ and $Tx_n \to y$ in $Z$ be the continuous embedding. Hence, $y = Tx$ and thus, $T$ is closed.

## 10.3 Comments

The theory of differential algebraic equations in finite dimensions is a very active field. The main motivation for studying these equations comes from the modelling of electrical circuits and from control theory (see e.g. [Dai89] and Exercise 10.5). The main reference for the statements presented in the first part of this lecture is the book by Kunkel and Mehrmann [KM06]. Of course, also in the finite-dimensional case Wong sequences can be used to determine the consistent initial values, see Exercise 10.1. For instance, in [BIT12] the connection between Wong sequences and the quasi-Weierstraß normal form for matrix pairs is studied. Of course, the theory is not restricted to linear and homogeneous problems. Indeed, in the non-homogeneous case it turns out that the set of consistent initial values also depends on the given right-hand side.

The theory of differential algebraic equations in infinite dimensions is less well studied than the finite-dimensional case. We refer to [TT96], where the theory of $C_0$-semigroups is used to deal with such equations. Moreover, we refer to [RT05; Rei07], where sequences of projectors are used to decouple the system. Moreover, there exist several references in the Russian literature, where the equations are called Sobolev type equations (see e.g. [SF03]). The results on infinite-dimensional problems presented here are based on [TW19; TW18; Tro19]. In [TW19] the focus was on systems with index 0 with an emphasis on exponential stability and dichotomy.

We also add the following remark concerning the result in Theorem 10.2.11. By Corollary 10.2.9 we know that $M_0\colon \operatorname{IV}(M_0, M_1) \to H$ is injective. If $\operatorname{IV}(M_0, M_1)$ is closed, it follows that the operator $C\colon \operatorname{dom}(C) \subseteq \operatorname{IV}(M_0, M_1) \to \operatorname{IV}(M_0, M_1)$ given by

$$\begin{aligned} \operatorname{dom}(C) &:= \{U_0 \in \operatorname{IV}(M_0, M_1)\,;\, M_1U_0 \in M_0\,[\operatorname{IV}(M_0, M_1)]\}\,, \\ CU_0 &:= M_0^{-1}M_1U_0 \quad (U_0 \in \operatorname{dom}(C)) \end{aligned}$$

is well-defined and closed. Using this operator, $C$, Theorem 10.2.11 states that if $\mathrm{IV}(M_0, M_1)$ is closed then $-C$ generates a $C_0$-semigroup on $\mathrm{IV}(M_0, M_1)$. The precise statement can be found in [Tro19, Theorem 5.7]. Moreover, $C$ is bounded if $\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ is closed (cf. Exercise 10.7).

## Exercises

**Exercise 10.1.** Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ such that $(M_0, M_1)$ is regular and define the Wong sequence $(\mathrm{IV}_j)_{j\in\mathbb{N}_0}$ associated with $(M_0, M_1)$. Moreover, let $P, Q \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{k\times k}$, and $N \in \mathbb{C}^{(n-k)\times(n-k)}$ be as in the quasi-Weierstraß normal form for $(M_0, M_1)$ with $N$ nilpotent (cf. Proposition 10.1.5). We decompose a vector $x \in \mathbb{C}^n$ into $\breve{x} \in \mathbb{C}^k$ and $\widehat{x} \in \mathbb{C}^{n-k}$ such that $x = (\breve{x}, \widehat{x})$. Prove that

$$x \in \mathrm{IV}_j \Leftrightarrow \widehat{Q^{-1}x} \in \operatorname{ran} N^j \quad (j \in \mathbb{N}_0).$$

Moreover, show that for each $z \in \rho(M_0, M_1)$ we have

$$\mathrm{IV}_j = \operatorname{ran}\big((zM_0 + M_1)^{-1}M_0\big)^j \quad (j \in \mathbb{N}_0).$$

**Exercise 10.2.** Let $E \in \mathbb{C}^{n\times n}$. We set $k := \mathrm{ind}(E, 1)$, where $1$ denotes the identity matrix in $\mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *Drazin inverse of $E$* if the following properties hold:

- $EX = XE$,
- $XEX = X$,
- $XE^{k+1} = E^k$.

Prove that each matrix $E \in \mathbb{C}^{n\times n}$ has a unique Drazin inverse.
Hint: For the existence consider the quasi-Weierstraß form for $(E, 1)$.

**Exercise 10.3.** Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with $(M_0, M_1)$ regular and $M_0M_1 = M_1M_0$. Denote by $M_0^{\mathrm{D}}$ the Drazin inverse of $M_0$ (see Exercise 10.2). Prove:

(a) $M_0^{\mathrm{D}}M_1 = M_1M_0^{\mathrm{D}}$,

(b) $\operatorname{ran} M_0^{\mathrm{D}}M_0 = \mathrm{IV}(M_0, M_1)$,

(c) For all $U_0 \in \mathrm{IV}(M_0, M_1)$ the solution $U$ of (10.2) is given by

$$U(t) = \mathrm{e}^{-tM_0^{\mathrm{D}}M_1}U_0 \quad (t \geqslant 0).$$

**Exercise 10.4.** Let $M_0, M_1 \in \mathbb{C}^{n\times n}$ with $(M_0, M_1)$ regular. Prove that there exist two matrices $E, A \in \mathbb{C}^{n\times n}$ with $(E, A)$ regular and $EA = AE$ such that

- $\mathrm{IV}(E, A) = \mathrm{IV}(M_0, M_1)$,

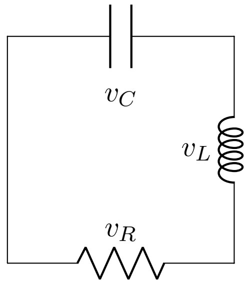


Figure 10.1: Electrical circuit

- $U$ solves the initial value problem (10.2) for the matrices $M_0, M_1$ if and only if $U$ solves the initial value problem (10.2) for the matrices $E, A$ with the same initial value $U_0 \in \mathrm{IV}(M_0, M_1)$.

**Exercise 10.5.** We consider the following electrical circuit (see Figure 10.1) with a resistor with resistance $R > 0$, an inductor with inductance $L > 0$ and a capacitor with capacitance $C > 0$. We denote the respective voltage drops by $v_R, v_L$ and $v_C$. Moreover, the current is denoted by $i$. The constitutive relations for resistor, inductor and capacitor are given by

$$\begin{aligned} Ri &= v_R, \\ Li' &= v_L, \\ Cv_C' &= i, \end{aligned}$$

respectively. Moreover, by Kirchhoff's second law we have

$$v_R + v_C + v_L = 0.$$

Write these equations as a differential algebraic equation and compute the index and the space of consistent initial values. Moreover, compute the solution for each consistent initial value for $R = 2$ and $C = L = 1$.

**Exercise 10.6.** Prove the assertions (a) to (c) in Lemma 10.2.3.

**Exercise 10.7.** Let $M_0, M_1 \in L(H)$.

(a) Assume that $\rho(M_0, M_1) \neq \varnothing$. Prove that for each $k \in \mathbb{N}$ the space $\mathrm{IV}_k$ is closed if and only if $M_0 [\mathrm{IV}_{k-1}]$ is closed.

(b) Assume that $(M_0, M_1)$ is regular with $\mathrm{ind}(M_0, M_1) \geqslant 1$. Prove that if $\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}$ is closed then the operator

$$M_0|_{\mathrm{IV}_{\mathrm{ind}(M_0,M_1)}} \colon \mathrm{IV}_{\mathrm{ind}(M_0,M_1)} \to M_0 \left[\mathrm{IV}_{\mathrm{ind}(M_0,M_1)-1}\right]$$

is an isomorphism.

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# 11 Exponential Stability of Evolutionary Equations

In this chapter we study the exponential stability of evolutionary equations. Roughly speaking, exponential stability of a well-posed evolutionary equation

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)\, U = F$$

means that exponentially decaying right-hand sides $F$ lead to exponentially decaying solutions $U$. The main problem in defining the notion of exponential decay for a solution of an evolutionary equation is the lack of continuity with respect to time, so a pointwise definition would not make sense in this framework. Instead, we will use our exponentially weighted spaces $L_{2,\nu}(\mathbb{R}; H)$, but this time for negative $\nu$, and define the exponential stability by the invariance of these spaces under the solution operator associated with the evolutionary equation under consideration.

## 11.1 The Notion of Exponential Stability

Throughout this section, let $H$ be a Hilbert space, $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ a material law and $A\colon \operatorname{dom}(A) \subseteq H \to H$ a skew-selfadjoint operator. Moreover, we assume that there exist $\nu_0 > \mathrm{s_b}\,(M)$ and $c > 0$ such that

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re} \geqslant \nu_0}).$$

By Picard's theorem (Theorem 6.2.1) we know that for $\nu \geqslant \nu_0$ the operator

$$S_\nu := \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} \in L(L_{2,\nu}(\mathbb{R}; H))$$

is causal and independent of the particular choice of $\nu$. We now define the notion of exponential stability.

**Definition.** We call the solution operators $(S_\nu)_{\nu \geqslant \nu_0}$ *exponentially stable with decay rate* $\rho_0 > 0$ if for all $\rho \in [0, \rho_0)$ and $\nu \geqslant \nu_0$ we have

$$S_\nu F \in L_{2,-\rho}(\mathbb{R}; H) \quad (F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,-\rho}(\mathbb{R}; H)).$$

*Remark* 11.1.1. We emphasise that the definition of exponential stability does not mean that the evolutionary equation is just solvable for some negative weights. Indeed, if we consider $H = \mathbb{C}$, $A = 0$ and $M(z) = 1$ for $z \in \mathbb{C}$ we obtain that the corresponding evolutionary equation

$$\partial_{t,\nu} U = F \tag{11.1}$$

is well-posed for each $\nu \neq 0$. However, we also place a demand for causality on our solution operator. Thus, we only have to consider parameters $\nu > 0$. We obtain the solution $U$ by

$$U(t) = \int_{-\infty}^{t} F(s)\,\mathrm{d}s.$$

As it turns out, the problem (11.1) is not exponentially stable. Indeed, for $F := \mathbb{1}_{[0,1]} \in \bigcap_{\nu\in\mathbb{R}} L_{2,\nu}(\mathbb{R})$ the solution $U$ is given by

$$U(t) = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } 0 \leqslant t \leqslant 1, \\ 1 & \text{if } t > 1, \end{cases}$$

which does not belong to the space $L_{2,-\rho}(\mathbb{R})$ for any $\rho > 0$.

We first show that the aforementioned notion of exponential stability also yields a pointwise exponential decay of solutions if we assume more regularity for our source term $F$.

**Proposition 11.1.2.** *Let $(S_\nu)_{\nu\geqslant\nu_0}$ be exponentially stable with decay rate $\rho_0 > 0$, $\nu \geqslant \nu_0$, $\rho \in [0, \rho_0)$ and $F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\partial_{t,-\rho})$. Then $U := S_\nu F$ is continuous and satisfies*

$$U(t)\mathrm{e}^{\rho t} \to 0 \quad (t \to \infty).$$

*Proof.* We first note that $\partial_{t,\nu}F = \partial_{t,-\rho}F$ by Exercise 11.1. Moreover, since $S_\nu$ is a material law operator (i.e., $S_\nu = S(\partial_{t,\nu})$ for some material law $S$; see Remark 6.3.3) we have

$$S_\nu \partial_{t,\nu} \subseteq \partial_{t,\nu} S_\nu.$$

Thus, in particular, we have

$$S_\nu \partial_{t,\nu} F = \partial_{t,\nu} S_\nu F = \partial_{t,\nu} U;$$

that is, $U \in \operatorname{dom}(\partial_{t,\nu})$. Moreover, since $\partial_{t,\nu}F = \partial_{t,-\rho}F \in L_{2,-\rho}(\mathbb{R}; H)$, we infer also $U, \partial_{t,\nu}U \in L_{2,-\rho}(\mathbb{R}; H)$ by exponential stability. By Exercise 11.1 this yields $U \in \operatorname{dom}(\partial_{t,-\rho})$ with $\partial_{t,-\rho}U = \partial_{t,\nu}U$. The assertion now follows from the Sobolev embedding theorem (Theorem 4.1.2 and Corollary 4.1.3). □

## 11.2 A Criterion for Exponential Stability of Parabolic-type Equations

In this section we will prove a useful criterion for exponential stability of a certain class of evolutionary equations. The easiest example we have in mind is the heat equation with homogeneous Dirichlet boundary conditions, which can be written as an evolutionary equation of the form (cf. Theorem 6.2.4)

$$\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix}$$

in $L_{2,\nu}(\mathbb{R};H)$, where $H = L_2(\Omega) \oplus L_2(\Omega)^d$ with $\Omega \subseteq \mathbb{R}^d$ open, and $a \in L(L_2(\Omega)^d)$ with

$$\operatorname{Re} a \geqslant c$$

for some $c > 0$ which models the heat conductivity, and $\nu > 0$.

**Theorem 11.2.1.** *Let $H_0, H_1$ be Hilbert spaces and $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ a densely defined closed linear operator which is boundedly invertible. Moreover, let $M_0 \in L(H_0)$ be selfadjoint with*

$$M_0 \geqslant c_0$$

*for some $c_0 > 0$ and $M_1\colon \operatorname{dom}(M_1) \subseteq \mathbb{C} \to L(H_1)$ be a material law satisfying $\mathrm{s_b}\,(M_1) < -\rho_1$ for some $\rho_1 > 0$ and*

$$\exists\, c_1 > 0\, \forall z \in \mathbb{C}_{\operatorname{Re}>-\rho_1} : \operatorname{Re} M_1(z) \geqslant c_1.$$

*Then*

$$S_\nu := \left(\overline{\partial_{t,\nu}\begin{pmatrix} M_0 & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{t,\nu})\end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0\end{pmatrix}}\right)^{-1} \in L\big(L_{2,\nu}(\mathbb{R}; H_0 \oplus H_1)\big)$$

*for each $\nu > 0$. Moreover, for all $\nu_0 > 0$ the family $(S_\nu)_{\nu \geqslant \nu_0}$ is exponentially stable with decay rate $\rho_0 := \min\Big\{\rho_1, c_1/\big(\|M_1\|^2_{\infty,\mathbb{C}_{\operatorname{Re}>-\rho_1}}\|M_0\|\big\|C^{-1}\big\|^2\big)\Big\}$.*

In order to prove this theorem we need a preparatory result.

**Lemma 11.2.2.** *Assume the hypotheses of Theorem 11.2.1. Then for each $z \in \mathbb{C}_{\operatorname{Re}>-\rho_0}$ the operator*

$$T(z) := \begin{pmatrix} zM_0 & 0 \\ 0 & M_1(z)\end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0\end{pmatrix} :\ \operatorname{dom}(C) \times \operatorname{dom}(C^*) \subseteq H_0 \oplus H_1 \to H_0 \oplus H_1$$

*is boundedly invertible. Moreover,*

$$\sup_{z \in \mathbb{C}_{\operatorname{Re}\geqslant -\rho}} \big\|T(z)^{-1}\big\| < \infty$$

*for each $\rho < \rho_0$.*

*Proof.* Let $z \in \mathbb{C}_{\operatorname{Re}\geqslant-\rho}$ for some $\rho < \rho_0$. We note that $M_1(z)$ is boundedly invertible with $\big\|M_1(z)^{-1}\big\| \leqslant 1/c_1$ (see Proposition 6.2.3(b)) and $(C^*)^{-1} = (C^{-1})^* \in L(H_0, H_1)$ (see Lemma 2.2.2 and Lemma 2.2.9). The beginning of the proof deals with a reformulation of $T(z)$. For this, let $u, f \in H_0$, $v, g \in H_1$. Then, by definition, $(u,v) \in \operatorname{dom}(T(z)) = \operatorname{dom}(C) \times \operatorname{dom}(C^*)$ and $T(z)(u,v) = (f,g)$ if and only if $v \in \operatorname{dom}(C^*)$ and $u \in \operatorname{dom}(C)$ together with

$$\begin{aligned} zM_0u - C^*v &= f \\ Cu + M_1(z)v &= g. \end{aligned}$$

Since both $C^*$ and $M_1(z)$ are continuously invertible, we obtain equivalently $u \in \mathrm{dom}(C)$ together with

$$\begin{aligned} z(C^*)^{-1}M_0u - v &= (C^*)^{-1}f \\ M_1(z)^{-1}Cu + v &= M_1(z)^{-1}g. \end{aligned}$$

Adding the latter two equations and retaining the first equation, we obtain the following equivalent system subject to the condition $u \in \mathrm{dom}(C)$

$$\begin{aligned} v &= z(C^*)^{-1}(M_0u - f) \in \mathrm{dom}(C^*), \\ (z(C^*)^{-1}M_0C^{-1} + M_1(z)^{-1})Cu &= M_1(z)^{-1}g + (C^*)^{-1}f. \end{aligned}$$

We now inspect the operator $S(z) \coloneqq z(C^{-1})^*M_0C^{-1} + M_1(z)^{-1} \in L(H_1)$. By Proposition 6.2.3 for $x \in H_1$ we estimate

$$\begin{aligned} \mathrm{Re}\,\langle x, S(z)x\rangle &= \mathrm{Re}\,\langle C^{-1}x, zM_0C^{-1}x\rangle + \mathrm{Re}\,\langle x, M_1(z)^{-1}x\rangle \\ &\geqslant -\rho\|M_0\|\left\|C^{-1}\right\|^2\|x\|^2 + \frac{c_1}{\|M_1(z)\|^2}\|x\|^2 \\ &\geqslant \underbrace{\Big(\frac{c_1}{\|M_1\|^2_{\infty,\mathbb{C}_{\mathrm{Re}>-\rho_1}}} - \rho\|M_0\|\left\|C^{-1}\right\|^2\Big)}_{=:\mu}\|x\|^2. \end{aligned}$$

Since $\rho < \rho_0$ and by the definition of $\rho_0$ we infer that $\mu > 0$. Hence, $S(z)$ is boundedly invertible with

$$\left\|S(z)^{-1}\right\| \leqslant \frac{1}{\mu}.$$

We now set

$$\begin{aligned} u &\coloneqq C^{-1}S(z)^{-1}\big((C^*)^{-1}f + M_1(z)^{-1}g\big) \in \mathrm{dom}(C), \\ v &\coloneqq z(C^*)^{-1}(M_0u - f) \in \mathrm{dom}(C^*). \end{aligned}$$

By the first part of the proof we have that $(u, v)$ is the unique solution of $T(z)(u, v) = (f, g)$. Moreover, we can estimate

$$\begin{aligned} \|u\| &\leqslant \left\|C^{-1}\right\|\frac{1}{\mu}\Big(\left\|(C^*)^{-1}\right\|\|f\| + \frac{1}{c_1}\|g\|\Big), \text{ and} \\ \|v\| &\leqslant \frac{1}{c_1}(\|g\| + \|Cu\|) \leqslant \frac{1}{c_1}\Big(\|g\| + \frac{1}{\mu}\big(\left\|(C^*)^{-1}\right\|\|f\| + \frac{1}{c_1}\|g\|\big)\Big), \end{aligned}$$

which proves that $T(z)$ is boundedly invertible with

$$\sup_{z\in\mathbb{C}_{\mathrm{Re}\geqslant-\rho}} \left\|T(z)^{-1}\right\| < \infty. \qquad \square$$

*Proof of Theorem 11.2.1.* Let $H \coloneqq H_0 \oplus H_1$. We set

$$M(z) \coloneqq \begin{pmatrix} M_0 & 0 \\ 0 & z^{-1}M_1(z) \end{pmatrix} \quad (z \in \operatorname{dom}(M_1) \setminus \{0\}).$$

Let $\nu > 0$. Then

$$\forall z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu} : \operatorname{Re} zM(z) \geqslant \min\{\nu c_0, c_1\}$$

and hence, the first assertion of the theorem follows from Theorem 6.2.1.
Next, we focus on exponential stability. For this note that $(S_\nu)_{\nu \geqslant \nu_0}$ is exponentially stable if $S_\nu \in L(L_{2,\nu}(\mathbb{R}; H) \cap L_{2,-\rho}(\mathbb{R}; H))$ for all $\rho \in [0, \rho_0)$ and $\nu \geqslant \nu_0$. For this, by Lemma 4.2.1, it suffices to show the corresponding estimate on $S_{\mathrm{c}}(\mathbb{R}; X)$. Therefore, we let $F \in S_{\mathrm{c}}(\mathbb{R}; H) \subseteq L_{2,\nu}(\mathbb{R}; H)$ and define

$$U \coloneqq \left(\overline{\partial_{t,\nu} \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{t,\nu}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\right)^{-1} F.$$

Since the estimate $\|U\|_{L_{2,\nu}} \leqslant \widetilde{C}\|F\|_{L_{2,\nu}}$ for some $\widetilde{C} \geqslant 0$ is a consequence of the continuity of the solution operator, it is left to show that for all $\rho \in [0, \rho_0)$ there exists $C \geqslant 0$ such that $\|U\|_{L_{2,-\rho}} \leqslant C\|F\|_{L_{2,-\rho}}$.
For this, we observe that since $F$ has compact support, we find $n \in \mathbb{N}$ such that $\operatorname{spt} F \subseteq \mathbb{R}_{\geqslant -n}$. Then $U \in L_{2,\nu}(\mathbb{R}_{\geqslant -n}; H)$ by causality. Since

$$\left(\overline{\partial_{t,\nu} \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{t,\nu}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\right)^{-1}$$

is a material law operator (induced by the material law $T(\cdot)^{-1}$ from Lemma 11.2.2), it commutes with the material law operator $\tau_{-n}$ (see Remark 5.3.3 for the commutation and Remark 6.3.3 as well as Example 5.3.4 for the material law operator properties). Thus, we obtain

$$\tau_{-n}U = \left(\overline{\partial_{t,\nu} \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{t,\nu}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\right)^{-1} \tau_{-n}F$$

and $\tau_{-n}F \in L_{2,-\rho}(\mathbb{R}_{\geqslant 0}; H)$ by assumption. Hence, Corollary 8.1.3 yields $\mathcal{L}\tau_{-n}F \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>-\rho}; H)$. Let $T(\cdot)$ be as in Lemma 11.2.2. We note here that thus

$$T(\cdot)^{-1}(\partial_{t,\nu}) = \left(\overline{\partial_{t,\nu} \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1(\partial_{t,\nu}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\right)^{-1}.$$

Since $T(\cdot)^{-1}$ has a bounded analytic extension to $\mathbb{C}_{\mathrm{Re}>-\rho}$, we obtain $T(\cdot)^{-1}\mathcal{L}\tau_{-n}F \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>-\rho}; H)$. Since $\tau_{-n}U \in L_{2,\nu}(\mathbb{R}_{\geqslant 0}; H)$ and

$$\big(\mathcal{L}\tau_{-n}U\big)(z) = T(z)^{-1}\big(\mathcal{L}\tau_{-n}F\big)(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}),$$

we conclude that $\tau_{-n}U \in L_{2,-\rho}(\mathbb{R}_{\geqslant 0}; H)$ by Corollary 8.1.3. The boundedness of $T(\cdot)^{-1}$ on $\mathbb{C}_{\mathrm{Re}>-\rho}$ implies the desired continuity estimate. In order to see this, let $C := \sup_{z\in\mathbb{C}_{\mathrm{Re}\geqslant-\rho}} \left\|T(z)^{-1}\right\|$. Using that $\mathcal{L}_{-\rho}$ is unitary from $L_{2,-\rho}(\mathbb{R}; H)$ to $L_2(\mathbb{R}; H)$, we compute

$$\|U\|_{L_{2,-\rho}} = \|\mathcal{L}_{-\rho}U\|_{L_2} = \left\|T(\mathrm{im} - \rho)^{-1}\mathcal{L}_{-\rho}F\right\|_{L_2} \leqslant C\|\mathcal{L}_{-\rho}F\|_{L_2} = C\|F\|_{L_{2,-\rho}},$$

which is the desired continuity estimate. □

## 11.3 Three Exponentially Stable Models for Heat Conduction

### The Classical Heat Equation

We recall the classical heat equation (cf. Theorem 6.2.4) on an open subset $\Omega \subseteq \mathbb{R}^d$ consisting of two equations, the heat-flux balance

$$\partial_t\theta + \operatorname{div} q = f$$

and Fourier's law

$$q = -a \operatorname{grad} \theta,$$

where $f$ is a given source term and $a \in L(L_2(\Omega)^d)$ is an operator modelling the heat conductivity of the underlying medium. We will impose Dirichlet boundary conditions which will be incorporated in our equation by replacing the operator grad by $\mathrm{grad}_0$ in Fourier's law (cf. Section 6.1).
In order to apply Theorem 11.2.1 we need that $\mathrm{grad}_0$ is boundedly invertible in some sense. This can be shown using *Poincaré's inequality.*

**Proposition 11.3.1** (Poincaré inequality)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab; that is, there exist $e \in \mathbb{R}^d$ with $\|e\| = 1$ and $a, b \in \mathbb{R}$, $a < b$ such that*

$$\Omega \subseteq \left\{x \in \mathbb{R}^d\,;\, a < \langle e, x\rangle < b\right\}.$$

*Then for each $u \in \mathrm{dom}(\mathrm{grad}_0)$ we have*

$$\|u\|_{L_2(\Omega)} \leqslant (b - a)\|\mathrm{grad}_0\, u\|_{L_2(\Omega)^d}.$$

*Proof.* Without loss of generality, let $e = (1, 0, \ldots, 0)$. Recall that, by definition, $C_c^\infty(\Omega)$ is a core for $\mathrm{grad}_0$. Thus, it suffices to prove the assertion for functions in $C_c^\infty(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. We identify $\varphi$ with its extension by 0 to the whole of $\mathbb{R}^d$. By the fundamental theorem of calculus, we may compute

$$\varphi(x) = \int_a^{x_1} \partial_1\varphi(s, x_2, \ldots, x_d)\,\mathrm{d}s \quad (x \in \Omega).$$

Hence, by the Cauchy–Schwarz inequality and Tonelli's theorem

$$\begin{aligned}\int_\Omega |\varphi(x)|^2 \,\mathrm{d}x &= \int_\Omega \left|\int_a^{x_1} \partial_1\varphi(s,x_2,\ldots,x_d)\,\mathrm{d}s\right|^2 \mathrm{d}x \\ &\leqslant \int_\Omega (b-a)\int_a^b (\partial_1\varphi(s,x_2,\ldots,x_d))^2\,\mathrm{d}s\,\mathrm{d}x = (b-a)^2\int_\Omega |\partial_1\varphi(x)|^2\,\mathrm{d}x \\ &\leqslant (b-a)^2\|\operatorname{grad}_0\varphi\|^2_{L_2(\Omega)^d},\end{aligned}$$

which shows the assertion. □

**Corollary 11.3.2.** *Under the assumptions of Proposition 11.3.1 the operator* $\operatorname{grad}_0$ *is one-to-one and* $\operatorname{ran}(\operatorname{grad}_0)$ *is closed.*

*Proof.* The injectivity follows immediately from Poincaré's inequality. To prove the closedness of $\operatorname{ran}(\operatorname{grad}_0)$, let $(u_k)_{k\in\mathbb{N}}$ in $\operatorname{dom}(\operatorname{grad}_0)$ with $\operatorname{grad}_0 u_k \to v$ in $L_2(\Omega)^d$ for some $v \in L_2(\Omega)^d$. By Poincaré's inequality, we infer that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $L_2(\Omega)$ and hence convergent to some $u \in L_2(\Omega)$. By the closedness of $\operatorname{grad}_0$ we obtain $u \in \operatorname{dom}(\operatorname{grad}_0)$ and $v = \operatorname{grad}_0 u \in \operatorname{ran}(\operatorname{grad}_0)$. □

We need another auxiliary result which is interesting in its own right.

**Lemma 11.3.3.** *Let $H$ be a Hilbert space and $V \subseteq H$ a closed subspace. We denote by*

$$\iota_V \colon V \to H, \quad x \mapsto x$$

*the canonical embedding of $V$ into $H$. Then $\iota_V\iota_V^* \colon H \to H$ is the orthogonal projection on $V$ and $\iota_V^*\iota_V \colon V \to V$ is the identity on $V$.*

*Proof.* The proof is left as Exercise 11.2. □

We now come to the exponential stability of the heat equation. First, we need to formulate both the heat-flux balance and Fourier's law as a suitable evolutionary equation. For doing so, we assume that $\Omega \subseteq \mathbb{R}^d$ is open and contained in a slab. Then $\operatorname{ran}(\operatorname{grad}_0)$ is closed by Corollary 11.3.2. It is clear that we can write Fourier's law as

$$q = -a\operatorname{grad}_0\theta = -a\iota_{\operatorname{ran}(\operatorname{grad}_0)}\iota^*_{\operatorname{ran}(\operatorname{grad}_0)}\operatorname{grad}_0\theta.$$

Hence, defining $\widetilde{q} \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}q$ and $\widetilde{a} \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}a\iota_{\operatorname{ran}(\operatorname{grad}_0)} \in L(\operatorname{ran}(\operatorname{grad}_0))$, we arrive at

$$\widetilde{q} = -\widetilde{a}\iota^*_{\operatorname{ran}(\operatorname{grad}_0)}\operatorname{grad}_0\theta.$$

Moreover, since $\operatorname{ran}(\operatorname{grad}_0)^\perp = \ker(\operatorname{div})$, we derive from the heat-flux balance

$$f = \partial_t\theta + \operatorname{div} q = \partial_t\theta + \operatorname{div}\iota_{\operatorname{ran}(\operatorname{grad}_0)}\widetilde{q}$$

and hence, assuming that $\widetilde{a}$ is invertible, we may write both equations with the unknowns $(\theta, \widetilde{q})$ as an evolutionary equation in $L_{2,\nu}(\mathbb{R}; H)$ for $\nu > 0$, where $H \coloneqq L_2(\Omega) \oplus \operatorname{ran}(\operatorname{grad}_0)$. This yields

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \widetilde{a}^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}\iota_{\operatorname{ran}(\operatorname{grad}_0)}\\ \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}\operatorname{grad}_0 & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ \widetilde{q}\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}. \quad (11.2)$$

For notational convenience, we set

$$C \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}\operatorname{grad}_0\colon \operatorname{dom}(\operatorname{grad}_0) \subseteq L_2(\Omega) \to \operatorname{ran}(\operatorname{grad}_0). \quad (11.3)$$

**Lemma 11.3.4.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab and $C$ as above. Then $C$ is densely defined, closed and boundedly invertible. Moreover*

$$C^* = -\operatorname{div}\iota_{\operatorname{ran}(\operatorname{grad}_0)}.$$

*Proof.* The proof is left as Exercise 11.3. □

**Proposition 11.3.5.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab, $a \in L(L_2(\Omega)^d)$, and $c_1 > 0$ such that*

$$\operatorname{Re} a \geqslant c_1.$$

*Then $\widetilde{a} \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} a \iota_{\operatorname{ran}(\operatorname{grad}_0)}$ is boundedly invertible and the solution operators associated with (11.2) are exponentially stable.*

*Proof.* For $x \in \operatorname{ran}(\operatorname{grad}_0)$ we have

$$\begin{aligned}\operatorname{Re}\langle x, \widetilde{a}x\rangle_{\operatorname{ran}(\operatorname{grad}_0)} &= \operatorname{Re}\left\langle \iota_{\operatorname{ran}(\operatorname{grad}_0)}x, a\iota_{\operatorname{ran}(\operatorname{grad}_0)}x\right\rangle_{L_2(\Omega)^d}\\ &\geqslant c_1\left\|\iota_{\operatorname{ran}(\operatorname{grad}_0)}x\right\|^2_{L_2(\Omega)^d} = c_1\|x\|^2_{\operatorname{ran}(\operatorname{grad}_0)},\end{aligned}$$

and thus, $\widetilde{a}$ is boundedly invertible. Hence, (11.2) is an evolutionary equation of the form considered in Theorem 11.2.1 with $M_0 \coloneqq 1$, $M_1(z) \coloneqq \widetilde{a}^{-1}$ for $z \in \mathbb{C}$ and $C$ given by (11.3). Since $\operatorname{Re}\widetilde{a}^{-1} \geqslant \frac{c_1}{\|\widetilde{a}\|^2}$, Theorem 11.2.1 is applicable and we derive the exponential stability. □

## The Heat Equation with Additional Delay

Again we consider the heat equation, but now we replace Fourier's law by

$$q = -a_1\operatorname{grad}_0\theta - a_2\tau_{-h}\operatorname{grad}_0\theta$$

for some operators $a_1, a_2 \in L(L_2(\Omega)^d)$ and $h > 0$. As above, we assume that $\Omega \subseteq \mathbb{R}^d$ is open and contained in a slab. We may introduce $\widetilde{q} \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}q$ and $\widetilde{a}_j \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)}a_j\iota_{\operatorname{ran}(\operatorname{grad}_0)} \in L(L_2(\Omega)^d)$ for $j \in \{1,2\}$. Moreover, we assume that there exists $c > 0$ such that

$$\operatorname{Re} a_1 \geqslant c.$$

By Lemma 7.3.1 there exists $\nu_0 > 0$ such that the operator $\widetilde{a}_1 + \widetilde{a}_2\tau_{-h}$ is boundedly invertible in $L_{2,\nu}(\mathbb{R}; \operatorname{ran}(\operatorname{grad}_0))$ and its inverse is uniformly strictly positive definite for each $\nu \geqslant \nu_0$. Hence, we may write the heat equation with additional delay as an evolutionary equation of the form

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & (\widetilde{a}_1 + \widetilde{a}_2\tau_{-h})^{-1}\end{pmatrix} + \begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}\right)\begin{pmatrix}\theta\\ \widetilde{q}\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix} \tag{11.4}$$

with $C$ given by (11.3).

**Proposition 11.3.6.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab, $h > 0$, $a_1, a_2 \in L(L_2(\Omega)^d)$, and $c > 0$ such that*

$$\operatorname{Re} a_1 \geqslant c$$

*and $\|a_2\| < c$. Then the solution operators $(S_\nu)_{\nu \geqslant \nu_0}$ associated with (11.4) are exponentially stable.*

*Proof.* Note that $\|\widetilde{a}_2\| \leqslant \|a_2\| < c$. We choose

$$0 < \rho_1 < \frac{1}{h}\log\frac{c}{\|\widetilde{a}_2\|}.$$

Then we estimate for $z \in \mathbb{C}_{\operatorname{Re} > -\rho_1}$

$$\operatorname{Re}\left\langle x, (\widetilde{a}_1 + \widetilde{a}_2 \mathrm{e}^{-zh})x\right\rangle_{\operatorname{ran}(\operatorname{grad}_0)} \geqslant (c - \|\widetilde{a}_2\|\mathrm{e}^{\rho_1 h})\|x\|^2_{\operatorname{ran}(\operatorname{grad}_0)}.$$

By the choice of $\rho_1$, we infer $\widetilde{c} := (c - \|\widetilde{a}_2\|\mathrm{e}^{\rho_1 h}) > 0$. Hence,

$$M_1(z) := \left(\widetilde{a}_1 + \widetilde{a}_2\mathrm{e}^{-hz}\right)^{-1} \quad (z \in \mathbb{C}_{\operatorname{Re} > -\rho_1})$$

is well-defined and satisfies

$$\operatorname{Re} M_1(z) \geqslant c_1 \quad (z \in \mathbb{C}_{\operatorname{Re} > -\rho_1})$$

for some $c_1 > 0$ by Proposition 6.2.3. Thus, Theorem 11.2.1 is applicable and yields the exponential stability of (11.4). $\square$

## A Dual Phase Lag Model

In this last variant of heat conduction, we replace Fourier's law by

$$(1 + s_q\partial_t)q = (1 + s_\theta\partial_t)\operatorname{grad}_0\theta,$$

where $s_q, s_\theta > 0$ are the so-called "phases" (cf. Section 7.4, where a different type of dual phase lag model is studied). The latter equation can be reformulated as

$$(1 + s_q\partial_{t,\nu})(1 + s_\theta\partial_{t,\nu})^{-1}q = \operatorname{grad}_0\theta$$

for $\nu > 0$. Assuming that $\Omega \subseteq \mathbb{R}^d$ is open and contained in a slab, and defining $\widetilde{q} \coloneqq \iota^*_{\mathrm{ran}(\mathrm{grad}_0)} q$, the dual phase lag model may be written as

$$\left( \partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (1 + s_q \partial_{t,\nu})(1 + s_\theta \partial_{t,\nu})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ \widetilde{q} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{11.5}$$

with $C$ given by (11.3).

**Proposition 11.3.7.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab, $\nu_0 > 0$. Moreover, let $s_\theta > s_q > 0$. Then the solution operators $(S_\nu)_{\nu \geqslant \nu_0}$ associated with (11.5) are exponentially stable.*

*Proof.* Again, we note that (11.5) is of the form considered in Theorem 11.2.1 with $M_0 \coloneqq 1$ and

$$M_1(z) \coloneqq \frac{1 + s_q z}{1 + s_\theta z} \quad (z \in \mathbb{C} \setminus \{-s_\theta^{-1}\}).$$

Setting $\mu \coloneqq \frac{s_q}{s_\theta} < 1$ we compute

$$\operatorname{Re} M_1(z) = \operatorname{Re} \left( \mu + \frac{(1-\mu)}{1 + s_\theta z} \right) = \mu + (1-\mu) \frac{1 + s_\theta \operatorname{Re} z}{|1 + s_\theta z|^2} \geqslant \mu \quad (z \in \mathbb{C}_{\mathrm{Re} > -s_\theta^{-1}}).$$

Thus, Theorem 11.2.1 is applicable and hence, the claim follows. □

## 11.4 Comments

The results of this chapter are based on the results obtained in [Tro18, Section 2]. There, Laplace transform techniques are used to characterise the exponential stability of evolutionary equations in a slightly more general setting. Moreover, besides parabolic-type equations, also hyperbolic-type equations are considered, such as the damped wave equation or the equations of visco-elasticity.

The exponential stability of partial differential equations is a well-studied field. In particular, in the framework of $C_0$-semigroups, where the exponential stability is defined via pointwise estimates due to the continuity of solutions, several results are known. We just mention Datko's theorem [Dat72] (see also [Are+11, Theorem 5.1.2]), which states that a $C_0$-semigroup is exponentially stable if and only if the solution operator associated with the equation

$$(\partial_{t,\nu} + A)\, U = F$$

leaves $L_p(\mathbb{R}_{\geqslant 0}; H)$ invariant for some (or equivalently all) $p \in [1, \infty)$. As it turns out, the latter is equivalent to the invariance of $L_{2,-\rho}(\mathbb{R}; H)$ for some $\rho > 0$ and thus, our notion of exponential stability coincides with the usual one used in the theory of $C_0$-semigroups. Another important theorem on the exponential stability of $C_0$-semigroups on Hilbert spaces is the Theorem of Gearhart–Prüß [Prü84] (see also [EN00, Chapter 5, Theorem 1.11]), where the exponential stability of a $C_0$-semigroup is characterised in terms of the resolvent of its generator.

Besides the exponential stability, which is the only type of stability studied so far within the current framework, different kinds of asymptotic behaviours were considered for $C_0$-semigroups. We just mention the celebrated Arendt–Batty–Lyubich–Vu theorem [AB88; LP88] on strong stability of $C_0$-semigroups or the Theorem of Borichev–Tomilov [BT10] on the polynomial stability of $C_0$-semigroups on Hilbert spaces.

## Exercises

**Exercise 11.1.** Let $H$ be a Hilbert space, $\nu, \rho \in \mathbb{R}$ and $u \in L_{1,\mathrm{loc}}(\mathbb{R}; H)$. Prove the following statements:

(a) If $u \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\partial_{t,\rho})$ then $\partial_{t,\nu} u = \partial_{t,\rho} u$.

(b) If $u \in \operatorname{dom}(\partial_{t,\nu})$ such that $u, \partial_{t,\nu} u \in L_{2,\rho}(\mathbb{R}; H)$ then $u \in \operatorname{dom}(\partial_{t,\rho})$.

**Exercise 11.2.** Prove Lemma 11.3.3.

**Exercise 11.3.** Let $H_0, H_1$ be Hilbert spaces and $A\colon \operatorname{dom}(A) \subseteq H_0 \to H_1$ a densely defined closed linear operator. Moreover, we assume that $A$ has closed range. Show that the adjoint of the operator $\iota_{\operatorname{ran}(A)}^* A\colon \operatorname{dom}(A) \subseteq H_0 \to \operatorname{ran}(A)$ is given by $A^* \iota_{\operatorname{ran}(A)}$. If additionally $A$ is one-to-one, show that $\iota_{\operatorname{ran}(A)}^* A$ is boundedly invertible.

**Exercise 11.4.** Let $\Omega \subseteq \mathbb{R}^d$ be open and contained in a slab. We consider the heat conduction with a memory term given by the equations

$$\begin{aligned} \partial_{t,\nu} \theta + \operatorname{div} q &= f, \\ q &= -(1 - k*) \operatorname{grad}_0 \theta, \end{aligned} \tag{11.6}$$

where $k \in L_{1,-\rho_1}(\mathbb{R}_{\geqslant 0}; \mathbb{R})$ for some $\rho_1 > 0$ with

$$\int_0^\infty |k(t)| \,\mathrm{d}t < 1.$$

Write (11.6) as a suitable evolutionary equation and prove that this equation is exponentially stable.

**Exercise 11.5.** Let $A \in \mathbb{C}^{n \times n}$ for some $n \in \mathbb{N}$ and consider the evolutionary equation

$$(\partial_{t,\nu} + A) U = F.$$

Prove that the solution operators associated with this problem are exponentially stable if and only if $A$ has only eigenvalues with strictly positive real part.

**Exercise 11.6.** Let $\Omega \subseteq \mathbb{R}^d$ be open.

(a) Let $\varphi \in C_{\mathrm{c}}^\infty(\Omega)^d$. Prove *Korn's inequality*

$$\|\operatorname{Grad} \varphi\|_{L_2(\Omega)^{d\times d}_{\mathrm{sym}}}^2 \geqslant \frac{1}{2} \sum_{j=1}^d \|\operatorname{grad} \varphi_j\|_{L_2(\Omega)^d}^2.$$

(b) Use Korn's inequality to prove that for $u \in L_2(\Omega)^d$ we have

$$u \in \operatorname{dom}(\operatorname{Grad}_0) \quad \iff \quad \forall j \in \{1, \ldots, d\} : \ u_j \in \operatorname{dom}(\operatorname{grad}_0).$$

Moreover, show that in either case

$$\frac{1}{2} \sum_{j=1}^{d} \|\operatorname{grad}_0 u_j\|^2_{L_2(\Omega)^d} \leqslant \|\operatorname{Grad}_0 u\|^2_{L_2(\Omega)^{d\times d}_{\mathrm{sym}}} \leqslant \sum_{j=1}^{d} \|\operatorname{grad}_0 u_j\|^2_{L_2(\Omega)^d}.$$

(c) Let now $\Omega$ be contained in a slab. Prove that $\operatorname{Grad}_0$ is one-to-one and has closed range.

**Exercise 11.7.** Let $\Omega \subseteq \mathbb{R}^d$ open and $a \in L(L_2(\Omega)^d)$ with $\operatorname{Re} a \geqslant c > 0$.

(a) Let $\nu > 0$ and $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$. Moreover, assume that $\Omega$ is contained in a slab and define $\widetilde{a} \coloneqq \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} a \iota_{\operatorname{ran}(\operatorname{grad}_0)}$. Let $\theta \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$, $q \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d)$ satisfy

$$\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

and $\widetilde{\theta} \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$, $\widetilde{q} \in L_{2,\nu}(\mathbb{R}; \operatorname{ran}(\operatorname{grad}_0))$ satisfy

$$\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{a}^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \iota_{\operatorname{ran}(\operatorname{grad}_0)} \\ \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} \operatorname{grad}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} \widetilde{\theta} \\ \widetilde{q} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Show that $(\theta, \iota^*_{\operatorname{ran}(\operatorname{grad}_0)} q) = (\widetilde{\theta}, \widetilde{q})$.

(b) Let $\Omega$ be bounded and consider the evolutionary equation

$$\left(\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Show that the associated solution operators are not exponentially stable.

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# 12 Boundary Value Problems and Boundary Value Spaces

This chapter is devoted to the study of inhomogeneous boundary value problems. For this, we shall reformulate the boundary value problem again into a form which fits within the general framework of evolutionary equations. In order to have an idea of the type of boundary values which make sense to study, we start off with a section that deals with the boundary values of functions in the domain of the gradient operator defined on a half space in $\mathbb{R}^d$ (for $d = 1$ we have $L_2(\mathbb{R}^{d-1}) = \mathbb{K}$).

## 12.1 The Boundary Values of $H^1(\mathbb{R}^{d-1} \times \mathbb{R}_{>0})$

In this section we let $\Omega \coloneqq \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$ and $f \in H^1(\Omega)$; our aim is to make sense of the function $\mathbb{R}^{d-1} \ni \check{x} \mapsto f(\check{x}, 0)$. Note that this makes no sense if we only assume $f \in L_2(\Omega)$ since $\mathbb{R}^{d-1} \times \{0\} = \partial\Omega$ is a set of ($d$-dimensional) Lebesgue-measure zero. However, if we assume $f$ to be weakly differentiable, something more can be said and the boundary values can be defined by means of a continuous extension of the so-called trace map. In order to properly formulate this, we need the following density result.

**Theorem 12.1.1.** *The set $\mathcal{D} \coloneqq \left\{\phi\colon \Omega \to \mathbb{K}\,;\, \exists \psi \in C_c^\infty(\mathbb{R}^d)\colon \psi|_\Omega = \phi\right\}$ is dense in the space $H^1(\Omega)$.*

We will need a density result for $H^1(\mathbb{R}^d)$ first.

**Lemma 12.1.2.** *$C_c^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$.*

*Proof.* Let $f \in H^1(\mathbb{R}^d)$. We first show that $f$ can be approximated by functions with compact support. For this let $\phi \in C_c^\infty(\mathbb{R}^d)$ with the properties $0 \leqslant \phi \leqslant 1$, $\phi = 1$ on $B\,(0, 1/2)$ and $\phi = 0$ on $\mathbb{R} \setminus B\,(0, 1)$. For all $k \in \mathbb{N}$ we put $\phi_k \coloneqq \phi(\cdot/k)$ and $f_k \coloneqq \phi_k f \in L_2(\mathbb{R}^d)$. Then $f_k$ has support contained in $B\,[0, k]$. The dominated convergence theorem implies that $f_k \to f$ in $L_2(\mathbb{R}^d)$ as $k \to \infty$. Next, let $\psi \in C_c^\infty(\mathbb{R}^d)^d$ and compute for all $k \in \mathbb{N}$

$$\begin{aligned} -\left\langle f_k, \operatorname{div}\psi\right\rangle &= -\left\langle \phi_k f, \operatorname{div}\psi\right\rangle = -\left\langle f, \phi_k \operatorname{div}\psi\right\rangle = -\left\langle f, \operatorname{div}\left(\phi_k\psi\right) - \left(\operatorname{grad}\phi_k\right)\cdot\psi\right\rangle \\ &= -\left\langle f, \operatorname{div}\left(\phi_k\psi\right)\right\rangle + \left\langle f\operatorname{grad}\phi_k, \psi\right\rangle = \left\langle (\operatorname{grad} f)\phi_k + \frac{1}{k} f (\operatorname{grad}\phi)(\cdot/k), \psi \right\rangle, \end{aligned}$$

which shows that $f_k \in \operatorname{dom}(\operatorname{grad}) = H^1(\mathbb{R}^d)$ and $\operatorname{grad} f_k = (\operatorname{grad} f)\phi_k + \frac{1}{k} f\,(\operatorname{grad}\phi)\,(\cdot/k)$. From this expression of $\operatorname{grad} f_k$ we observe $\operatorname{grad} f_k \to \operatorname{grad} f$ in $L_2(\mathbb{R}^d)^d$ by dominated convergence. Hence, $f_k \to f$ in $\operatorname{dom}(\operatorname{grad}) = H^1(\mathbb{R}^d)$.

To conclude the proof of this lemma it suffices to revisit Exercise 3.2. For this, let $(\psi_k)_k$ in $C_c^\infty(\mathbb{R}^d)$ be a $\delta$-sequence. Then, by Exercise 3.2, we infer $\psi_k * f \to f$ in $L_2(\mathbb{R}^d)$ as $k \to \infty$ and hence, by Exercise 12.1, it follows also that $\operatorname{grad}(\psi_k * f) = \psi_k * \operatorname{grad} f \to \operatorname{grad} f$ (note the component-wise definition of the convolution). A combination of the first part of this proof together with an estimate for the support of the convolution (see again Exercise 3.2) yields the assertion. □

*Proof of Theorem 12.1.1.* Let $f \in H^1(\Omega)$. The approximation of $f$ by functions in $\mathcal{D}$ is done in two steps. First, we shift $f$ in the negative $e_d$-direction to avoid the boundary, and then we convolve the shifted $f$ to obtain smooth approximants in $\mathcal{D}$.
Let $\widetilde{f} \in L_2(\mathbb{R}^d)$ be the extension of $f$ by zero. Put $e_d \coloneqq (\delta_{jd})_{j\in\{1,\ldots d\}}$, the $d$-th unit vector. Then for all $\tau > 0$ we have $\Omega + \tau e_d \subseteq \Omega$ and, thus by Exercise 12.2, we deduce $f_\tau \coloneqq \widetilde{f}(\cdot + \tau e_d)|_\Omega \to f$ in $H^1(\Omega)$ as $\tau \to 0$. Thus, it suffices to approximate $f_\tau$ for $\tau > 0$. Let $\tau > 0$ and let $(\psi_k)_k$ in $C_c^\infty(\mathbb{R}^d)$ be a $\delta$-sequence. Then $\psi_k * \widetilde{f}(\cdot + \tau e_d) \in H^1(\mathbb{R}^d)$, by Exercise 12.1. Define $f_{k,\tau} \coloneqq \big(\psi_k * \widetilde{f}(\cdot + \tau e_d)\big)|_\Omega$. Then we obtain that $f_{k,\tau} \to f_\tau$ in $H^1(\Omega)$ as $k \to \infty$. Indeed, the only thing left to prove is that $\operatorname{grad} f_{k,\tau} \to \operatorname{grad} f_\tau$ in $L_2(\Omega)^d$ as $k \to \infty$. For this, we denote by $g$ the extension of $\operatorname{grad} f$ by 0. Since $g \in L_2(\mathbb{R}^d)^d$ it suffices to show that $\operatorname{grad} f_{k,\tau} = \psi_k * g_\tau$ on $\Omega$ for all large enough $k \in \mathbb{N}$, where $g_\tau = g(\cdot + \tau e_d)$. Let $k > \frac{1}{\tau}$. Then for all $x \in \Omega$ and $y \in \operatorname{spt} \psi_k \subseteq [-1/k, 1/k]^d$ we infer $x - y + \tau e_d \in \Omega$. In particular, $f(\cdot - y + \tau e_d) \in H^1(\Omega)$ and $\operatorname{grad} f(\cdot - y + \tau e_d) = g(\cdot - y + \tau e_d)$. Take $\eta \in C_c^\infty(\Omega)^d$ and compute

$$
\begin{aligned}
-\left\langle f_{k,\tau}, \operatorname{div}\eta\right\rangle_{L_2(\Omega)} &= -\int_\Omega \int_{\mathbb{R}^d} \psi_k(x-y)\widetilde{f}(y+\tau e_d)^* \,\mathrm{d}y \operatorname{div}\eta(x)\,\mathrm{d}x \\
&= -\int_\Omega \int_{\mathbb{R}^d} \psi_k(y)\widetilde{f}(x-y+\tau e_d)^* \,\mathrm{d}y \operatorname{div}\eta(x)\,\mathrm{d}x \\
&= -\int_\Omega \int_{[-1/k,1/k]^d} \psi_k(y) f(x-y+\tau e_d)^* \,\mathrm{d}y \operatorname{div}\eta(x)\,\mathrm{d}x \\
&= -\int_{[-1/k,1/k]^d} \psi_k(y) \left\langle f(\cdot - y + \tau e_d), \operatorname{div}\eta\right\rangle_{L_2(\Omega)} \,\mathrm{d}y \\
&= \int_{[-1/k,1/k]^d} \psi_k(y) \left\langle g(\cdot - y + \tau e_d), \eta\right\rangle_{L_2(\Omega)^d} \,\mathrm{d}y \\
&= \left\langle \psi_k * g_\tau, \eta\right\rangle_{L_2(\Omega)^d}.
\end{aligned}
$$

As $\psi_k * \widetilde{f}(\cdot + \tau e_d) \in H^1(\mathbb{R}^d)$, we conclude the proof using Lemma 12.1.2. □

With these preparations at hand, we can define the boundary trace of $H^1(\Omega)$.

**Theorem 12.1.3.** *The operator*

$$
\begin{aligned}
\gamma\colon \mathcal{D} \subseteq H^1(\Omega) &\to L_2(\mathbb{R}^{d-1}) \\
f &\mapsto \big(\mathbb{R}^{d-1} \ni \check{x} \mapsto f(\check{x}, 0)\big)
\end{aligned}
$$

*is continuous, densely defined and, thus, admits a unique continuous extension to* $H^1(\Omega)$ *again denoted by* $\gamma$. *Moreover, we have*

$$\|\gamma f\|_{L_2(\mathbb{R}^{d-1})} \leqslant \left(2\|f\|_{L_2(\Omega)}\|\operatorname{grad} f\|_{L_2(\Omega)^d}\right)^{\frac{1}{2}} \leqslant \|f\|_{H^1(\Omega)} \quad (f \in H^1(\Omega)).$$

*Proof.* Note that $\gamma$ is densely defined by Theorem 12.1.1. Let $f \in C_c^\infty(\mathbb{R}^d)$ and $\check{x} \in \mathbb{R}^{d-1}$. Let $R > 0$ be such that $\operatorname{spt} f \subseteq B(0, R)$. Then

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} |f(\check{x}, 0)|^2 \,\mathrm{d}\check{x} &= -\int_{\mathbb{R}^{d-1}} \int_0^R \partial_d |f(\check{x}, \hat{x})|^2 \,\mathrm{d}\hat{x} \,\mathrm{d}\check{x} \\ &= -\int_\Omega \big(f(x)^* \partial_d f(x) + \partial_d f^*(x) f(x)\big) \,\mathrm{d}x \\ &\leqslant 2\|f\|_{L_2(\Omega)} \|\operatorname{grad} f\|_{L_2(\Omega)^d}.\end{aligned}$$

The remaining inequality follows from $2ab \leqslant a^2 + b^2$ for all $a, b \in \mathbb{R}$. □

Except for one spatial dimension, where the boundary trace can be obtained by point evaluation, the boundary trace $\gamma$ does not map onto the whole of $L_2(\mathbb{R}^{d-1})$. Hence, in order to define the space of all possible boundary values for a function in $H^1$ one uses a quotient construction: we set

$$H^{1/2}(\mathbb{R}^{d-1}) := \{\gamma f\, ;\, f \in H^1(\Omega)\}$$

and endow $H^{1/2}(\mathbb{R}^{d-1})$ with the norm

$$\|\gamma f\|_{H^{1/2}(\mathbb{R}^{d-1})} := \inf\left\{\|g\|_{H^1(\Omega)}\, ;\, g \in H^1(\Omega), \gamma g = \gamma f\right\}.$$

It is not difficult to see that $H^{1/2}(\mathbb{R}^{d-1})$ is unitarily equivalent to $(\ker \gamma)^\perp$, where the orthogonal complement is computed with respect to the scalar product in $H^1(\Omega)$. Thus, $H^{1/2}(\mathbb{R}^{d-1})$ is a Hilbert space.

*Remark* 12.1.4. The norm defined on the space $H^{1/2}(\mathbb{R}^{d-1})$ given above is not the standard norm defined on this space. Indeed, following [Neč12, Section 2.3.8] the usual norm is given by

$$\left(\|u\|^2_{L_2(\mathbb{R}^{d-1})} + \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|u(x) - u(y)|^2}{|x - y|^d} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2}$$

for $u \in H^{1/2}(\mathbb{R}^{d-1})$. However, this norm turns out to be equivalent to the norm given above, see e.g. [Tro14, Section 4].
As the notation of this space suggests, it is can also defined as an interpolation space between $H^1(\mathbb{R}^{d-1})$ and $L_2(\mathbb{R}^{d-1})$, see [LM72, Theorem 15.1].

## 12.2 The Boundary Values of $H(\mathrm{div}, \mathbb{R}^{d-1} \times \mathbb{R}_{>0})$

Let $\Omega := \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. There is also a space of corresponding boundary traces for the divergence operator. Similarly to the boundary values for the domain of the gradient operator, $H^1(\Omega)$, the construction of the boundary trace for $H(\mathrm{div})$-vector fields rests on a density result. The proof can be done along the lines of Theorem 12.1.1 and will be addressed in Exercise 12.3.

**Theorem 12.2.1.** *$\mathcal{D}^d$ is dense in $H(\mathrm{div}, \Omega)$.*

Equipped with this result, we can describe all possible boundary values of $H(\mathrm{div}, \Omega)$. It will turn out that vector fields in $H(\mathrm{div}, \Omega)$ have a well-defined *normal* trace, which for $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$ is just the negative of the last coordinate of the vector field.

**Theorem 12.2.2.** *The operator*

$$\begin{aligned}\gamma_{\mathrm{n}} \colon \mathcal{D}^d \subseteq H(\mathrm{div}, \Omega) &\to \Big(H^{1/2}(\mathbb{R}^{d-1})\Big)' =: H^{-1/2}(\mathbb{R}^{d-1}) \\ q &\mapsto \big(\mathbb{R}^{d-1} \ni \breve{x} \mapsto -q_d(\breve{x}, 0)\big),\end{aligned}$$

*is densely defined, continuous with norm bounded by 1 and has dense range. Thus $\gamma_{\mathrm{n}}$ admits a unique extension to $H(\mathrm{div}, \Omega)$ again denoted by $\gamma_{\mathrm{n}}$. Here, $-q_d$ is the negative of the d-th component of $q$ pointing into the outward normal direction of $\Omega$ and $-q_d$ is identified with the linear functional*

$$H^{1/2}(\mathbb{R}^d) \ni \gamma f \mapsto \langle -q_d(\cdot, 0), \gamma f\rangle_{L_2(\mathbb{R}^{d-1})}.$$

*Moreover, for all $f \in \mathrm{dom}(\mathrm{grad})$ and $q \in \mathrm{dom}(\mathrm{div})$ we have*

$$\langle \mathrm{div}\, q, f\rangle + \langle q, \mathrm{grad}\, f\rangle = (\gamma_{\mathrm{n}} q)(\gamma f), \tag{12.1}$$

*where we denoted the extension of $\gamma_{\mathrm{n}}$ again by $\gamma_{\mathrm{n}}$.*

*Proof.* Let $f \in \mathcal{D}$ and $q \in \mathcal{D}^d$. Then integration by parts yields

$$\begin{aligned}\langle \mathrm{div}\, q, f\rangle + \langle q, \mathrm{grad}\, f\rangle = \int_\Omega \mathrm{div}(q^* f) &= \int_{\mathbb{R}^{d-1}} \langle q^*(\breve{x}, 0) f(\breve{x}, 0), -e_d\rangle \,\mathrm{d}\breve{x} \\ &= -\int_{\mathbb{R}^{d-1}} \gamma q_d^* \gamma f = \langle \gamma_{\mathrm{n}} q, \gamma f\rangle_{L_2(\mathbb{R}^{d-1})} = (\gamma_{\mathrm{n}} q)(\gamma f).\end{aligned}$$

Hence,

$$\Big|\langle \gamma_{\mathrm{n}} q, \gamma f\rangle_{L_2(\mathbb{R}^{d-1})}\Big| \leqslant \|q\|_{H(\mathrm{div})} \|f\|_{H^1}.$$

Since $\mathcal{D}$ is dense in $H^1(\Omega)$, the inequality remains true for all $f \in H^1(\Omega)$. Thus,

$$\Big|\langle \gamma_{\mathrm{n}} q, \gamma f\rangle_{L_2(\mathbb{R}^{d-1})}\Big| \leqslant \|q\|_{H(\mathrm{div})} \|f\|_{H^1} \quad (f \in H^1(\Omega)).$$

Computing the infimum over all $g \in H^1(\Omega)$ with $\gamma g = \gamma f$, we deduce

$$\left|\langle \gamma_{\mathrm{n}} q, \gamma f\rangle_{L_2(\mathbb{R}^{d-1})}\right| \leqslant \|q\|_{H(\mathrm{div})}\|\gamma f\|_{H^{1/2}(\mathbb{R}^{d-1})} \quad (f \in H^1(\Omega)).$$

Therefore $\gamma_{\mathrm{n}} q \in H^{-1/2}(\mathbb{R}^{d-1})$ and $\|\gamma_{\mathrm{n}} q\|_{H^{-1/2}} \leqslant \|q\|_{H(\mathrm{div})}$, which shows continuity of $\gamma_{\mathrm{n}}$. It is left to show that $\gamma_{\mathrm{n}}$ has dense range. For this, take $\gamma f \in H^{1/2}(\mathbb{R}^{d-1})$ for some $f \in H^1(\Omega)$ such that

$$\langle \gamma_{\mathrm{n}} g, \gamma f\rangle_{L_2(\mathbb{R}^{d-1})} = 0$$

for all $g \in \mathcal{D}^d$. Next, take $\widetilde{g} \in C_{\mathrm{c}}^\infty(\mathbb{R}^{d-1})$ and $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ with $\psi(0) = 1$. Then we set $g\colon \Omega \ni (\breve{x}, \widehat{x}) \mapsto -e_d \widetilde{g}(\breve{x})\psi(\widehat{x}) \in \mathcal{D}^d$ and note that $\gamma_{\mathrm{n}} g = \widetilde{g}$. Hence

$$\langle \gamma f, \widetilde{g}\rangle_{L_2(\mathbb{R}^{d-1})} = 0 \quad (\widetilde{g} \in C_{\mathrm{c}}^\infty(\mathbb{R}^{d-1})).$$

Thus, $\gamma f = 0$, which implies that the range of $\gamma_{\mathrm{n}}$ is dense, as $H^{1/2}(\mathbb{R}^{d-1})$ is a Hilbert space. The remaining formula (12.1) follows by continuously extending both the left- and right-hand side of the integration by parts formula from the beginning of the proof. Note that for this, we have used both Theorem 12.1.1 and Theorem 12.2.1. □

**Corollary 12.2.3.** *Let $f \in H^1(\Omega)$, $q \in H(\mathrm{div}, \Omega)$. Then $f \in \mathrm{dom}(\mathrm{grad}_0)$ if and only if $\gamma f = 0$, and $q \in \mathrm{dom}(\mathrm{div}_0)$ if and only if $\gamma_{\mathrm{n}} q = 0$.*

*Proof.* We only show the statement for $q$. The proof for $f$ is analogous. If $q \in \mathrm{dom}(\mathrm{div}_0)$, then there exists a sequence $(\psi_n)_n$ in $C_{\mathrm{c}}^\infty(\Omega)^d$ such that $\psi_n \to q$ in $H(\mathrm{div}, \Omega)$ as $n \to \infty$. Thus, by continuity of $\gamma_{\mathrm{n}}$, we infer $0 = \gamma_{\mathrm{n}}\psi_n \to \gamma_{\mathrm{n}} q$. Assume on the other hand that $q \in \mathrm{dom}(\mathrm{div})$ with $\gamma_{\mathrm{n}} q = 0$. Using (12.1), we obtain for all $f \in \mathrm{dom}(\mathrm{grad})$

$$\langle \mathrm{div}\, q, f\rangle + \langle q, \mathrm{grad}\, f\rangle = 0.$$

This equality implies that $q \in \mathrm{dom}(\mathrm{grad}^*) = \mathrm{dom}(\mathrm{div}_0)$, which shows the remaining assertion. □

The remaining part of this section is devoted to showing that the continuous extension of $\gamma_{\mathrm{n}}$ maps onto $H^{-1/2}(\mathbb{R}^{d-1})$. For this we require the following observation, which will also be needed later on.

**Proposition 12.2.4.** *Let $U \subseteq \mathbb{R}^d$ be open. Then*

$$H_0(\mathrm{div}, U)^{\perp_{H(\mathrm{div},U)}} = \left\{q \in H(\mathrm{div}, U)\,;\, \mathrm{div}\, q \in H^1(U), q = \mathrm{grad}\,\mathrm{div}\, q\right\}.$$

*Proof.* Let $q \in H(\mathrm{div}, U)$. Then $q \in H_0(\mathrm{div}, U)^{\perp_{H(\mathrm{div},U)}}$ if and only if for all $r \in H_0(\mathrm{div}, U)$ we have

$$0 = \langle r, q\rangle_{H(\mathrm{div},U)} = \langle r, q\rangle_{L_2(U)^d} + \langle \mathrm{div}\, r, \mathrm{div}\, q\rangle_{L_2(U)} = \langle r, q\rangle_{L_2(U)^d} + \langle \mathrm{div}_0\, r, \mathrm{div}\, q\rangle_{L_2(U)}.$$

The latter, in turn, is equivalent to $\mathrm{div}\, q \in \mathrm{dom}(\mathrm{div}_0^*) = \mathrm{dom}(\mathrm{grad}) = H^1(U)$ and $-\mathrm{grad}\,\mathrm{div}\, q = \mathrm{div}_0^*\,\mathrm{div}\, q = -q$. □

**Theorem 12.2.5.** *$\gamma_{\mathrm{n}}$ maps onto $H^{-1/2}(\mathbb{R}^{d-1})$. In particular, we have*

$$\|q\|_{H(\operatorname{div},\Omega)} \leqslant \|\gamma_{\mathrm{n}} q\|_{H^{-1/2}(\mathbb{R}^{d-1})}$$

*for all $q \in H_0(\operatorname{div},\Omega)^{\perp_{H(\operatorname{div},\Omega)}}$.*

*Proof.* By Theorem 12.2.2 it suffices to show that $\gamma_{\mathrm{n}}$ has closed range. For this, it suffices to show that there exists $c > 0$ such that

$$\|q\|_{H(\operatorname{div},\Omega)} \leqslant c\|\gamma_{\mathrm{n}} q\|_{H^{-1/2}(\mathbb{R}^{d-1})}$$

for all $q \in \ker(\gamma_{\mathrm{n}})^{\perp_{H(\operatorname{div},\Omega)}}$. By Corollary 12.2.3, we obtain $\ker(\gamma_{\mathrm{n}}) = H_0(\operatorname{div},\Omega)$. Hence, by Proposition 12.2.4, we deduce that $q \in \ker(\gamma_{\mathrm{n}})^{\perp_{H(\operatorname{div},\Omega)}}$ if and only if $q \in \operatorname{dom}(\operatorname{grad}\operatorname{div})$ and $q = \operatorname{grad}\operatorname{div} q$. So, assume that $q \in \operatorname{dom}(\operatorname{grad}\operatorname{div})$ with $q = \operatorname{grad}\operatorname{div} q$. Then (12.1) applied to $q \in \operatorname{dom}(\operatorname{div})$ and $f = \operatorname{div} q \in \operatorname{dom}(\operatorname{grad})$ yields

$$\begin{aligned}(\gamma_{\mathrm{n}} q)(\gamma \operatorname{div} q) &= \langle \operatorname{div} q, \operatorname{div} q\rangle + \langle q, \operatorname{grad}\operatorname{div} q\rangle = \langle \operatorname{div} q, \operatorname{div} q\rangle + \langle q, q\rangle \\ &= \|q\|^2_{H(\operatorname{div},\Omega)},\end{aligned}$$

where we used $\operatorname{grad}\operatorname{div} q = q$. Hence

$$\begin{aligned}\|q\|^2_{H(\operatorname{div},\Omega)} &\leqslant \|\gamma \operatorname{div} q\|_{H^{1/2}}\|\gamma_{\mathrm{n}} q\|_{H^{-1/2}} \leqslant \|\operatorname{div} q\|_{H^1(\Omega)}\|\gamma_{\mathrm{n}} q\|_{H^{-1/2}} \\ &= \|q\|_{H(\operatorname{div},\Omega)}\|\gamma_{\mathrm{n}} q\|_{H^{-1/2}}\end{aligned}$$

where we again used that $\operatorname{grad}\operatorname{div} q = q$. This yields the assertion. □

## 12.3 Inhomogeneous Boundary Value Problems

Let $\Omega := \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. With the notion of traces we now have a tool at hand that allows us to formulate inhomogeneous boundary value problems. Here we focus on the scalar wave type equation for given Neumann data $\widetilde{g} \in H^{-1/2}(\mathbb{R}^{d-1})$. We shall address other boundary value problems in the exercises. Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L\big(L_2(\Omega) \times L_2(\Omega)^d\big)$ be a material law with $\mathrm{s_b}(M) < \nu_0$ for some $\nu_0 \in \mathbb{R}$. We assume that $M$ satisfies the positive definiteness condition in Theorem 6.2.1; that is, we assume there exists $c > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}$ we have $\operatorname{Re} zM(z) \geqslant c$. For $\nu \geqslant \nu_0$ we want to solve

$$\begin{cases}\left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix} v \\ q\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix} & \text{on } \Omega, \\ \gamma_{\mathrm{n}} q(t,\cdot) = \widetilde{g} & \text{on } \partial\Omega \text{ for all } t \in \mathbb{R}.\end{cases}$$

Let us reformulate this problem. Let $\phi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \phi \leqslant 1$ with $\phi = 1$ on $[0,\infty)$ and $\phi = 0$ on $(-\infty,-1]$. We define the function $g := \big(t \mapsto \phi(t)\widetilde{g} \in H^{-1/2}(\mathbb{R}^{d-1})\big) \in \bigcap_{\nu>0} L_{2,\nu}(\mathbb{R}; H^{-1/2}(\mathbb{R}^{d-1}))$ and consider

$$\begin{cases}\left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix} v \\ q\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix} & \text{on } \Omega, \\ \gamma_{\mathrm{n}} q(t) = g(t) & \text{for all } t > 0.\end{cases} \tag{12.2}$$

instead.

**Theorem 12.3.1.** *Let $\nu \geqslant \max\{\nu_0, 0\}, \nu \neq 0$. Then (12.2) admits a unique solution $(v,q) \in H_\nu^1\Big(\mathbb{R}; \mathrm{dom}\Big(\begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix}\Big)\Big)$.*

*Proof.* We start with the existence part. By Theorem 12.2.5, we find $\widetilde{G} \in H(\mathrm{div}, \Omega)$ such that $\gamma_{\mathrm{n}}\widetilde{G} = \widetilde{g}$; set $G := \phi(\cdot)\widetilde{G} \in H_\nu^3(\mathbb{R}; H(\mathrm{div}, \Omega))$. Consider the following evolutionary equation

$$\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div}_0 \\ \mathrm{grad} & 0 \end{pmatrix}}\right) \begin{pmatrix} u \\ r \end{pmatrix} = \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} 0 \\ -G \end{pmatrix} + \begin{pmatrix} -\mathrm{div}\, G \\ 0 \end{pmatrix}.$$

Note that the right-hand side is in $H_\nu^2(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d)$. By Theorem 6.2.1, we obtain

$$\begin{aligned}\begin{pmatrix} u \\ r \end{pmatrix} &= \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div}_0 \\ \mathrm{grad} & 0 \end{pmatrix}}\right)^{-1} \left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} 0 \\ -G \end{pmatrix} + \begin{pmatrix} -\mathrm{div}\, G \\ 0 \end{pmatrix}\right) \\ &\in H_\nu^1(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d) \cap L_{2,\nu}\Big(\mathbb{R}; \mathrm{dom}\Big(\begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix}\Big)\Big).\end{aligned}$$

Indeed, since the solution operator commutes with $\partial_{t,\nu}$ and the right-hand side lies in $H_\nu^2$, it even follows that $\begin{pmatrix} u \\ r \end{pmatrix} \in H_\nu^2(\mathbb{R}; L_2(\Omega) \times L_2(\Omega)^d)$. From the equality

$$\left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div}_0 \\ \mathrm{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ r \end{pmatrix} = \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} 0 \\ -G \end{pmatrix} + \begin{pmatrix} -\mathrm{div}\, G \\ 0 \end{pmatrix}$$

it follows that

$$\begin{pmatrix} u \\ r \end{pmatrix} \in H_\nu^1\Big(\mathbb{R}; \mathrm{dom}\Big(\begin{pmatrix} 0 & \mathrm{div}_0 \\ \mathrm{grad} & 0 \end{pmatrix}\Big)\Big).$$

Also, we deduce that

$$\left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ r+G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $r \in H_\nu^1(\mathbb{R}; \mathrm{dom}(\mathrm{div}_0))$, by Corollary 12.2.3 and Theorem 4.1.2 we obtain

$$\gamma_{\mathrm{n}}\left((r+G)(t)\right) = \gamma_{\mathrm{n}} G(t) = g(t) \quad (t \in \mathbb{R}).$$

Hence, $(u, r+G)$ solves (12.2).
Next we address the uniqueness result. For this we note that a straightforward computation shows

$$\begin{pmatrix} v \\ q-G \end{pmatrix} = \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div}_0 \\ \mathrm{grad} & 0 \end{pmatrix}}\right)^{-1} \left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} 0 \\ -G \end{pmatrix} + \begin{pmatrix} -\mathrm{div}\, G \\ 0 \end{pmatrix}\right),$$

which coincides with the formula for $(u, r+G)$. □

The upshot of the rationale exemplified in the proof is that inhomogeneous boundary value problems can be reduced to an evolutionary equation of the standard form with non-vanishing right-hand side. Of course the treatment of inhomogeneous Dirichlet data works along similar lines.

## 12.4 Abstract Boundary Data Spaces

Of course inhomogeneous boundary value problems can be addressed for other domains $\Omega$ than the half space $\mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Classically, some more specific properties need to be imposed on the description of the boundary $\partial\Omega$. In this section, however, we deviate from the classical perspective in as much as we like to consider *arbitrary* open sets $\Omega \subseteq \mathbb{R}^d$. For this we introduce

$$\begin{aligned} \mathrm{BD}(\mathrm{div}) &= \{q \in H(\mathrm{div}, \Omega)\,;\, \mathrm{div}\, q \in \mathrm{dom}(\mathrm{grad}), \mathrm{grad}\,\mathrm{div}\, q = q\}\,, \\ \mathrm{BD}(\mathrm{grad}) &= \left\{u \in H^1(\Omega)\,;\, \mathrm{grad}\, u \in \mathrm{dom}(\mathrm{div}), \mathrm{div}\,\mathrm{grad}\, u = u\right\}. \end{aligned}$$

By Proposition 12.2.4 and Exercise 6.7, these spaces are closed subspaces of $H(\mathrm{div}, \Omega)$ and $H^1(\Omega)$, respectively, and therefore Hilbert spaces. Indeed,

$$\mathrm{BD}(\mathrm{div}) = H_0(\mathrm{div}, \Omega)^{\perp_{H(\mathrm{div},\Omega)}}$$

and

$$\mathrm{BD}(\mathrm{grad}) = H_0^1(\Omega)^{\perp_{H^1(\Omega)}}.$$

Now, we are in a position to solve inhomogeneous boundary value problems, where the trace mappings $\gamma$ and $\gamma_{\mathrm{n}}$ are replaced by the canonical orthogonal projections $\pi_{\mathrm{BD}(\mathrm{grad})}$ and $\pi_{\mathrm{BD}(\mathrm{div})}$ respectively; see Exercise 12.4. We devote the rest of this section to describe the relationship between the classical trace spaces introduced before and the BD-spaces. In the perspective outlined here, there is not much of a difference between Neumann boundary values and Dirichlet boundary values. The next result is an incarnation of this.

**Proposition 12.4.1.** *We have*

$$\mathrm{grad}[\mathrm{BD}(\mathrm{grad})] \subseteq \mathrm{BD}(\mathrm{div}) \quad \textit{and} \quad \mathrm{div}[\mathrm{BD}(\mathrm{div})] \subseteq \mathrm{BD}(\mathrm{grad}).$$

*Moreover, the mappings*

$$\begin{aligned} \mathrm{grad}_{\mathrm{BD}} \colon \mathrm{BD}(\mathrm{grad}) &\to \mathrm{BD}(\mathrm{div}), \\ u &\mapsto \mathrm{grad}\, u \end{aligned}$$

*and*

$$\begin{aligned} \mathrm{div}_{\mathrm{BD}} \colon \mathrm{BD}(\mathrm{div}) &\to \mathrm{BD}(\mathrm{grad}), \\ q &\mapsto \mathrm{div}\, q \end{aligned}$$

*are unitary, and* $\mathrm{grad}_{\mathrm{BD}}^* = \mathrm{div}_{\mathrm{BD}}$.

*Proof.* Let $\phi \in \mathrm{BD}(\mathrm{grad})$. Then $\mathrm{grad}\,\phi \in H(\mathrm{div}, \Omega)$ and $\mathrm{div}\,\mathrm{grad}\,\phi = \phi$. This implies $\mathrm{div}\,\mathrm{grad}\,\phi \in \mathrm{dom}(\mathrm{grad})$ and $\mathrm{grad}\,\mathrm{div}\,\mathrm{grad}\,\phi = \mathrm{grad}\,\phi$, which yields $\mathrm{grad}\,\phi \in \mathrm{BD}(\mathrm{div})$. Thus, $\mathrm{grad}_{\mathrm{BD}}$ is defined everywhere; interchanging the roles of grad and div, we obtain $\mathrm{div}_{\mathrm{BD}}$ is also defined everywhere. We infer $\mathrm{div}_{\mathrm{BD}}\,\mathrm{grad}_{\mathrm{BD}} = 1_{\mathrm{BD}(\mathrm{grad})}$ and $\mathrm{grad}_{\mathrm{BD}}\,\mathrm{div}_{\mathrm{BD}} =$

$1_{\mathrm{BD(div)}}$ and thus $\mathrm{grad}_{\mathrm{BD}}$ is bijective with $\mathrm{grad}_{\mathrm{BD}}^{-1} = \mathrm{div}_{\mathrm{BD}}$. It remains to show that $\mathrm{grad}_{\mathrm{BD}}$ preserves the norm. For this we compute

$$\begin{aligned}
\langle \mathrm{grad}_{\mathrm{BD}}\,\phi, \mathrm{grad}_{\mathrm{BD}}\,\phi\rangle_{\mathrm{BD(div)}} &= \langle \mathrm{grad}\,\phi, \mathrm{grad}\,\phi\rangle_{H(\mathrm{div})} \\
&= \langle \mathrm{grad}\,\phi, \mathrm{grad}\,\phi\rangle_{L_2(\Omega)^d} + \langle \mathrm{div}\,\mathrm{grad}\,\phi, \mathrm{div}\,\mathrm{grad}\,\phi\rangle_{L_2(\Omega)} \\
&= \langle \mathrm{grad}\,\phi, \mathrm{grad}\,\phi\rangle_{L_2(\Omega)^d} + \langle \phi, \phi\rangle_{L_2(\Omega)} \\
&= \langle \phi, \phi\rangle_{\mathrm{dom(grad)}} = \langle \phi, \phi\rangle_{\mathrm{BD(grad)}}\,,
\end{aligned}$$

which implies that $\mathrm{grad}_{\mathrm{BD}}$ is unitary. Hence, $\mathrm{div}_{\mathrm{BD}} = \mathrm{grad}_{\mathrm{BD}}^{-1} = \mathrm{grad}_{\mathrm{BD}}^{*}$. □

It is also possible to show an 'integration by parts' formula analogous to (12.1) for the abstract situation:

**Proposition 12.4.2.** *Let $u \in H^1(\Omega)$ and $q \in H(\mathrm{div}, \Omega)$. Then*

$$\begin{aligned}
\langle \mathrm{div}\,q, u\rangle_{L_2(\Omega)} + \langle q, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} &= \left\langle \mathrm{div}_{\mathrm{BD}}\,\pi_{\mathrm{BD(div)}}q, \pi_{\mathrm{BD(grad)}}u\right\rangle_{\mathrm{BD(grad)}} \\
&= \left\langle \pi_{\mathrm{BD(div)}}q, \mathrm{grad}_{\mathrm{BD}}\,\pi_{\mathrm{BD(grad)}}u\right\rangle_{\mathrm{BD(div)}}\,.
\end{aligned}$$

*Proof.* We decompose $u = u_0 + u_1$ and $q = q_0 + q_1$ with $u_0 \in H_0^1(\Omega)$, $q_0 \in H_0(\mathrm{div}, \Omega)$, $u_1 = \pi_{\mathrm{BD(grad)}}u$ and $q_1 = \pi_{\mathrm{BD(div)}}q$. Then we obtain

$$\begin{aligned}
&\langle \mathrm{div}\,q, u\rangle_{L_2(\Omega)} + \langle q, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} \\
&= \langle \mathrm{div}_0\,q_0, u\rangle_{L_2(\Omega)} + \langle \mathrm{div}\,q_1, u\rangle_{L_2(\Omega)} + \langle q_0, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} + \langle q_1, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} \\
&= \langle q_0, -\,\mathrm{grad}\,u\rangle_{L_2(\Omega)^d} + \langle \mathrm{div}\,q_1, u\rangle_{L_2(\Omega)} + \langle q_0, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} + \langle q_1, \mathrm{grad}\,u\rangle_{L_2(\Omega)^d} \\
&= \langle \mathrm{div}\,q_1, u_0\rangle_{L_2(\Omega)} + \langle \mathrm{div}\,q_1, u_1\rangle_{L_2(\Omega)} + \langle q_1, \mathrm{grad}\,u_0\rangle_{L_2(\Omega)^d} + \langle q_1, \mathrm{grad}\,u_1\rangle_{L_2(\Omega)^d} \\
&= \langle q_1, -\,\mathrm{grad}_0\,u_0\rangle_{L_2(\Omega)^d} + \langle \mathrm{div}\,q_1, u_1\rangle_{L_2(\Omega)} + \langle q_1, \mathrm{grad}_0\,u_0\rangle_{L_2(\Omega)^d} + \langle q_1, \mathrm{grad}\,u_1\rangle_{L_2(\Omega)^d} \\
&= \langle \mathrm{div}\,q_1, u_1\rangle_{L_2(\Omega)} + \langle q_1, \mathrm{grad}\,u_1\rangle_{L_2(\Omega)^d} \\
&= \langle \mathrm{div}\,q_1, u_1\rangle_{L_2(\Omega)} + \langle \mathrm{grad}\,\mathrm{div}\,q_1, \mathrm{grad}\,u_1\rangle_{L_2(\Omega)^d} = \langle \mathrm{div}\,q_1, u_1\rangle_{\mathrm{BD(grad)}}\,.
\end{aligned}$$

The remaining equality follows from the unitarity of $\mathrm{grad}_{\mathrm{BD}}$ and $\mathrm{grad}_{\mathrm{BD}}^{-1} = \mathrm{div}_{\mathrm{BD}}$ by Proposition 12.4.1. □

In view of Proposition 12.4.2 the proper replacement of $\gamma_{\mathrm{n}}$ appears to be $\mathrm{div}_{\mathrm{BD}}\,\pi_{\mathrm{BD(div)}}$ instead of just $\pi_{\mathrm{BD(div)}}$. Next, we show the equivalence of the trace spaces for the half space and the abstract ones introduced in this section.

**Theorem 12.4.3.** *Let $\Omega \coloneqq \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Then $\gamma|_{\mathrm{BD(grad)}} : \mathrm{BD(grad)} \to H^{1/2}(\mathbb{R}^{d-1})$ and $\gamma_{\mathrm{n}}|_{\mathrm{BD(div)}} : \mathrm{BD(div)} \to H^{-1/2}(\mathbb{R}^{d-1})$ are unitary mappings.*

*Proof.* We begin with $\gamma_{\mathrm{n}}$. We have shown in Theorem 12.2.2 that $\gamma_{\mathrm{n}}|_{\mathrm{BD(div)}}$ is continuous and in Theorem 12.2.5 it has been shown that $(\gamma_{\mathrm{n}}|_{\mathrm{BD(div)}})^{-1}$ is continuous. Also the two norm inequalities have been established.
The injectivity of $\gamma|_{\mathrm{BD(grad)}}$ follows from $\ker\gamma = H_0^1(\Omega)$ by Corollary 12.2.3. All that remains simply relies upon recalling that $H^{1/2}(\mathbb{R}^{d-1})$ is isomorphic to $(\ker\gamma)^{\perp}$ with the orthogonal complement computed in $H^1(\Omega)$. □

## 12.5 Robin Boundary Conditions

The classical Robin boundary conditions involve both traces, the Dirichlet trace $\gamma$ and the Neumann trace $\gamma_{\mathrm{n}}$. To motivate things, let us again have a look at the case $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. We consider the boundary condition for given $q \in H(\operatorname{div}, \Omega)$ and $u \in H^1(\Omega)$

$$\gamma_{\mathrm{n}} q + \mathrm{i}\gamma u = 0,$$

in the sense that

$$(\gamma_{\mathrm{n}} q)(v) = \langle -\mathrm{i}\gamma u, v\rangle_{L_2(\mathbb{R}^{d-1})} \quad (v \in H^{1/2}(\mathbb{R}^{d-1})).$$

Note that this is an implicit regularity statement as $\gamma_{\mathrm{n}} q \in H^{-1/2}(\mathbb{R}^{d-1})$ is representable as an $L_2(\mathbb{R}^{d-1})$ function. The next result asserts that an evolutionary equation with a spatial operator of the type $\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$ with the above Robin boundary condition fits into the setting rendered by Theorem 6.2.1. In other words:

**Theorem 12.5.1.** *Let $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Then the operator $A \colon \operatorname{dom}(A) \subseteq L_2(\Omega)^{d+1} \to L_2(\Omega)^{d+1}$ with $A \subseteq \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$ with domain*

$$\operatorname{dom}(A) = \big\{(u, q) \in H^1(\Omega) \times H(\operatorname{div}, \Omega)\,;\, \gamma_{\mathrm{n}} q + \mathrm{i}\gamma u = 0\big\}$$

*is skew-selfadjoint.*

*Proof.* Let $(u, q), (v, r) \in H^1(\Omega) \times H(\operatorname{div}, \Omega)$. Then, by (12.1) we obtain

$$\begin{aligned}
&\left\langle \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} u \\ q \end{pmatrix}, \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u \\ q \end{pmatrix}, \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle \\
&= \langle \operatorname{div} q, v\rangle + \langle \operatorname{grad} u, r\rangle + \langle u, \operatorname{div} r\rangle + \langle q, \operatorname{grad} v\rangle = (\gamma_{\mathrm{n}} q)(\gamma v) + ((\gamma_{\mathrm{n}} r)(\gamma u))^*
\end{aligned}$$

If, in addition, $(u, q) \in \operatorname{dom}(A)$, we obtain

$$\begin{aligned}
&\left\langle A \begin{pmatrix} u \\ q \end{pmatrix}, \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u \\ q \end{pmatrix}, \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} \right\rangle \\
&= (\gamma_{\mathrm{n}} q)(\gamma v) + ((\gamma_{\mathrm{n}} r)(\gamma u))^* = \langle -\mathrm{i}\gamma u, \gamma v\rangle_{L_2(\mathbb{R}^{d-1})} + ((\gamma_{\mathrm{n}} r)(\gamma u))^* \\
&= \langle \gamma u, \mathrm{i}\gamma v\rangle_{L_2(\mathbb{R}^{d-1})} + ((\gamma_{\mathrm{n}} r)(\gamma u))^* = ((\mathrm{i}\gamma v + \gamma_{\mathrm{n}} r)(\gamma u))^*.
\end{aligned}$$

Since for every $u \in \mathcal{D}$, we find $q \in \mathcal{D}^d$ such that $(u, q) \in \operatorname{dom}(A)$,

$$\gamma[\mathcal{D}] \subseteq \{\gamma u\,;\, \exists q \in H(\operatorname{div}, \Omega) \colon (u, q) \in \operatorname{dom}(A)\}.$$

Thus, the set on the right-hand side is dense in $H^{1/2}(\mathbb{R}^{d-1})$. This in turn implies that $(v, r) \in \operatorname{dom}(A^*)$ if and only if $\mathrm{i}\gamma v + \gamma_{\mathrm{n}} r = 0$, and in this case we have $A^*(v, r) = -A(v, r)$. This implies that $A$ is skew-selfadjoint. $\square$

*Remark* 12.5.2. The factor i in front of $\gamma u$ is chosen as a mere convenience in order to render the corresponding operator $A$ in Theorem 12.5.1 skew-selfadjoint. It is also possible to choose $\beta \in L(H^{1/2}(\partial\Omega))$ with $-\operatorname{Re}\beta \geqslant 0$ instead of i. Then one obtains for all $U \in \operatorname{dom}(A)$ and $V \in \operatorname{dom}(A^*)$ the estimates $\operatorname{Re}\langle U, AU\rangle \geqslant 0$ and $\operatorname{Re}\langle V, A^*V\rangle \geqslant 0$. Appealing to Remark 6.3.2, it can be shown that the corresponding evolutionary equation

$$(\partial_{t,\nu}M(\partial_{t,\nu}) + A)U = F$$

for a suitable material law $M$ as in Theorem 6.2.1 is well-posed.

Next, one could argue that in the case for arbitrary $\Omega$, the condition

$$\mathrm{i}\pi_{\mathrm{BD(grad)}}u + \operatorname{div}_{\mathrm{BD}} \pi_{\mathrm{BD(div)}}q = 0 \tag{12.3}$$

amounts to a generalisation of the Robin boundary condition just considered. However, this is not true as the following proposition shows.

**Proposition 12.5.3.** *Let $u \in H^1(\Omega), q \in H(\operatorname{div}, \Omega)$. Moreover, we set $\kappa\colon \mathrm{BD(grad)} \to L_2(\mathbb{R}^{d-1})$ with $\kappa v = \gamma v$ for $v \in \mathrm{BD(grad)}$. Then $\gamma_{\mathrm{n}} q + \mathrm{i}\gamma u = 0$ if and only if*

$$\operatorname{div}_{\mathrm{BD}} \pi_{\mathrm{BD(div)}}q + \mathrm{i}\kappa^*\kappa\pi_{\mathrm{BD(grad)}}u = 0.$$

*Proof.* We first observe that $\kappa\pi_{\mathrm{BD(grad)}}w = \gamma w$ for each $w \in H^1(\Omega)$.
Assume now that $\gamma_{\mathrm{n}} q + \mathrm{i}\gamma u = 0$ and let $v \in \mathrm{BD(grad)}$. Then we compute, using Proposition 12.4.2 and (12.1)

$$\begin{aligned}
\left\langle \mathrm{i}\kappa^*\kappa\pi_{\mathrm{BD(grad)}}u, v\right\rangle_{\mathrm{BD(grad)}} &= \left\langle \mathrm{i}\kappa\pi_{\mathrm{BD(grad)}}u, \kappa v\right\rangle_{L_2(\mathbb{R}^{d-1})} \\
&= \langle \mathrm{i}\gamma u, \gamma v\rangle_{L_2(\mathbb{R}^{d-1})} \\
&= -(\gamma_{\mathrm{n}} q)(\gamma v) \\
&= \langle -\operatorname{div} q, v\rangle_{L_2(\Omega)} + \langle -q, \operatorname{grad} v\rangle_{L_2(\Omega)^d} \\
&= \left\langle -\operatorname{div}_{\mathrm{BD}} \pi_{\mathrm{BD(div)}}q, v\right\rangle_{\mathrm{BD(grad)}},
\end{aligned}$$

which proves one of the asserted implications.
Assume now that $\operatorname{div}_{\mathrm{BD}} \pi_{\mathrm{BD(div)}}q + \mathrm{i}\kappa^*\kappa\pi_{\mathrm{BD(grad)}}u = 0$ and let $v \in H^{1/2}(\mathbb{R}^{d-1})$. We take $w \in H^1(\Omega)$ with $\gamma w = v$ and compute

$$\begin{aligned}
(\gamma_{\mathrm{n}} q)(v) &= \langle \operatorname{div} q, w\rangle_{L_2(\Omega)} + \langle q, \operatorname{grad} w\rangle_{L_2(\Omega)^d} \\
&= \left\langle \operatorname{div}_{\mathrm{BD}} \pi_{\mathrm{BD(div)}}q, \pi_{\mathrm{BD(grad)}}w\right\rangle_{\mathrm{BD(grad)}} \\
&= \left\langle -\mathrm{i}\kappa^*\kappa\pi_{\mathrm{BD(grad)}}u, \pi_{\mathrm{BD(grad)}}w\right\rangle_{\mathrm{BD(grad)}} \\
&= \left\langle -\mathrm{i}\kappa\pi_{\mathrm{BD(grad)}}u, \kappa\pi_{\mathrm{BD(grad)}}w\right\rangle_{L_2(\mathbb{R}^{d-1})} \\
&= \langle -\mathrm{i}\gamma u, v\rangle_{L_2(\mathbb{R}^{d-1})},
\end{aligned}$$

which shows the remaining implication. □

## 12.6 Comments

The concept of abstract trace spaces has been introduced in [PTW16] in order to study a multi-dimensional analogue for port-Hamiltonian systems. Also concerning differential equations at the boundary (so-called impedance type boundary conditions), the concept of abstract boundary value spaces has been employed, see [Pic+16].
A comparison between abstract and classical trace spaces has been provided in [EGW18; Tro14] particularly concerning $H^{-1/2}(\mathbb{R}^{d-1})$. A good introduction for trace mappings for more complicated geometries can be found e.g. in [Are+15]. The trace operator can also be suitably established for $H(\mathrm{curl},\Omega)$-regular vector fields given that $\Omega$ is a so-called Lipschitz domain, see [BCS02].

## Exercises

**Exercise 12.1.** Let $\phi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$, $f \in L_2(\mathbb{R}^d)$. Show that

$$\phi * f \colon x \mapsto \int_{\mathbb{R}^d} \phi(x-y) f(y)\,\mathrm{d}y$$

belongs to $H^1(\mathbb{R}^d)$ and that $\mathrm{grad}\,(\phi * f) = (\mathrm{grad}\,\phi) * f$. If, in addition, $f \in H^1(\mathbb{R}^d) = \mathrm{dom}(\mathrm{grad})$, then $\mathrm{grad}(\phi * f) = \phi * \mathrm{grad}\, f$, where the convolution is always taken component wise.

**Exercise 12.2.** Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in L_2(\Omega)$ and denote by $\widetilde{f} \in L_2(\mathbb{R}^d)$ the extension of $f$ by zero. Let $v \in \mathbb{R}^d$, $\tau > 0$ and define $f_\tau \coloneqq \widetilde{f}(\cdot + \tau v)|_\Omega$.

(a) Show that $f_\tau \to f$ in $L_2(\Omega)$ as $\tau \to 0$.

(b) Let now $f \in H^1(\Omega)$ and $\Omega + \tau v \subseteq \Omega$ for all $\tau > 0$. Show that $f_\tau \to f$ in $H^1(\Omega)$ as $\tau \to 0$.

**Exercise 12.3.** Prove Theorem 12.2.1.

**Exercise 12.4.** Let $\Omega \subseteq \mathbb{R}^d$ be open, $M\colon \mathrm{dom}(M) \subseteq \mathbb{C} \to L\big(L_2(\Omega) \times L_2(\Omega)^d\big)$ with $\mathrm{s_b}\,(M) < \nu_0$ for some $\nu_0 \in \mathbb{R}$, $c > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re} \geqslant \nu_0}$ we have $\mathrm{Re}\, zM(z) \geqslant c$, $\nu \geqslant \max\{\nu_0, 0\}$ and $\nu \neq 0$. Show that there exists a unique

$$\begin{pmatrix} v \\ q \end{pmatrix} \in H_\nu^1\Big(\mathbb{R}; \mathrm{dom}\Big(\begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix}\Big)\Big)$$

satisfying

$$\begin{cases} \left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \Omega, \\ \pi_{\mathrm{BD(grad)}} v(t) = \phi(t) f & \text{for all } t \in \mathbb{R}, \end{cases}$$

for some bounded $\phi \in C^\infty(\mathbb{R})$ with $\inf \mathrm{spt}\, \phi > -\infty$ and $f \in \mathrm{BD}(\mathrm{grad})$.

**Exercise 12.5.** Let $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Show that there exists a continuous linear operator $E\colon H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that $E(\phi)|_\Omega = \phi$ for each $\phi \in H^1(\Omega)$.

**Exercise 12.6** (Korn's second inequality)**.** Let $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Using Exercise 12.5 show that there exists $c > 0$ such that for all $\phi \in H^1(\Omega)^d$ we have

$$\|\phi\|_{H^1(\Omega)^d} \leqslant c\left(\|\phi\|_{L_2(\Omega)^d} + \|\operatorname{Grad}\phi\|_{L_2(\Omega)^{d\times d}}\right).$$

Thus, describe the space of boundary values of dom(Grad).
*Hint:* Prove a corresponding result for $\Omega = \mathbb{R}^d$ first after having shown that $C_{\mathrm{c}}^\infty(\mathbb{R}^d)^d$ forms a dense subset of both $H^1(\Omega)^d$ and dom(Grad).

**Exercise 12.7.** Let $\Omega \subseteq \mathbb{R}^3$ be open. Compute $\mathrm{BD}(\mathrm{curl}) \coloneqq H_0(\mathrm{curl},\Omega)^{\perp_{H(\mathrm{curl},\Omega)}}$ and show that $\mathrm{curl}\colon \mathrm{BD}(\mathrm{curl}) \to \mathrm{BD}(\mathrm{curl})$ is well-defined, unitary and skew-selfadjoint.

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# 13 Continuous Dependence on the Coefficients I

The power of the functional analytic framework for evolutionary equations lies in its variety. In fact, as we have outlined in earlier lectures, it is possible to formulate many differential equations in the form

$$(\partial_t M(\partial_t) + A)\, U = F.$$

In this chapter we want to use this versatility and address continuity of the above expression (or more precisely of the solution operator) in $M(\partial_t)$. To see this more clearly, fix $F$ and take a sequence of material laws $(M_n)_n$. We will address the following question: what are the conditions or notions of convergence of $(M_n)_n$ to some $M$ in order that $(U_n)_n$ with $U_n$ given as the solution of

$$(\partial_t M_n(\partial_t) + A)\, U_n = F$$

converges to $U$, which satisfies

$$(\partial_t M(\partial_t) + A)\, U = F?$$

In the first of two chapters on this subject, we shall specialise to $A = 0$; that is, we will discuss ordinary differential equations with infinite-dimensional state space. To begin with, we address the convergence of material laws pointwise in the Fourier–Laplace transformed domain and its relation to the convergence of material laws evaluated at the time derivative.

## 13.1 Convergence of Material Laws

Throughout, let $H$ be a Hilbert space. We briefly recall that a sequence $(T_n)_n$ in $L(H)$ converges in the *strong operator topology* to some $T \in L(H)$ if for all $x \in H$ we have

$$T_n x \to T x \quad (n \to \infty).$$

$(T_n)_n$ is said to converge in the *weak operator topology* to $T \in L(H)$ if for all $x, y \in H$ we have

$$\langle y, T_n x\rangle \to \langle y, T x\rangle \quad (n \to \infty).$$

We denote the set of material laws on $H$ with abscissa of boundedness less than or equal to $\nu_0 \in \mathbb{R}$ by

$$\mathcal{M}(H, \nu_0) \coloneqq \{M \colon \operatorname{dom}(M) \to L(H)\,;\, M \text{ material law}, \mathrm{s_b}(M) \leqslant \nu_0\}.$$

*Remark* 13.1.1. Let $\nu_0 \in \mathbb{R}$, $\nu > \nu_0$. Then $\mathcal{M}(H, \nu_0)$ is an algebra and $\mathcal{M}(H, \nu_0) \ni M \mapsto M(\partial_{t,\nu}) \in L\big(L_{2,\nu}(\mathbb{R}; H)\big)$ is an algebra homomorphism which is one-to-one by Theorem 8.2.1.

**Definition.** Let $\nu_0 \in \mathbb{R}$. A sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(H, \nu_0)$ is called *bounded* if

$$\sup_{n\in\mathbb{N}} \|M_n\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu_0}} < \infty.$$

**Theorem 13.1.2.** *Let $\nu_0 \in \mathbb{R}$, $(M_n)_n$ in $\mathcal{M}(H, \nu_0)$ be bounded. Assume that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ the sequence $(M_n(z))_n$ converges in the weak operator topology of $L(H)$ with limit $M(z)$ and let $\nu > \nu_0$. Then $M \in \mathcal{M}(H, \nu_0)$ and $M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$ as $n \to \infty$ in the weak operator topology of $L\big(L_{2,\nu}(\mathbb{R}, H)\big)$.*
*If, in addition, $(M_n(z))_n$ converges in the strong operator topology of $L(H)$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, then, as $n \to \infty$, $M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$ in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}, H)\big)$.*

*Proof.* Let $z_0 \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, $r \in (0, \operatorname{Re} z_0 - \nu_0)$. For $x, y \in H$, by Cauchy's integral formula, we deduce

$$\langle y, M_n(z_0)x\rangle = \frac{1}{2\pi\mathrm{i}} \int_{\partial B(z_0,r)} \frac{\langle y, M_n(z)x\rangle_H}{z - z_0} \,\mathrm{d}z \quad (n \in \mathbb{N}).$$

Using boundedness of $(M_n)_n$, Lebesgue's dominated convergence theorem yields

$$\langle y, M(z_0)x\rangle = \frac{1}{2\pi\mathrm{i}} \int_{\partial B(z_0,r)} \frac{\langle y, M(z)x\rangle_H}{z - z_0} \,\mathrm{d}z.$$

Since

$$\|\langle y, M(z)x\rangle\|_H \leqslant \|x\|_H \|y\|_H \sup_{n\in\mathbb{N}} \|M_n\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu_0}} \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}), \tag{13.1}$$

$\langle y, M(\cdot)x\rangle_H$ is holomorphic in a neighbourhood of $z_0$. By Exercise 5.3 we obtain that $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ is holomorphic. In fact, the estimate (13.1) even implies that $M \in \mathcal{M}(H, \nu_0)$.
If $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ and $(M_n(z))_n$ even converges in the strong operator topology, then the limit is clearly $M(z)$.
The convergence statements for $(M_n(\partial_{t,\nu}))_n$ (in the weak and strong operator topology) are then implied by Fourier–Laplace transformation. □

*Remark* 13.1.3. In Theorem 13.1.2, it suffices to assume that $(M_n(z))_n$ converges only for $z$ belonging to a countable subset of $\mathbb{C}_{\mathrm{Re}>\nu_0}$ with an accumulation point in $\mathbb{C}_{\mathrm{Re}>\nu_0}$.

The next statement is essential for the convergence statement for "ordinary" evolutionary equations.

**Proposition 13.1.4.** *Let $(T_n)_n$ be a sequence in $L(H)$ converging in the strong operator topology to some $T \in L(H)$ with $0 \in \bigcap_{n\in\mathbb{N}} \rho(T_n)$, $\sup_{n\in\mathbb{N}} \big\|T_n^{-1}\big\| < \infty$ and $\operatorname{ran}(T) \subseteq H$ dense. Then $T$ is continuously invertible and $(T_n^{-1})_n$ converges to $T^{-1}$ in the strong operator topology.*

*Proof.* We set $K \coloneqq \sup_{n\in\mathbb{N}} \left\|T_n^{-1}\right\|$. We show that $T$ is continuously invertible first. For this, let $x \in H$. Then

$$\|x\| = \left\|T_n^{-1}T_n x\right\| \leqslant K\|T_n x\| \to K\|Tx\| \quad (n\to\infty).$$

Hence, $T$ is one-to-one and it follows that $\operatorname{ran}(T) \subseteq H$ is closed. Hence, $0 \in \rho(T)$. For $x \in H$ we conclude

$$\left\|T_n^{-1}x - T^{-1}x\right\| = \left\|T_n^{-1}(T-T_n)T^{-1}x\right\| \leqslant K\left\|(T-T_n)T^{-1}x\right\| \to 0 \quad (n\to\infty). \qquad \square$$

We are now in the position to obtain the first result on continuous dependence.

**Theorem 13.1.5.** *Let $\nu_0 \in \mathbb{R}$, $(M_n)_n$ a bounded sequence in $\mathcal{M}(H,\nu_0)$, $c > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have*

$$\operatorname{Re} zM_n(z) \geqslant c.$$

*If $(M_n(z))_n$ converges in the strong operator topology for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ then for the limit $M(z)$ we have $M \in \mathcal{M}(H,\nu_0)$ with $\operatorname{Re} zM(z) \geqslant c$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ and for $\nu > \nu_0$ we have*

$$\big(\partial_{t,\nu}M_n(\partial_{t,\nu})\big)^{-1} \to \big(\partial_{t,\nu}M(\partial_{t,\nu})\big)^{-1}$$

*in the strong operator topology.*

*Proof.* By Theorem 13.1.2, we observe $M \in \mathcal{M}(H,\nu_0)$. Let $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Then we have $\operatorname{Re} zM(z) = \lim_{n\to\infty} \operatorname{Re} zM_n(z) \geqslant c$ and hence $zM(z)$ is continuously invertible. Since $0 \in \bigcap_{n\in\mathbb{N}} \rho(zM_n(z))$ and $\left\|(zM_n(z))^{-1}\right\| \leqslant 1/c$ by Proposition 6.2.3(b), we deduce by Proposition 13.1.4 applied to $T_n = zM_n(z)$ that $(zM_n(z))^{-1} \to (zM(z))^{-1}$ in the strong operator topology. By Theorem 13.1.2, for $\nu > \nu_0$ we infer $\big(\partial_{t,\nu}M_n(\partial_{t,\nu})\big)^{-1} \to \big(\partial_{t,\nu}M(\partial_{t,\nu})\big)^{-1}$ in the strong operator topology. $\square$

## 13.2 A Leading Example

We want to illustrate the findings of the previous section with the help of an ordinary differential equation. Also, we shall provide an argument on the limitations of the theory presented above. Let $(\Omega,\Sigma,\mu)$ be a finite measure space.
Note that for $V \in L_\infty(\mu)$ with associated multiplication operator $V(\mathrm{m})$ as in Theorem 2.4.2 we have that

$$M\colon z \mapsto 1 + z^{-1}V(\mathrm{m}) \in L(L_2(\mu))$$

is a material law with $\mathrm{s_b}\,(M) = 0$ unless $V = 0$ (in case $V = 0$ we have $\mathrm{s_b}\,(M) = -\infty$). The corresponding evolutionary equation is given by

$$\partial_{t,\nu}u + V(\mathrm{m})u = f.$$

We want to study sequences of material laws of this form; that is, material laws induced by sequences $(V_n)_n$ in $L_\infty(\mu)$. First, we provide the following characterisation of the convergence of multiplication operators. We recall that for a Banach space $X$ the weak$^*$ topology $\sigma(X',X)$ on $X'$ is the coarsest topology such that all the mappings $X' \ni x' \mapsto x'(x)$ $(x \in X)$ are continuous.

**Proposition 13.2.1.** *Let $(V_n)_n$ in $L_\infty(\mu)$ and $V \in L_\infty(\mu)$. Then the following statements hold.*

(a) $V_n(\mathrm{m}) \to V(\mathrm{m})$ *in* $L(L_2(\mu))$ *if and only if* $V_n \to V$ *in* $L_\infty(\mu)$.

(b) $V_n(\mathrm{m}) \to V(\mathrm{m})$ *in the strong operator topology of* $L(L_2(\mu))$ *if and only if* $(V_n)$ *is bounded in* $L_\infty(\mu)$ *and* $V_n \to V$ *in* $L_1(\mu)$.

(c) $V_n(\mathrm{m}) \to V(\mathrm{m})$ *in the weak operator topology of* $L(L_2(\mu))$ *if and only if* $V_n \to V$ *in the weak* topology* $\sigma\big(L_\infty(\mu), L_1(\mu)\big)$.

*Proof.* (a) This is a direct consequence of Proposition 2.4.4.
(b) Assume $V_n \to V$ in $L_1(\mu)$ and that $(V_n)_n$ is bounded in $L_\infty(\mu)$. Then $(V_n - V)_n$ is also bounded in $L_\infty(\mu)$. For $f \in L_\infty(\mu) \subseteq L_2(\mu)$ we obtain

$$\begin{aligned}\|V_n(\mathrm{m})f - V(\mathrm{m})f\|_{L_2(\mu)}^2 &= \int_\Omega |V_n - V|^2 |f|^2 \,\mathrm{d}\mu \\ &\leqslant \sup_{n\in\mathbb{N}} \|V_n - V\|_{L_\infty(\mu)} \|f\|_{L_\infty(\mu)}^2 \int_\Omega |V_n - V| \, d\mu \to 0.\end{aligned}$$

Since $L_\infty(\mu)$ is dense in $L_2(\mu)$ and $(V_n(\mathrm{m}) - V(\mathrm{m}))_n$ is bounded by Proposition 2.4.4, we obtain $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$.
Now, let $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$. Then $(V_n(\mathrm{m}))_n$ is bounded in $L(L_2(\mu))$ by the uniform boundedness principle. Now Proposition 2.4.4 yields boundedness of $(V_n)_n$ in $L_\infty(\mu)$. Moreover, since $\mathbb{1}_\Omega \in L_2(\mu)$, we deduce $V_n = V_n(\mathrm{m})\mathbb{1}_\Omega \to V(\mathrm{m})\mathbb{1}_\Omega = V$ in $L_2(\mu)$. Since $L_2(\mu)$ embeds continuously into $L_1(\mu)$ we obtain $V_n \to V$ in $L_1(\mu)$.
(c) The assertion follows easily upon realising that $\phi \in L_1(\mu)$ if and only if there exists $\psi_1, \psi_2 \in L_2(\mu)$ such that $\phi = \psi_1\psi_2$. □

With the latter result at hand together with the results in the previous section, we easily deduce the next theorem on continuous dependence on the coefficients.

**Theorem 13.2.2.** *Let $(V_n)_n$ in $L_\infty(\mu)$ be bounded, $V \in L_\infty(\mu)$, and $V_n \to V$ in $L_1(\mu)$. Then there exists $\nu > 0$ such that*

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)^{-1} \to \big(\partial_{t,\nu} + V(\mathrm{m})\big)^{-1}$$

*in the strong operator topology of* $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu))\big)$.

Note that the convergence statement can be improved, see Exercise 13.3.

*Proof.* By Proposition 13.2.1(b) we obtain $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$. Note that for $\nu \geqslant 1 + \sup_{n\in\mathbb{N}} \|V_n\|_{L_\infty(\mu)}$ we have

$$\operatorname{Re}(z + V_n(\mathrm{m})) \geqslant 1 \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu}, n \in \mathbb{N}).$$

Now Theorem 13.1.5 applied to $M_n(z) = 1 + z^{-1}V_n(\mathrm{m})$ yields the assertion. □

*Remark* 13.2.3. Theorem 13.2.2 can be generalized in the following way. Let $(B_n)_n$ in $L(H)$, $B \in L(H)$, $B_n \to B$ in the strong operator topology. Then there txists $\nu > 0$ such that

$$\left(\partial_{t,\nu} + B_n\right)^{-1} \to \left(\partial_{t,\nu} + B\right)^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu))\big)$.

In Theorem 13.2.2 we did assume strong convergence of the sequence of multiplication operators $(V_n(\mathrm{m}))_n$. A natural question to ask is whether the stated result can be improved to $(V_n)_n$ converging in the weak* topology $\sigma\big(L_\infty(\mu), L_1(\mu)\big)$ only. The answer is neither 'yes' nor 'no', but rather 'not quite', as we will show in the following. We start with a result on weak* limits of scaled periodic functions, which will serve as the prototypical example for a sequence converging in the weak* topology of $L_\infty$.

**Theorem 13.2.4.** *Let $f \in L_\infty(\mathbb{R}^d)$ be* $[0,1)^d$-periodic*; that is,*

$$f(\cdot + k) = f \quad (k \in \mathbb{Z}^d).$$

*Then*

$$f(n\cdot) \to \int_{[0,1)^d} f(x)\,\mathrm{d}x \mathbb{1}_{\mathbb{R}^d}$$

*in the weak* topology $\sigma\big(L_\infty(\mathbb{R}^d), L_1(\mathbb{R}^d)\big)$ as $n \to \infty$.*

*Proof.* Without loss of generality, we may assume $\int_{[0,1)^d} f(x)\,\mathrm{d}x = 0$. By the density of simple functions in $L_1(\mathbb{R}^d)$ and the boundedness of $(f(n\cdot))_n$ in $L_\infty(\mathbb{R}^d)$, it suffices to show

$$\int_Q f(nx)\,\mathrm{d}x \to 0 \quad (n \to \infty)$$

for $Q = [a,b] \coloneqq [a_1,b_1] \times \ldots \times [a_d,b_d]$ where $a = (a_1,\ldots,a_d), b = (b_1,\ldots,b_d) \in \mathbb{R}^d$. By translation and the periodicity of $f$ we may assume $a = 0$. Thus, it suffices to show

$$\int_{[0,b]} f(nx)\,\mathrm{d}x \to 0 \quad (n \to \infty)$$

for all $b \in (0,\infty)^d$. So, let $b = (b_1,\ldots,b_d) \in (0,\infty)^d$. Let $n \in \mathbb{N}$. Then we find $z \in \mathbb{N}^d$ and $\zeta \in [0,1)^d$ such that $nb = z + \zeta$. We compute

$$\begin{aligned}
&\int_{[0,b]} f(nx)\,\mathrm{d}x \\
&= \frac{1}{n^d}\int_{[0,nb]} f(x)\,\mathrm{d}x \\
&= \frac{1}{n^d}\int_{[0,z_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x + \frac{1}{n^d}\int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x.
\end{aligned}$$

We now estimate

$$\left|\frac{1}{n^d}\int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x\right| \leqslant \frac{1}{n^d}\int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} |f(x)|\,\mathrm{d}x$$

$$\begin{aligned}&\leqslant \frac{1}{n^d}\int_{(0,1]\times[0,nb_2]\times\ldots\times[0,nb_d]} \mathrm{d}x\|f\|_{L_\infty(\mu)}\\&= \frac{1}{n}b_2\cdot\ldots\cdot b_d\|f\|_{L_\infty(\mu)}.\end{aligned}$$

Continuing in this manner and using $z_j \leqslant nb_j$ for all $j \in \{1,\ldots,d\}$, we obtain

$$\left|\int_{[0,b]} f(nx)\,\mathrm{d}x\right| \leqslant \frac{1}{n^d}\left|\int_{[0,z]} f(x)\,\mathrm{d}x\right| + \frac{1}{n}\sum_{j=1}^{d}\frac{b_1\cdot\ldots\cdot b_d}{b_j}\|f\|_{L_\infty(\mu)}.$$

Since $f$ is $[0,1)^d$-periodic and $z \in \mathbb{N}^d$ we observe

$$\int_{[0,z]} f(x)\,\mathrm{d}x = \prod_{j=1}^{d} z_j \int_{[0,1)^d} f(x)\,\mathrm{d}x = 0.$$

Thus,

$$\left|\int_{[0,b]} f(nx)\,\mathrm{d}x\right| \leqslant \frac{1}{n}\sum_{j=1}^{d}\frac{b_1\cdot\ldots\cdot b_d}{b_j}\|f\|_{L_\infty(\mu)},$$

which tends to 0 as $n \to \infty$. □

*Remark* 13.2.5. Note that Theorem 13.2.4 also yields

$$f(n\cdot) \to \int_{[0,1)^d} f(x)\,\mathrm{d}x\mathbb{1}_\Omega$$

in the weak* topology $\sigma(L_\infty(\Omega), L_1(\Omega))$ for all measurable subsets $\Omega \subseteq \mathbb{R}^d$ with non-zero Lebesgue measure.

We now present an example which shows that weak* convergence of $(V_n)_n$ does not yield the result of Theorem 13.2.2.

**Example 13.2.6.** Let $(\Omega,\Sigma,\mu) = ((0,1), \mathcal{B}((0,1)), \lambda|_{(0,1)})$. For $n \in \mathbb{N}$ let $V_n$ be given by $V_n(x) := \sin(2\pi nx)$ for $x \in (0,1)$. Then, by Theorem 13.2.4, we obtain $V_n \to 0$ in $\sigma\big(L_\infty((0,1)), L_1((0,1))\big)$ as $n \to \infty$. Let $\nu > 1$. Then $\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)$ is continuously invertible as an operator in $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$. Let $\widetilde{f} \in C([0,1])$ and denote $f\colon (t,x) \mapsto \mathbb{1}_{\mathbb{R}_{\geq 0}}(t)\widetilde{f}(x)$. Then $f \in L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$. The solution $u_n \in L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ of

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)u_n = f$$

is given by the variations of constants formula; that is,

$$u_n(t,x) = \mathbb{1}_{[0,\infty)}(t)\int_0^t \exp\big(-(t-s)\sin(2\pi nx)\big)\,\mathrm{d}s\widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).$$

Thus, if a variant of Theorem 13.2.2 were true also in this case, $(u_n)_n$ needs to converge (in some sense) to the solution $u$ of

$$\partial_{t,\nu}u = f,$$

which is given by

$$u(t,x) = \mathbb{1}_{[0,\infty)}(t)t\widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).$$

However, by Theorem 13.2.4, for $x \in (0,1)$ we deduce

$$\int_0^t \exp\bigl(-(t-s)\sin(2\pi nx)\bigr)\,\mathrm{d}s \to \int_0^t J(-(t-s))\,\mathrm{d}s \quad (n \to \infty)$$

in $\sigma\bigl(L_\infty((0,1)), L_1((0,1))\bigr)$ for each $t \geqslant 0$, where

$$J(s) \coloneqq \int_0^1 \exp\bigl(s\sin(2\pi x)\bigr)\,\mathrm{d}x \quad (s \in \mathbb{R})$$

denotes the 0-th order modified Bessel function of the first kind, cf. [AS64, p. 9.6.19]. Moreover, for $\varphi \in C_\mathrm{c}^\infty(\mathbb{R})$, $A \in \mathcal{B}((0,1))$ and using domianted convergence we obtain

$$\begin{aligned}
&\langle u_n, \varphi \mathbb{1}_A\rangle_{L_{2,\nu}(\mathbb{R};L_2((0,1)))} \\
&= \int_0^\infty \int_0^1 \int_0^t \exp\bigl(-(t-s)\sin(2\pi nx)\bigr)\,\mathrm{d}s\widetilde{f}(x)\mathbb{1}_A(x)\,\mathrm{d}x\varphi(t)\mathrm{e}^{-2\nu t}\,\mathrm{d}t \\
&\to \int_0^\infty \int_0^1 \int_0^t J(-(t-s))\,\mathrm{d}s\widetilde{f}(x)\mathbb{1}_A(x)\,\mathrm{d}x\varphi(t)\mathrm{e}^{-2\nu t}\,\mathrm{d}t \\
&= \langle \widetilde{u}, \varphi\mathbb{1}_A\rangle_{L_{2,\nu}(\mathbb{R};L_2((0,1)))}
\end{aligned}$$

with

$$\widetilde{u}(t,x) \coloneqq \mathbb{1}_{[0,\infty)}(t)\int_0^t J(-(t-s))\,\mathrm{d}s\widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).$$

Since $(u_n)_n$ is bounded in $L_{2,\nu}\bigl(\mathbb{R}; L_2((0,1))\bigr)$ and $\{\varphi\mathbb{1}_A\,;\, A \in \mathcal{B}((0,1)),\, \varphi \in C_\mathrm{c}^\infty(\mathbb{R})\}$ is total in $L_{2,\nu}\bigl(\mathbb{R}; L_2((0,1))\bigr)$ by Lemma 3.1.10, we infer $u_n \to \widetilde{u}$ weakly in $L_{2,\nu}\bigl(\mathbb{R}; L_2((0,1))\bigr)$ as $n \to \infty$. In particular, $\widetilde{u} \neq u$. Furthermore, $\widetilde{u}$ is *not* of the form

$$\int_0^t \exp\bigl(-(t-s)\widetilde{V}(x)\bigr)\,\mathrm{d}s\widetilde{f}(x)$$

for some $\widetilde{V} \in L_\infty((0,1))$ and hence, we *cannot* hope for $\widetilde{u}$ to satisfy an equation of the type

$$\bigl(\partial_{t,\nu} + \widetilde{V}(\mathrm{m})\bigr)\widetilde{u} = f.$$

As we shall see next, in the framework of evolutionary equations it is possible to derive an equation involving suitable limits of $(V_n)_n$ and $f$ as a right-hand side.

## 13.3 Convergence in the Weak Operator Topology

In this section, we consider a particular class of material laws and characterise convergence of the solution operators of the corresponding evolutionary equations in the weak operator topology. The main theorem that will serve to compute the limit equation satisfied by $\widetilde{u}$ in Example 13.2.6 reads as follows.

**Theorem 13.3.1.** *Let $H$ be a Hilbert space, $(B_n)_n$ a bounded sequence in $L(H)$ and $\nu > \sup_{n\in\mathbb{N}} \|B_n\|$. Then $\big((\partial_{t,\nu} + B_n)^{-1}\big)_n$ converges in the weak operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$ if and only if for all $k \in \mathbb{N}$ the sequence $(B_n^k)_n$ converges in the weak operator topology of $L(H)$. In either case, we have*

$$(\partial_{t,\nu} + B_n)^{-1} \to \sum_{k=0}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k \partial_{t,\nu}^{-1}$$

*in the weak operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$, where $C_k \in L(H)$ denotes the weak limit of $(B_n^k)_n$ for $k \in \mathbb{N}$ and $C_0 \coloneqq 1_H$.*

*Remark* 13.3.2. In the situation of Theorem 13.3.1, let $B_n^k \to C_k$ in the weak operator topology for all $k \in \mathbb{N}$. Let $L \coloneqq \sup_{n\in\mathbb{N}} \|B_n\|$, $\nu > 2L$, and $f \in L_{2,\nu}(\mathbb{R}; H)$. By Theorem 13.3.1, if $(\partial_{t,\nu} + B_n)u_n = f$ for all $n \in \mathbb{N}$, then $(u_n)_n$ converges weakly in $L_{2,\nu}(\mathbb{R}; H)$ to some element $\widetilde{u} \in L_{2,\nu}(\mathbb{R}; H)$. In order to determine the differential equation satisfied by $\widetilde{u}$, we make the following observations: by weak convergence,

$$\|C_k\| \leqslant \liminf_{n\to\infty} \left\|B_n^k\right\| \leqslant L^k.$$

Hence, since $\left\|\partial_{t,\nu}^{-1}\right\|_{L_{2,\nu}} \leqslant \frac{1}{\nu}$ (see Section 3.2) we infer that

$$\sum_{k=1}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k$$

converges in $L(L_{2,\nu}(\mathbb{R}; H))$ and

$$\left\|\sum_{k=1}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k\right\| \leqslant \sum_{k=1}^{\infty} \left\|\partial_{t,\nu}^{-1}\right\|^k \|C_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Hence, since $C_0 = 1_H$ we deduce that $\sum_{k=0}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k$ is boundedly invertible by the Neumann series. Thus, we obtain

$$\begin{aligned} f &= \partial_{t,\nu} \left(\sum_{k=0}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k\right)^{-1} \widetilde{u} = \partial_{t,\nu} \left(1_H + \sum_{k=1}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k\right)^{-1} \widetilde{u} \\ &= \partial_{t,\nu} \sum_{\ell=0}^{\infty} \left(-\sum_{k=1}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k\right)^{\ell} \widetilde{u} = \partial_{t,\nu}\widetilde{u} + \partial_{t,\nu} \sum_{\ell=1}^{\infty} \left(-\sum_{k=1}^{\infty} \big(-\partial_{t,\nu}^{-1}\big)^k C_k\right)^{\ell} \widetilde{u}. \end{aligned}$$

Before we prove Theorem 13.3.1 we revisit Example 13.2.6.

**Example 13.3.3** (Example 13.2.6 continued)**.** By Theorem 13.3.1, we need to compute the limit of $(\sin^k(2\pi n\cdot))_n$ in the weak* topology of $L_\infty((0,1))$ for all $k \in \mathbb{N}$. By Theorem 13.2.4, we obtain for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} \sin^k(2\pi n\cdot) = \int_0^1 \sin^k(2\pi\xi)\,\mathrm{d}\xi \mathbb{1}_{(0,1)} = \begin{cases} \frac{(2m)!}{(m!2^m)^2} \mathbb{1}_{(0,1)}, & k = 2m \text{ for some } m \in \mathbb{N}, \\ 0, & k \text{ odd}, \end{cases}$$

in $\sigma\big(L_\infty((0,1)), L_1((0,1))\big)$. Hence, $u_n \to \widetilde{u}$ weakly, where $\widetilde{u}$ satisfies

$$\partial_{t,\nu}\widetilde{u} + \partial_{t,\nu}\sum_{\ell=1}^{\infty}\left(-\sum_{m=1}^{\infty}\partial_{t,\nu}^{-2m}\frac{(2m)!}{(m!2^m)^2}\right)^{\ell}\widetilde{u} = f$$

for $\nu > 2$ by Remark 13.3.2.

*Proof of Theorem 13.3.1.* Before we prove the equivalence, we make some observations. Since $\nu > \sup_{n\in\mathbb{N}}\|B_n\| =: L$, by a Neumann series argument we deduce that

$$\big(\partial_{t,\nu} + B_n\big)^{-1} = \sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}B_n\big)^k\partial_{t,\nu}^{-1} = \sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}\big)^k B_n^k\partial_{t,\nu}^{-1}.$$

The series $\sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}\big)^k B_n^k\partial_{t,\nu}^{-1}$ is absolutely convergent in $L(L_{2,\nu}(\mathbb{R};H))$. Also note that for $M_n\colon \mathbb{C}_{\mathrm{Re}>L} \ni z \mapsto \sum_{k=0}^{\infty}(-\frac{1}{z})^k B_n^k\frac{1}{z}$ we have $M_n \in \mathcal{M}(H,\nu)$.
Assume now that $(B_n^k)_n$ converges in the weak operator topology to some $C_k$ for all $k \in \mathbb{N}$. A little computation reveals that as $n \to \infty$,

$$M_n(z) \to \sum_{k=0}^{\infty}\left(-\frac{1}{z}\right)^k C_k\frac{1}{z} =: M(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>L})$$

in the weak operator topology, where the series on the right-hand side converges in $L(H)$ since

$$\|C_k\| \leqslant \liminf_{n\to\infty}\left\|B_n^k\right\| \leqslant L^k \quad (k \in \mathbb{N}).$$

Moreover, since $\nu > L$, the sequence $(M_n)_n$ is bounded in $\mathcal{M}(H,\nu)$ and thus, $M \in \mathcal{M}(H,\nu)$ and

$$M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$$

in the weak operator topology by Theorem 13.1.2.
Now, we assume that $\big((\partial_{t,\nu} + B_n)^{-1}\big)_n$ converges in the weak operator topology. Then $(M_n(\partial_{t,\nu}))_n$ converges in the weak operator topology. Let $k \in \mathbb{N}$. We need to show that for all $\phi, \psi \in H$ the sequence $(\left\langle \phi, B_n^k\psi\right\rangle_H)_n$ is convergent to some number $c_{k,\phi,\psi}$ as $n \to \infty$. The Riesz representation theorem then yields the existence of $C_k \in L(H)$ with $\langle \phi, C_k\psi\rangle = c_{k,\phi,\psi}$. So, let $\phi, \psi \in H$. Moreover, we consider the functions $m_n$ and $h_n$ given by

$$m_n(z) := \sum_{k=0}^{\infty}(-z)^k z\left\langle \phi, B_n^k\psi\right\rangle_H \quad (z \in B(0,1/L), n \in \mathbb{N})$$

and

$$h_n(z) := \langle \phi, M_n(z)\psi\rangle_H = \sum_{k=0}^{\infty}\frac{1}{z}\left(-\frac{1}{z}\right)^k\left\langle \phi, B_n^k\psi\right\rangle_H \quad (z \in \mathbb{C}_{\mathrm{Re}>L}, n \in \mathbb{N}).$$

Clearly, $m_n$ and $h_n$ are holomorphic on their respective domains for each $n \in \mathbb{N}$ and the sequences $(m_n)_n$ and $(h_n)_n$ are uniformly bounded on compact subsets (in other words they form normal families). Moreover,

$$m_n(z) = h_n\Big(\frac{1}{z}\Big) \quad \big(z \in B\big(1/(2L), 1/(2L)\big), n \in \mathbb{N}\big).$$

We aim to show that the coefficients of the power series of $m_n$ converge as $n$ tends to infinity. The proof will be done in two steps. In step 1, we will prove that the sequence $(h_n)_n$ converges to a holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ uniformly on compact sets. Then, in the second step, we will use this to deduce that $(m_n)_n$ also converges uniformly on compact sets and prove the assertion with the help of Cauchy's integral formula.
*Step 1:* By Proposition 5.3.2, $(M_n(\mathrm{im}+\nu))_n$ converges in the weak operator topology of $L(L_2(\mathbb{R};H))$. For $f, g \in L_2(\mathbb{R})$ we thus obtain that

$$\big(\langle f, h_n(\mathrm{im}+\nu)g\rangle_{L_2(\mathbb{R})}\big)_n = \big(\langle f\phi, M_n(\mathrm{im}+\nu)g\psi\rangle_{L_2(\mathbb{R};H)}\big)_n$$

is convergent. Thus, using $L_2(\mathbb{R}) \cdot L_2(\mathbb{R}) = L_1(\mathbb{R})$, we obtain that

$$\Psi\colon L_1(\mathbb{R}) \ni u \mapsto \lim_{n\to\infty} \Big(\int_{\mathbb{R}} h_n(\mathrm{i}t+\nu)u(t)\,\mathrm{d}t\Big) \in \mathbb{C}$$

defines a linear functional, which is continuous, since

$$\sup_{n\in\mathbb{N}}\sup_{t\in\mathbb{R}} \|M_n(\mathrm{i}t+\nu)\|_{L(H)} = \sup_{n\in\mathbb{N}} \|M_n(\mathrm{im}+\nu)\|_{L(L_2(\mathbb{R};H))} < \infty$$

by boundedness of $(B_n)_n$. Hence, since $L_1(\mathbb{R})' = L_\infty(\mathbb{R})$, we find a unique $\widetilde{h} \in L_\infty(\mathbb{R})$ with

$$\lim_{n\to\infty} \int_{\mathbb{R}} h_n(\mathrm{i}t+\nu)u(t)\,\mathrm{d}t = \int_{\mathbb{R}} \widetilde{h}(t)u(t)\,\mathrm{d}t \quad (u \in L_1(\mathbb{R})).$$

We now show that every subsequence $(h_{n_k})_k$ of $(h_n)_n$ has a subsequence $(h_{n_{k_l}})_l$ which converges locally uniformly to a holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ such that $h(\mathrm{i}\cdot+\nu) = \widetilde{h}$ a.e., and that this implies that the limit $h$ does not depend on the subsequences. Then we conclude that $(h_n)_n$ itself converges locally uniformly to $h$.
So, let $(h_{n_k})_k$ be a subsequence of $(h_n)$. By Montel's theorem (see [Sim15, Theorem 6.2.2]), we find a subsequence $(h_{n_{k_l}})_l$ of $(h_{n_k})_k$ such that $h_{n_{k_l}} \to h$ as $l \to \infty$ uniformly on compact subsets of $\mathbb{C}_{\mathrm{Re}>L}$ for some holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$. In particular, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}} h_{n_{k_l}}(\mathrm{i}t+\nu)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} h(\mathrm{i}t+\nu)\varphi(t)\,\mathrm{d}t \quad (\varphi \in C_{\mathrm{c}}(\mathbb{R}))$$

by dominated convergence and hence, $h(\mathrm{i}t+\nu) = \widetilde{h}(t)$ for almost every $t \in \mathbb{R}$. This shows that the limit $h$ is independent of choice of the subsequences $(h_{n_k})_k$ and $(h_{n_{k_l}})_l$. Indeed, if $\widehat{h}\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ is the limit of another subsubsequence of $(h_n)_n$ as above, then

$\widehat{h}(\mathrm{i}\cdot+\nu) = \widetilde{h} = h(\mathrm{i}\cdot+\nu)$ a.e. Since $\widehat{h}$ and $h$ are holomorphic, the identity theorem yields $\widehat{h} = h$.

Now, assume for a contradiction that $(h_n)_n$ does not converge locally uniformly to $h$. Then we find a subsequence $(h_{n_k})_k$ of $(h_n)_n$, a compact set $K \subseteq \mathbb{C}_{\mathrm{Re}>L}$ and $\varepsilon > 0$ such that

$$\|h_{n_k} - h\|_{\infty,K} \geqslant \varepsilon \quad (k \in \mathbb{N}). \tag{13.2}$$

However, the subsequence $(h_{n_k})_k$ has a subsequence $(h_{n_{k_l}})_l$ which converges locally uniformly to $h$, contradicting (13.2). Thus, $(h_n)_n$ itself converges locally uniformly to $h$, and, in particular, $h_n \to h$ pointwise on $\mathbb{C}_{\mathrm{Re}>L}$.

*Step 2:* By what we have shown in Step 1, the sequence $(m_n)_{n\in\mathbb{N}}$ converges pointwise on $B\big(1/(2L), 1/(2L)\big)$. Since $(m_n)_n$ is also uniformly bounded on compact subsets of $B(0, 1/L)$, we derive that $(m_n)_n$ converges uniformly on compact subsets of $B(0, 1/L)$ by Vitali's theorem (see [Sim15, Theorem 6.2.8]). Choosing $0 < r < 1/L$, we thus obtain by Cauchy's integral formula

$$\left\langle \phi, B_n^k \psi \right\rangle_H = (-1)^k \frac{1}{2\pi\mathrm{i}} \int_{\partial B(0,r)} \frac{m_n(z)}{z^{k+2}} \,\mathrm{d}z.$$

Thus $(B_n^k)_n$ converges in the weak operator topology as $n \to \infty$. □

## 13.4 Comments

The problems discussed here are contained in [Wau16; Wau14] for both the weak and the strong operator topology. The case of differential-algebraic equations has been invoked as well.

The appearance of memory effects; that is, the occurence of higher order integral operators due to a weak convergence of the coefficients has been first observed by Tartar and can, for instance, be found in [Tar09]. The limit equation, however, is described by a convolution term rather than a power series of integral operators. It is, however, possible to reformulate these resulting equations into one another, see [Wau11].

The last characterisation of weak convergence in Theorem 13.3.1 was formulated for the first time in [PTW15].

## Exercises

**Exercise 13.1.** Let $(V_n)_n$ in $L_\infty(\mathbb{R}^d)$ and $V \in L_\infty(\mathbb{R}^d)$. Characterise convergence of $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$ in terms of convergence of $(V_n)_n$ similar to as was done in Proposition 13.2.1.

**Exercise 13.2.** Show that there exists an unbounded sequence $(V_n)_n$ in $L_\infty((0,1))$ and $V \in L_\infty((0,1))$ with $V_n \to V$ in $L_1((0,1))$.

**Exercise 13.3.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space, $(V_n)_n$ a bounded sequence in $L_\infty(\mu)$ and assume that $V_n \to V$ in $L_1(\mu)$ for some $V \in L_\infty(\mu)$. Show that there exists $\nu > 0$ such that

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)^{-1} \to \big(\partial_{t,\nu} + V(\mathrm{m})\big)^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu)), H^1_\nu(\mathbb{R}; L_2(\mu))\big)$.

**Exercise 13.4.** Let $D = \bigcup_{n\in\mathbb{Z}} [n + 1/2, n + 1]$, $V_n := \mathbb{1}_D(n\cdot)$. For suitable $\nu > 0$ compute the limit of

$$\big((\partial_{t,\nu} + V_n(\mathrm{m}))^{-1}\big)_n$$

in the weak operator topology of $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$.

**Exercise 13.5.** Let $H$ be a Hilbert space, $c > 0$ and $c \leqslant B_n = B_n^* \in L(H)$ for all $n \in \mathbb{N}$. Characterise, in terms of convergence of $(B_n)_n$ in a suitable sense, that

$$\big((\partial_{t,\nu} B_n)^{-1}\big)_n$$

converges in the weak operator topology. In the case of convergence, find its limit and a sufficient condition for which there exists a $B \in L(H)$ such that

$$(\partial_{t,\nu} B_n)^{-1} \to (\partial_{t,\nu} B)^{-1}$$

in the weak operator topology.

**Exercise 13.6.** Let $H$ be a Hilbert space. Show that $B_{L(H)} := \{B \in L(H)\,;\, \|B\| \leqslant 1\}$ is a compact subset under the weak operator topology. If, in addition, $H$ is separable, show that $B_{L(H)}$ is also metrisable under the weak operator topology.

**Exercise 13.7.** Let $H$ be a separable Hilbert space, $(B_n)_n$ in $L(H)$ bounded. Show that there exists a subsequence $(B_{n_k})_k$ of $(B_n)_n$, a material law $M\colon \operatorname{dom}(M) \to L(H)$ and $\nu > 0$ such that given $f \in L_{2,\nu}(\mathbb{R}; H)$ and $(u_k)_k$ in $L_{2,\nu}(\mathbb{R}; H)$ with

$$\partial_{t,\nu} u_k + B_{n_k} u_k = f \quad (k \in \mathbb{N}),$$

we deduce that $(u_k)_k$ converges weakly to some $u \in L_{2,\nu}(\mathbb{R}; H)$ with the property that

$$\partial_{t,\nu} M(\partial_{t,\nu}) u = f.$$

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# 14 Continuous Dependence on the Coefficients II

This chapter is concerned with the study of problems of the form

$$(\partial_{t,\nu} M_n(\partial_{t,\nu}) + A)\, U_n = F$$

for a suitable sequence of material laws $(M_n)_n$ when $A \neq 0$. The aim of this chapter will be to provide the conditions required for convergence of the material law sequence to imply the existence of a limit material law $M$ such that the limit $U = \lim_{n\to\infty} U_n$ exists and satisfies

$$(\partial_{t,\nu} M(\partial_{t,\nu}) + A)\, U = F.$$

Additionally, for material laws of the form $M_n(\partial_{t,\nu}) = M_{0,n} + \partial_{t,\nu}^{-1} M_{1,n}$ it will be desirable to have the respective limit material law satisfy $M(\partial_{t,\nu}) = M_0 + \partial_{t,\nu}^{-1} M_1$ for some $M_0, M_1 \in L(H)$. This cannot be expected (as we have seen in the guiding example in the previous chapter) if $A$ is a bounded operator, the Hilbert space $H$ is infinite-dimensional, and the material law sequence only converges pointwise in the weak operator topology. It will turn out, however, that if $A$ is “strictly unbounded” then a suitable result can hold, even if we only assume weak convergence of the material law operators.

## 14.1 A Convergence Theorem

The main convergence theorem of this chapter will be presented next.

**Theorem 14.1.1.** *Let $H$ be a Hilbert space, $\nu_0 \in \mathbb{R}$, $(M_n)_n$ in $\mathcal{M}(H, \nu_0)$ and $M \in \mathcal{M}(H, \nu_0)$. Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\operatorname{Re} z M_n(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu_0}).$$

*Let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint and assume $\operatorname{dom}(A) \hookrightarrow H$ compactly. If $M_n(z) \to M(z)$ as $n \to \infty$ in the weak operator topology for all $z \in \mathbb{C}_{\operatorname{Re}>\nu_0}$, then*

$$\left(\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A}\right)^{-1} \to \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1}$$

*in the strong operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$ for each $\nu > \nu_0$.*

For the proof of this theorem, we need a lemma first.

**Lemma 14.1.2.** *Let $H$ be a Hilbert space, $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint, $c > 0$, $(T_n)_n$ in $L(H)$ with $\operatorname{Re} T_n \geqslant c$ for all $n \in \mathbb{N}$, and $T \in L(H)$. Assume $\operatorname{dom}(A) \hookrightarrow H$ compactly and $T_n \to T$ in the weak operator topology. Then $0 \in \bigcap_{n\in\mathbb{N}} \rho(T_n+A) \cap \rho(T+A)$ and*

$$(T_n + A)^{-1} \to (T + A)^{-1}$$

*in the norm topology of $L(H)$.*

*Proof.* From $\operatorname{Re} T_n \geqslant c$ it follows that $0 \in \rho(T_n + A)$ $(n \in \mathbb{N})$ and $\big((T_n + A)^{-1}\big)_n$ is bounded in $L(H)$. Indeed, since $B \coloneqq T_n + A$ satisfies $\operatorname{Re} B = \operatorname{Re} T_n \geqslant c$ and $\operatorname{dom}(B) = \operatorname{dom}(A) = \operatorname{dom}(B^*)$ due to the skew-selfadjointness of $A$, Proposition 6.3.1 yields the assertion. Moreover, since

$$A(T_n + A)^{-1} = 1 - T_n(T_n + A)^{-1}$$

for all $n \in \mathbb{N}$, it follows that $\big((T_n + A)^{-1}\big)_n$ is also bounded in $L(H, \operatorname{dom}(A))$ by the boundedness of $(T_n)_n$ in $L(H)$. Due to the convergence of $(T_n)_n$ to $T$, it follows that $\operatorname{Re} T \geqslant c$, and thus, $(T+A)^{-1} \in L(H, \operatorname{dom}(A))$. Before we come to a proof of the desired result, we will prove an auxiliary observation.

Claim: for all $(f_n)_n$ in $H$ weakly converging to $f$, we have $(T_n + A)^{-1} f_n \to (T + A)^{-1} f$ in the norm topology of $H$.

For proving the claim, let $(f_n)_n$ in $H$ be weakly convergent to some $f$. Consider $u_n \coloneqq (T_n + A)^{-1} f_n$. Then $(u_n)_n$ is bounded in $\operatorname{dom}(A)$, since $\big((T_n + A)^{-1}\big)_n$ is bounded in $L(H, \operatorname{dom}(A))$ and $(f_n)_n$ is bounded in $H$. Hence, there exists a subsequence $(u_{n_k})_k$ which weakly converges to some $u$ in $\operatorname{dom}(A)$. Since $\operatorname{dom}(A) \hookrightarrow H$ compactly, we infer $u_{n_k} \to u$ in the norm topology of $H$. Hence, in the equality

$$T_{n_k} u_{n_k} + A u_{n_k} = f_{n_k},$$

as $T_{n_k} \to T$ in the weak operator topology and $u_{n_k} \to u$ in $H$, we may let $k \to \infty$ and obtain for the weak limits

$$Tu + Au = f;$$

that is, $u = (T + A)^{-1} f$. Having identified the limit, a contradiction argument (here a so-called 'subsequence argument', see Exercise 14.3) concludes that $(u_n)_n$ itself converges weakly in $\operatorname{dom}(A)$ and strongly in $H$ to $u$. Thus, the claim is proved.

Next, assume by contradiction that $\big((T_n + A)^{-1}\big)_n$ does not converge in operator norm to $(T + A)^{-1}$. Then we find an $\varepsilon > 0$ and a strictly increasing sequence of integers, $(n_k)_k$, and a sequence of unit vectors $(f_{n_k})_k$ in $H$ such that

$$\left\| (T_{n_k} + A)^{-1} f_{n_k} - (T + A)^{-1} f_{n_k} \right\| \geqslant \varepsilon. \tag{14.1}$$

By possibly taking another subsequence, we may assume without loss of generality that $(f_{n_k})_k$ converges weakly to some $f \in H$. By the claim proved above, we deduce $(T_{n_k} + A)^{-1} f_{n_k} \to (T + A)^{-1} f$ and $(T + A)^{-1} f_{n_k} \to (T + A)^{-1} f$, both in the norm topology of $H$ as $k \to \infty$. Thus, we may let $k \to \infty$ in (14.1), and obtain the desired contradiction. □

*Proof of Theorem 14.1.1.* By Theorem 13.1.2 it suffices to show that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$

$$(zM_n(z) + A)^{-1} \to (zM(z) + A)^{-1} \quad (n \to \infty)$$

in the strong operator topology. This, however, follows from Lemma 14.1.2 applied to $T_n = zM_n(z)$. □

*Remark* 14.1.3. Note that we only used convergence in the strong operator topology in the proof of Theorem 14.1.1. However, the assertion in Lemma 14.1.2 is about convergence in the norm topology. The reason that we cannot assert the convergence claimed in Theorem 14.1.1 in the norm topology is that the compact embedding of $\mathrm{dom}(A) \hookrightarrow H$ only works locally for fixed $z$, and not uniformly in $z$. This situation can, however, be rectified. We refer to Exercise 14.1 for this.

## 14.2 The Theorem of Rellich and Kondrachov

In order to apply Theorem 14.1.1, we need to provide a setting where the condition on the compactness of the embedding is satisfied. In fact, it is true that $H^1(\Omega)$ embeds compactly into $L_2(\Omega)$ given $\Omega \subseteq \mathbb{R}^d$ is bounded and has 'continuous boundary', see e.g. [Are+15, Theorem 7.11]. In this chapter, we restrict ourselves to a proof of a less general statement.
A preparatory result needed to prove the compact embedding theorem is given next.

**Proposition 14.2.1.** *Let $I \subseteq \mathbb{R}$ be an open, bounded, non-empty interval. Then the mapping $H^1(\mathbb{R}) \ni f \mapsto f|_I \in H^1(I)$ is well-defined, continuous and onto. Moreover, there exists a continuous right inverse $H^1(I) \to H^1(\mathbb{R})$.*

For the proof of this proposition, we need an auxiliary result first.

**Lemma 14.2.2.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected. Moreover, let $u \in H^1(\Omega)$ with* $\mathrm{grad}\, u = 0$*. Then $u$ is constant.*

We leave the proof of this lemma as Exercise 14.2.

*Proof of Proposition 14.2.1.* The mapping $H^1(\mathbb{R}) \to H^1(I), f \mapsto f|_I$ is readily confirmed to be continuous. It remains to prove that it is onto. Let $I = (a, b)$, $u \in H^1(I)$ and define the function $v$ by

$$v(t) := \int_a^t \partial u(s)\,\mathrm{d}s \quad (t \in (a, b)).$$

Clearly, $v \in L_2((a, b))$ and we compute for each $\varphi \in C_c^\infty((a, b))$

$$\begin{aligned}\langle v, \varphi' \rangle_{L_2((a,b))} &= \int_a^b \left( \int_a^t \partial u(s)\,\mathrm{d}s \right)^* \varphi'(t)\,\mathrm{d}t = \int_a^b \int_s^b \varphi'(t)\,\mathrm{d}t\, \partial u(s)^*\,\mathrm{d}s \\ &= - \langle \partial u, \varphi \rangle_{L_2((a,b))} .\end{aligned}$$

This shows $v \in H^1((a,b))$ with $\partial v = \partial u$. Hence, by Lemma 14.2.2 there exists a constant $c \in \mathbb{C}$ with $u = c + v$. We now define $f$ by

$$f(t) \coloneqq \begin{cases} 0 & \text{if } t < a-1 \text{ or } t > b+1, \\ ct + c(1-a) & \text{if } a-1 \leqslant t \leqslant a, \\ u(t) & \text{if } a < t < b, \\ -(c+v(b))t + (c+v(b))(1+b) & \text{if } b \leqslant t \leqslant b+1. \end{cases}$$

We then easily see that $f \in H^1(\mathbb{R})$ and clearly $f|_{(a,b)} = u$. In order to see that $u \mapsto f$ is continuous, we need to establish that the value $c$ depends continuously on $u$. This, however, follows from the estimate

$$\begin{aligned} |c| &= \frac{1}{\sqrt{b-a}} \left( \int_a^b |c|^2 \right)^{1/2} \leqslant \frac{1}{\sqrt{b-a}} (\|u\|_{L_2(a,b)} + \|v\|_{L_2(a,b)}) \\ &\leqslant \frac{1}{\sqrt{b-a}} (\|u\|_{L_2(a,b)} + (b-a)\|\partial u\|_{L_2(a,b)}) \leqslant \frac{\sqrt{2}\max\{1,(b-a)\}}{\sqrt{b-a}} \|u\|_{H^1(a,b)}. \qquad \square \end{aligned}$$

**Theorem 14.2.3.** *Let $I \subseteq \mathbb{R}$ be an open bounded interval. Then $H^1(I) \hookrightarrow L_2(I)$ compactly.*

*Proof.* By Proposition 14.2.1, we find a continuous mapping $E\colon H^1(I) \to H^1(\mathbb{R})$ such that for all $u \in H^1(I)$ we have $E(u)|_I = u$. Moreover, by Exercise 4.3 the mapping $H^1(\mathbb{R}) \hookrightarrow C^{1/2}(\mathbb{R})$ is continuous. Thus,

$$H^1(I) \xrightarrow{E} H^1(\mathbb{R}) \hookrightarrow C^{1/2}(\mathbb{R}) \to C^{1/2}(I),$$

is a composition of continuous mappings, where the last mapping is the restriction to $I$. Since $C^{1/2}(I) \hookrightarrow C(I)$ compactly by the Arzelà–Ascoli theorem, and $C(I) \hookrightarrow L_2(I)$ continuously, we infer $H^1(I) \hookrightarrow L_2(I)$ compactly. $\square$

We now have the opportunity to study the limit behaviour of a periodic mixed type problem.

**Example 14.2.4** (Highly oscillatory problems)**.** Let $s_1, s_2\colon \mathbb{R} \to [0,1]$ be 1-periodic, measurable functions. Then for $\nu > 0$, we set

$$S^{(n)} \coloneqq \left( \overline{\partial_{t,\nu} \begin{pmatrix} s_1(n\mathrm{m}) & 0 \\ 0 & s_2(n\mathrm{m}) \end{pmatrix} + \begin{pmatrix} 1-s_1(n\mathrm{m}) & 0 \\ 0 & 1-s_2(n\mathrm{m}) \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}} \right)^{-1},$$

where $\partial = \operatorname{div}$ and $\partial_0 = \operatorname{grad}_0$ are regarded as operators in $L_2((0,1))$ with respective domains $H^1((0,1))$ and $H_0^1((0,1))$. Then, by Theorem 14.2.3, the operator $A \coloneqq \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}$ satisfies the assumptions of Theorem 14.1.1. Moreover, Theorem 13.2.4 implies that the remaining assumptions of Theorem 14.1.1 are satisfied. Hence, we deduce that $\big(S^{(n)}\big)_n$ converges in the strong operator topology of $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ to the limit

$$\left( \overline{\partial_{t,\nu} \begin{pmatrix} \int_0^1 s_1 & 0 \\ 0 & \int_0^1 s_2 \end{pmatrix} + \begin{pmatrix} 1-\int_0^1 s_1 & 0 \\ 0 & 1-\int_0^1 s_2 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial_0 & 0 \end{pmatrix}} \right)^{-1}.$$

Next, we aim to provide an application to more than one spatial dimension. For this, we will also need a corresponding compactness statement. This is the subject of the rest of this section.

**Theorem 14.2.5** (Rellich–Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ compactly.*

*Proof.* Without loss of generality (by shifting and shrinking of $\Omega$ and extending by 0), we may assume that $\Omega = (0,1)^d$. We carry out the proof by induction on the spatial dimension $d$. The case $d = 1$ has been dealt with in Theorem 14.2.3. Assume the statement is true for some $d-1$. Using that $C_c^\infty((0,1)^d)$ is dense in $H_0^1((0,1)^d)$, we infer the continuity of the injection

$$\begin{aligned} R\colon H_0^1((0,1)^d) &\to H^1\big(\mathbb{R}; L_2((0,1)^{d-1})\big) \cap L_2\big(\mathbb{R}; H_0^1((0,1)^{d-1})\big) \\ \phi &\mapsto \big(t \mapsto \big(\omega \mapsto \phi(t,\omega)\big)\big), \end{aligned}$$

where we identify $\phi$ with its extension to $\mathbb{R}^d$ by 0. The range space is endowed with the usual sum scalar product.

Let $(\phi_n)_n$ be a weakly convergent null-sequence in $H_0^1((0,1)^d)$. In particular, $(R\phi_n)_n$ is bounded in $H^1\big(\mathbb{R}; L_2((0,1)^{d-1})\big)$ and hence, it is also bounded in $C_\mathrm{b}\big(\mathbb{R}; L_2\,(0,1)^{d-1}\big)$ by Theorem 4.1.2 (and Corollary 4.1.3); that is,

$$\sup_{t\in[0,1],n\in\mathbb{N}} \|\phi_n(t,\cdot)\|_{L_2((0,1)^{d-1})} < \infty. \tag{14.2}$$

Let $f \in L_2((0,1)^{d-1})$. Then $(\phi_{n,f})_n$ given by

$$\phi_{n,f}\colon t \mapsto \langle \phi_n(t,\cdot), f\rangle_{L_2((0,1)^{d-1})}$$

is a weakly convergent null-sequence in $H^1((0,1))$. We obtain by Theorem 14.2.3 that $\phi_{n,f} \to 0$ in $L_2((0,1))$ as $n \to \infty$. By separability of $L_2((0,1)^{d-1})$ we find $D \subseteq L_2((0,1)^{d-1})$ countable and dense, a subsequence (again labeled by $n$) and a null-set $N \subseteq \mathbb{R}$ such that $\phi_{n,f}(t) \to 0$ for all $t \in \mathbb{R}\setminus N$ and $f \in D$ as $n \to \infty$. By (14.2), we deduce $\phi_{n,f}(t) \to 0$ for all $t \in \mathbb{R}\setminus N$ and $f \in L_2((0,1)^{d-1})$ as $n \to \infty$, or, in other words, $\phi_n(t,\cdot) \to 0$ weakly in $L_2((0,1)^{d-1})$ for each $t \in \mathbb{R}\setminus N$ as $n \to \infty$.

Next, we show that there exists a null set $N \subseteq N_1 \subseteq \mathbb{R}$ such that $\phi_n(t,\cdot) \to 0$ in $L_2((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$. For this, since $(R\phi_n)_n$ in $L_2\big(\mathbb{R}; H_0^1((0,1)^{d-1})\big)$ is bounded, we find a null set $N \subseteq N_1 \subseteq \mathbb{R}$ such that $(\phi_n(t,\cdot))_n$ is bounded in $H_0^1((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$. Let $t \in \mathbb{R}\setminus N_1$. Then there exists a further subsequence $(\phi_{n_k}(t,\cdot))_k$ which converges weakly in $H_0^1((0,1)^{d-1})$. By the induction hypothesis, $(\phi_{n_k}(t,\cdot))_{n_k}$ converges strongly in $L_2((0,1)^{d-1})$, and since we have already seen that it is a weak null-sequence in $L_2((0,1)^{d-1})$, we derive $\phi_{n_k}(t,\cdot) \to 0$ in $L_2((0,1)^{d-1})$. By a subsequence argument we derive that

$$\phi_n(t,\cdot) \to 0$$

in $L_2((0,1)^{d-1})$ for all $t \in \mathbb{R}\setminus N_1$.

Now, for $n \in \mathbb{N}$ we deduce

$$\|\phi_n\|^2_{L_2((0,1)^d)} = \int_0^1 \|\phi_n(t,\cdot)\|^2_{L_2((0,1)^{d-1})} \,\mathrm{d}t \to 0,$$

where we have used dominated convergence, which is possible due to (14.2). $\square$

## 14.3 The Periodic Gradient

In this section we investigate the gradient on periodic functions on $\mathbb{R}^d$. Throughout, we set $Y := [0,1)^d$.

**Definition** (Periodic Gradient)**.** We define

$$C^\infty_\sharp(Y) := \left\{\phi|_Y \,;\, \phi \in C^\infty(\mathbb{R}^d),\ \phi(\cdot + k) = \phi\ (k \in \mathbb{Z}^d)\right\}$$

and

$$\begin{aligned} \operatorname{grad}_{\sharp,\infty} \colon C^\infty_\sharp(Y) \subseteq L_2(Y) &\to L_2(Y)^d \\ \phi &\mapsto \operatorname{grad}\phi. \end{aligned}$$

Moreover, we set $\operatorname{div}_\sharp := -\operatorname{grad}^*_{\sharp,\infty}$ and $\operatorname{grad}_\sharp := -\operatorname{div}^*_\sharp = \overline{\operatorname{grad}_{\sharp,\infty}}$.

*Remark* 14.3.1. The operators just introduced can easily be shown to lie between the operator realisations we have introduced in earlier chapters. Indeed, it is easy to see that

$$\operatorname{div}_0 \subseteq \operatorname{div}_\sharp \text{ and } \operatorname{grad}_0 \subseteq \operatorname{grad}_\sharp$$

and, consequently, we also have

$$\operatorname{grad}_\sharp \subseteq \operatorname{grad} \text{ and } \operatorname{div}_\sharp \subseteq \operatorname{div}.$$

The corresponding domains for the operators $\operatorname{grad}_\sharp$ and $\operatorname{div}_\sharp$ will be denoted by $H^1_\sharp(Y)$ and $H_\sharp(\operatorname{div}, Y)$, respectively.
For the next results, we define the periodic extension operator. For $\phi \in L_2(Y)^m$ we put

$$\phi_{\mathrm{pe}}(x+k) := \phi(x)$$

for almost every $x \in Y$ and all $k \in \mathbb{Z}^d$.

We start with the following two observations.

**Lemma 14.3.2.** *Let $f \in L_2(Y)$ and $(\rho_k)_k$ be a $\delta$-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ (cf. Exercise 3.1). Define*

$$f_k := (\rho_k * f_{\mathrm{pe}})|_Y \quad (k \in \mathbb{N}).$$

*Then $f_k \in C^\infty_\sharp(Y)$ for each $k \in \mathbb{N}$ and $f_k \to f$ in $L_2(Y)$ as $k \to \infty$.*

*Proof.* It follows as in Exercise 3.2 that $\rho_k * f_{\mathrm{pe}}$ is in $C^\infty$. Moreover, one easily sees that $\rho_k * f_{\mathrm{pe}}$ is $[0,1)^d$-periodic, and hence, $f_k \in C_\sharp^\infty(Y)$ for each $k \in \mathbb{N}$. For the convergence we observe

$$\big(\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})\big)(x) = f_k(x) \quad (x \in Y, k \in \mathbb{N}).$$

Moreover, by Exercise 3.2 we have $\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}}) \to \mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}}$ in $L_2(\mathbb{R}^d)$ as $k \to \infty$, and thus,

$$f_k = \big(\rho_k * (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})\big)|_Y \to (\mathbb{1}_{Y+B(0,1)} f_{\mathrm{pe}})|_Y = f \quad (k \to \infty) \quad \text{in } L_2(Y). \qquad \square$$

**Lemma 14.3.3.** *$C_\sharp^\infty(Y)^d$ is a core for $\operatorname{div}_\sharp$ and $\operatorname{div}_\sharp \Psi = \operatorname{div} \Psi$ for each $\Psi \in C_\sharp^\infty(Y)^d$.*

*Proof.* First we note that $C_\sharp^\infty(Y)^d \subseteq \operatorname{dom}(\operatorname{div}_\sharp)$ and $\operatorname{div}_\sharp \Psi = \operatorname{div} \Psi$ for $\Psi \in C_\sharp^\infty(Y)^d$. To see this, for $\phi \in C_\sharp^\infty(Y), \Psi \in C_\sharp^\infty(Y)^d$ we compute

$$\begin{aligned}\langle \operatorname{grad} \phi, \Psi \rangle_{L_2(Y)^d} &= \int_Y \langle \operatorname{grad} \phi(x), \Psi(x) \rangle_{\mathbb{K}^d} \,\mathrm{d}x = -\int_Y \phi(x)^* \operatorname{div} \Psi(x) \,\mathrm{d}x \\ &= \langle \phi, -\operatorname{div} \Psi \rangle_{L_2(Y)}\end{aligned}$$

by integration by parts (note that the boundary values cancel out due to the periodicity of $\phi$ and $\Psi$). Now, let $q \in \operatorname{dom}(\operatorname{div}_\sharp)$ and $(\rho_k)_k$ be a $\delta$-sequence in $C_c^\infty(\mathbb{R}^d)$. For $k \in \mathbb{N}$ we define

$$q_k := (\rho_k * q_{\mathrm{pe}})|_Y,$$

and obtain $q_k \in C_\sharp^\infty(Y)$ and $q_k \to q$ in $L_2(Y)^d$ as $k \to \infty$ by Lemma 14.3.2. It is left to show that $\operatorname{div} q_k \to \operatorname{div}_\sharp q$ in $L_2(Y)$ as $k \to \infty$. For doing so, we show that $\operatorname{div} q_k = \big(\rho_k * (\operatorname{div}_\sharp q)_{\mathrm{pe}}\big)|_Y$, which would then yield the assertion again by Lemma 14.3.2. So, let $k \in \mathbb{N}$ and $\phi \in C_\sharp^\infty(Y)$. We compute

$$\begin{aligned}\langle q_k, \operatorname{grad} \phi \rangle_{L_2(Y)^d} &= \int_Y \left\langle \int_{\mathbb{R}^d} \rho_k(y) q_{\mathrm{pe}}(x-y) \,\mathrm{d}y, \operatorname{grad} \phi(x) \right\rangle_{\mathbb{K}^d} \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q_{\mathrm{pe}}(x-y), \operatorname{grad} \phi(x) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_{Y-y} \langle q_{\mathrm{pe}}(x), (\operatorname{grad} \phi)_{\mathrm{pe}}(x+y) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q(x), (\operatorname{grad} \phi)_{\mathrm{pe}}(x+y) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle q(x), (\operatorname{grad} \phi_{\mathrm{pe}}(\cdot + y))(x) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} \rho_k(y) \int_Y \langle \operatorname{div}_\sharp q(x), \phi_{\mathrm{pe}}(x+y) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} \rho_k(y) \int_{Y+y} \langle (\operatorname{div}_\sharp q)_{\mathrm{pe}}(x-y), \phi_{\mathrm{pe}}(x) \rangle_{\mathbb{K}^d} \,\mathrm{d}x \,\mathrm{d}y \\ &= -\left\langle \big(\rho_k * (\operatorname{div}_\sharp q)_{\mathrm{pe}}\big)|_Y, \phi \right\rangle_{L_2(Y)},\end{aligned}$$

where we have used perodicity as well as $\phi_{\mathrm{pe}}(\cdot + y) \in C_\sharp^\infty(Y)$. $\square$

*Remark* 14.3.4. The proof of Lemma 14.3.3 reveals that every $q \in \ker(\operatorname{div}_\sharp)$ can be approximated by elements in $C_\sharp^\infty(Y)^d \cap \ker(\operatorname{div}_\sharp)$.

**Proposition 14.3.5.** *Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, $u \in H_\sharp^1(Y)$ and $q \in H_\sharp(\operatorname{div}, Y)$. Then $u_{\mathrm{pe}}|_\Omega \in H^1(\Omega)$, $q_{\mathrm{pe}}|_\Omega \in H(\operatorname{div}, \Omega)$ and*

$$\operatorname{grad}(u_{\mathrm{pe}}|_\Omega) = \big(\operatorname{grad}_\sharp u\big)_{\mathrm{pe}}|_\Omega \text{ and } \operatorname{div}(q_{\mathrm{pe}}|_\Omega) = (\operatorname{div}_\sharp q)_{\mathrm{pe}}|_\Omega.$$

*Proof.* Let first $\phi \in C_\sharp^\infty(Y)$. Then by definition $\phi_{\mathrm{pe}} \in C^\infty(\mathbb{R}^d)$ and we easily see

$$\operatorname{grad}\phi_{\mathrm{pe}} = (\operatorname{grad}\phi)_{\mathrm{pe}} = (\operatorname{grad}_\sharp \phi)_{\mathrm{pe}}.$$

Moreover, since $\Omega$ is bounded, we infer $\phi_{\mathrm{pe}} \in H^1(\Omega)$. By definition of $H_\sharp^1(Y)$ we find a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C_\sharp^\infty(Y)$ such that $\phi_k \to u$ in $L_2(Y)$ and $\operatorname{grad}_\sharp \phi_k \to \operatorname{grad}_\sharp u$ in $L_2(Y)^d$ as $k \to \infty$. Since

$$L_2(Y) \to L_2(\Omega), \quad f \mapsto f_{\mathrm{pe}}$$

is bounded due to the boundedness of $\Omega$, we also derive $\phi_{k,\mathrm{pe}} \to u_{\mathrm{pe}}$ in $L_2(\Omega)$ and $(\operatorname{grad}_\sharp \phi_k)_{\mathrm{pe}} \to (\operatorname{grad}_\sharp u)_{\mathrm{pe}}$ in $L_2(\Omega)^d$ as $k \to \infty$. By what we have shown above, we infer

$$\operatorname{grad}\phi_{k,\mathrm{pe}} = (\operatorname{grad}_\sharp \phi_k)_{\mathrm{pe}} \to (\operatorname{grad}_\sharp u)_{\mathrm{pe}} \quad (k \to \infty)$$

in $L_2(\Omega)^d$, and thus, $u_{\mathrm{pe}} \in H^1(\Omega)$ with $\operatorname{grad} u_{\mathrm{pe}} = (\operatorname{grad}_\sharp u)_{\mathrm{pe}}$ by the closedness of grad. The proof for $q$ follows by the same argument with Lemma 14.3.3 as an additional resource. $\square$

The extension result just established yields the following compactness statement.

**Theorem 14.3.6** (Rellich–Kondrachov II)**.** *The embedding $H_\sharp^1(Y) \hookrightarrow L_2(Y)$ is compact.*

*Proof.* Let $(\phi_n)_n$ be a bounded sequence in $H_\sharp^1(Y)$. Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded such that $\overline{Y} \subseteq \Omega$. By Proposition 14.3.5, we deduce that $(\phi_{n,\mathrm{pe}}|_\Omega)_n$ is bounded in $H^1(\Omega)$. Let $\psi \in C_c^\infty(\Omega)$ with $\psi = 1$ on $\overline{Y}$. Then $(\psi\phi_{n,\mathrm{pe}})_n$ is bounded in $H_0^1(\Omega)$. By Theorem 14.2.5, we find an $L_2(\Omega)$-convergent subsequence. This sequence also converges in $L_2(Y)$. Since $\psi = 1$ on $Y$, we obtain the assertion. $\square$

Next, we provide a Poincaré-type inequality for the periodic gradient.

**Proposition 14.3.7.** *There exists $c \geqslant 0$ such that for all $u \in H_\sharp^1(Y)$*

$$\left\| u - \int_Y u \right\|_{L_2(Y)} \leqslant c \big\|\operatorname{grad}_\sharp u\big\|_{L_2(Y)^d}.$$

*In particular,* $\operatorname{ran}(\operatorname{grad}_\sharp) \subseteq L_2(Y)^d$ *is closed,* $\ker(\operatorname{grad}_\sharp) = \operatorname{lin}\{\mathbb{1}_Y\}$ *and the operator*

$$\operatorname{grad}_\sharp \colon H_\sharp^1(Y) \cap \{\mathbb{1}_Y\}^\perp \to \operatorname{ran}(\operatorname{grad}_\sharp)$$

*is an isomorphism.*

*Proof.* The proof is left as Exercise 14.4. □

We are now in a position to formulate the particular example we have in mind. Problems of this type with highly oscillatory coefficients are also referred to as *homogenisation problems*. We refer to the comments section for more details on this.

**Example 14.3.8** (Homogenisation problem for the wave equation)**.** Let $c > 0$, $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be bounded, measurable, $a(x) = a(x)^* \geqslant c$ for all $x \in \mathbb{R}^d$. Furthermore, assume that $a$ is $[0,1)^d$-periodic. Let $\nu > 0$, $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ and for $n \in \mathbb{N}$ consider the problem of finding $u_n \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ such that

$$\partial_{t,\nu}^2 u_n - \operatorname{div}_\sharp a(n\mathrm{m}) \operatorname{grad}_\sharp u_n = f. \tag{14.3}$$

We have already established that there exists a uniquely determined solution, $u_n$. Employing the same trick as in Section 11.3, we rewrite (14.3) using $v_n := \partial_{t,\nu} u_n$, the canonical embedding $\iota_\sharp\colon \operatorname{ran}(\operatorname{grad}_\sharp) \hookrightarrow L_2(Y)^d$ as well as $q_n := -\iota_\sharp^* a(n\mathrm{m})\iota_\sharp \iota_\sharp^* \operatorname{grad}_\sharp u_n$ to obtain

$$\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_\sharp \iota_\sharp\\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}\right)\begin{pmatrix}v_n\\ q_n\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

Note that we have used that $\left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)\colon \operatorname{ran}(\operatorname{grad}_\sharp) \to \operatorname{ran}(\operatorname{grad}_\sharp)$ is continuously invertible and strictly positive definite (uniformly in $n$); see Proposition 11.3.5. Also note that $\iota_\sharp^* a(n\mathrm{m})\iota_\sharp$ is selfadjoint. As in Exercise 11.3 we see that $\left(\iota_\sharp^* \operatorname{grad}_\sharp\right)^* = -\operatorname{div}_\sharp \iota_\sharp$. Thus, the operator

$$S^{(n)} := \overline{\left(\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_\sharp \iota_\sharp\\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}\right)}^{-1}$$

is well-defined and bounded in $L_{2,\nu}\big(\mathbb{R}; L_2(Y) \times \operatorname{ran}(\operatorname{grad}_\sharp)\big)$. We aim to find the limit of $(S^{(n)})_n$ as $n \to \infty$. For this, we want to apply Theorem 14.1.1. We readily see using Theorem 14.3.6 and Exercise 14.5 that

$$A := \begin{pmatrix}0 & \operatorname{div}_\sharp \iota_\sharp\\ \iota_\sharp^* \operatorname{grad}_\sharp & 0\end{pmatrix}$$

satisfies the assumptions in Theorem 14.1.1. Thus, it is left to analyse $\left(\left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1}\right)_n$. This is the subject of the next section. For this reason, we define

$$\mathfrak{a}_n := \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1} \quad (n \in \mathbb{N}).$$

## 14.4 The Limit of $(\mathfrak{a}_n)_n$

In this section, we shall apply our earlier findings to higher-dimensional problems. Again, we fix $Y := [0,1)^d$ as well as $\iota_\sharp\colon \operatorname{ran}(\operatorname{grad}_\sharp) \hookrightarrow L_2(Y)^d$, the canonical embedding. Before we are able to present the central result of this section, we need a preliminary result. Throughout, let $a\colon \mathbb{R}^d \to \mathbb{K}^{d\times d}$ be measurable, bounded and $[0,1)^d$-periodic such that $\operatorname{Re} a(x) \geqslant c$ for each $x \in \mathbb{R}^d$ for some $c > 0$.

**Lemma 14.4.1.** *Let $\xi \in \mathbb{K}^d$. Then there exists a unique $v_\xi \in L_2(Y)^d$ with $v_\xi - \xi \in$ $\operatorname{ran}(\operatorname{grad}_\sharp)$ and $a(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$.*

*Proof.* Take $w \in H^1_\sharp(Y)$ such that

$$\operatorname{grad}_\sharp w = -\iota_\sharp \left(\iota_\sharp^* a(\mathrm{m})\iota_\sharp\right)^{-1} \iota_\sharp^* a(\mathrm{m})\xi = -\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi.$$

This is possible, since the right-hand side belongs to $\operatorname{ran}(\operatorname{grad}_\sharp)$ by definition. We put $v_\xi \coloneqq \operatorname{grad}_\sharp w + \xi$. Then $v_\xi - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and we have

$$\begin{aligned}\iota_\sharp^* a(\mathrm{m})v_\xi &= \iota_\sharp^* a(\mathrm{m})\left(\operatorname{grad}_\sharp w + \xi\right) = \iota_\sharp^* a(\mathrm{m})\left(-\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi + \xi\right) \\ &= -\iota_\sharp^* a(\mathrm{m})\iota_\sharp \mathfrak{a}_n \iota_\sharp^* a(\mathrm{m})\xi + \iota_\sharp^* a(\mathrm{m})\xi = 0.\end{aligned}$$

The latter gives $a(\mathrm{m})v_\xi \in \operatorname{ran}(\operatorname{grad}_\sharp)^\perp = \ker(\operatorname{div}_\sharp)$. For the uniqueness, we assume $v \in \operatorname{ran}(\operatorname{grad}_\sharp)$ with $a(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$. Then

$$(\iota_\sharp^* a(\mathrm{m})\iota_\sharp)\iota_\sharp^* v = \iota_\sharp^* a(\mathrm{m})v = 0,$$

which implies $\iota_\sharp^* v = 0$ since $\iota_\sharp^* a(\mathrm{m})\iota_\sharp$ is invertible. Thus $v = 0$. □

The previous result induces the linear mapping

$$a_{\mathrm{hom}} \colon \mathbb{K}^d \ni \xi \mapsto \int_Y a v_\xi \in \mathbb{K}^d,$$

where $v_\xi \in L_2(Y)^d$ is the unique vector field from Lemma 14.4.1.

*Remark* 14.4.2. We gather some elementary facts on $a_{\mathrm{hom}}$.

(a) We have $(a^*)_{\mathrm{hom}} = a^*_{\mathrm{hom}}$. In particular, if $a$ is pointwise selfadjoint then so is $a_{\mathrm{hom}}$. Indeed, let $\xi, \zeta \in \mathbb{K}^d$ and $v_\xi$ and $v_\zeta \in L_2(Y)^d$ be the corresponding functions for $a^*$ and $a$, respectively, according to Lemma 14.4.1. Then there exist $w_\xi, w_\zeta \in$ $\operatorname{dom}(\operatorname{grad}_\sharp)$ with $v_\xi - \xi = \operatorname{grad}_\sharp w_\xi$ and $v_\zeta - \zeta = \operatorname{grad}_\sharp w_\zeta$. We compute

$$\begin{aligned}\langle (a^*)_{\mathrm{hom}}\xi, \zeta\rangle_{\mathbb{K}^d} &= \int_Y \left\langle \left(a^* v_\xi\right)(y), v_\zeta(y) - \operatorname{grad}_\sharp w_\zeta(y)\right\rangle_{\mathbb{K}^d} \,\mathrm{d}y \\ &= \int_Y \langle \left(a^* v_\xi\right)(y), v_\zeta(y)\rangle_{\mathbb{K}^d} \,\mathrm{d}y - \int_Y \left\langle \left(a^* v_\xi\right)(y), \operatorname{grad}_\sharp w_\zeta(y)\right\rangle_{\mathbb{K}^d} \,\mathrm{d}y \\ &= \int_Y \langle v_\xi(y), \left(a v_\zeta\right)(y)\rangle_{\mathbb{K}^d} \,\mathrm{d}y - \left\langle a^* v_\xi, \operatorname{grad}_\sharp w_\zeta\right\rangle_{L_2(Y)^d} \\ &= \int_Y \langle v_\xi(y), \left(a v_\zeta\right)(y)\rangle_{\mathbb{K}^d} \,\mathrm{d}y \\ &= \int_Y \left\langle \operatorname{grad}_\sharp w_\xi(y) + \xi, \left(a v_\zeta\right)(y)\right\rangle_{\mathbb{K}^d} \,\mathrm{d}y \\ &= \int_Y \langle \xi, \left(a v_\zeta\right)(y)\rangle_{\mathbb{K}^d} \,\mathrm{d}y = \langle \xi, a_{\mathrm{hom}}\zeta\rangle_{\mathbb{K}^d}.\end{aligned}$$

(b) $\operatorname{Re} a_{\mathrm{hom}}$ is strictly positive definite. As above, one shows

$$\operatorname{Re}\langle \xi, a_{\mathrm{hom}}\xi\rangle_{\mathbb{K}^d} = \operatorname{Re}\int_Y \langle v_\xi(y), (av_\xi)(y)\rangle_{\mathbb{K}^d}\,\mathrm{d}y \geqslant c\|v_\xi\|^2_{L_2(Y)^d} \quad (\xi\in\mathbb{K}^d)$$

and since the right-hand side is strictly positive if $\xi \neq 0$ by Lemma 14.4.1, we derive the assertion.

The construction of $a_{\mathrm{hom}}$ now allows us to formulate the main result of this section.

**Theorem 14.4.3.** *We have*

$$\mathfrak{a}_n = \left(\iota_\sharp^* a(n\mathrm{m})\iota_\sharp\right)^{-1} \to \left(\iota_\sharp^* a_{\mathrm{hom}}\iota_\sharp\right)^{-1} =: \mathfrak{a}_{\mathrm{hom}} \quad (n\to\infty)$$

*in the weak operator topology of $L(\operatorname{ran}(\operatorname{grad}_\sharp))$.*

The proof of Theorem 14.4.3 requires some more preparation. One of the results needed is a variant of Theorem 13.2.4 for $L_2(Y)$. However, it will be beneficial to finish Example 14.3.8 first.

**Example 14.4.4** (Example 14.3.8 continued)**.** The operator sequence $(S^{(n)})_n$ converges in the strong operator topology of $L\big(L_{2,\nu}\big(\mathbb{R}; L_2(Y)\times\operatorname{ran}(\operatorname{grad}_\sharp)\big)\big)$ to the following limit

$$\left(\overline{\partial_{t,\nu}\begin{pmatrix}1 & 0\\ 0 & \mathfrak{a}_{\mathrm{hom}}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}_\sharp \iota_\sharp\\ \iota_\sharp^*\operatorname{grad}_\sharp & 0\end{pmatrix}}\right)^{-1}.$$

**Lemma 14.4.5.** *Let $f\colon \mathbb{R}^d\to\mathbb{K}$ be measurable and $[0,1)^d$-periodic. Let $\Omega\subseteq\mathbb{R}^d$ be open, bounded and assume $f|_Y\in L_2(Y)$. Then*

$$f(n\cdot) \to \Big(\int_Y f\Big)\mathbb{1}_\Omega$$

*weakly in $L_2(\Omega)$ as $n\to\infty$.*

*Proof.* Due to the boundedness of $\Omega$ we find a finite set $F\subseteq\mathbb{Z}^d$ such that $\Omega\subseteq\bigcup_{k\in F} k+Y$. Thus, by periodicity, it suffices to restrict our attention to the case when $\Omega = Y$. First of all we show that $(f(n\cdot))_n$ is a bounded sequence in $L_2(Y)$. For this, we compute

$$\int_Y |f(nx)|^2\,\mathrm{d}x = \frac{1}{n^d}\int_{nY}|f(y)|^2\,\mathrm{d}y = \frac{1}{n^d}n^d\int_Y |f(y)|^2\,\mathrm{d}y = \|f\|^2_{L_2(Y)},$$

where we used periodicity again. This chain of equalities also shows with the help of a density argument, that it suffices to assume $f|_Y$ to be a simple function. Note that in this case $f\in L_\infty(\mathbb{R}^d)$.
Finally, we note that by Theorem 13.2.4

$$\langle f(n\cdot), g\rangle_{L_2(Y)} \to \left\langle \Big(\int_Y f\Big)\mathbb{1}_Y, g\right\rangle_{L_2(Y)} \quad (n\to\infty)$$

for each $g\in L_2(Y)\subseteq L_1(Y)$, which implies the desired assertion. $\square$

**Lemma 14.4.6.** *Let $(q_n)_n$ and $(r_n)_n$ be weakly convergent sequences in a Hilbert space $H$ with weak limits $q, r \in H$, respectively. Moreover, let $X \subseteq H$ be a closed subspace and $\iota\colon X \to H$ the canonical embedding. Assume that*

$$q_n \in X \text{ for each } n \in \mathbb{N} \text{ and } \left(\iota^* r_n\right)_n \text{ is strongly convergent in } X.$$

*Then*

$$\lim_{n\to\infty} \langle r_n, q_n\rangle_H = \langle r, q\rangle_H .$$

*Proof.* Since $\iota^*\colon H \to X$ is continuous it is also weakly continuous, and thus,

$$\iota^* r_n \to \iota^* r \quad (n \to \infty)$$

strongly in $X$. For $n \in \mathbb{N}$ we compute

$$\langle r_n, q_n\rangle_H = \langle r_n, \iota\iota^* q_n\rangle_H = \langle \iota^* r_n, \iota^* q_n\rangle_X \to \langle \iota^* r, \iota^* q\rangle_X .$$

Since $X$ is a closed subspace, it is also weakly closed and thus $q \in X$ which yields

$$\langle \iota^* r, \iota^* q\rangle_X = \langle r, q\rangle_H . \qquad \square$$

The next theorem is a version of the so-called 'div-curl lemma'.

**Theorem 14.4.7.** *Let $(q_n)_n$ and $(r_n)_n$ be weakly convergent sequences in $L_2(Y)^d$ to some $q, r \in L_2(Y)^d$, respectively. Assume that*

$$q_n \in \operatorname{ran}(\operatorname{grad}_\sharp) \text{ for each } n \in \mathbb{N} \text{ and } \left(\iota_\sharp^* r_n\right)_n \text{ is strongly convergent in } \operatorname{ran}(\operatorname{grad}_\sharp).$$

*Then*

$$\int_Y \langle r_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x \to \int_Y \langle r(x), q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x$$

*for all $\phi \in C_\mathrm{c}^\infty(Y)$ as $n \to \infty$.*

*Proof.* Let $\phi \in C_\mathrm{c}^\infty(Y)$, $n \in \mathbb{N}$. Since $q_n \in \operatorname{ran}(\operatorname{grad}_\sharp)$, we find a unique $w_n \in H_\sharp^1(Y)$ with $w_n \in \{\mathbb{1}_Y\}^\perp = \ker(\operatorname{grad}_\sharp)^\perp$ such that

$$\operatorname{grad}_\sharp w_n = q_n.$$

Moreover, since $\operatorname{grad}_\sharp\colon H_\sharp^1(Y) \cap \{\mathbb{1}_Y\}^\perp \to \operatorname{ran}(\operatorname{grad}_\sharp)$ is an isomorphism by Proposition 14.3.7, we infer that $(w_n)_n$ is a weakly convergent sequence in $H_\sharp^1(Y)$ and denote its weak limit by $w \in H_\sharp^1(Y)$. By Theorem 14.3.6, we deduce $w_n \to w$ strongly in $L_2(Y)^d$. Moreover, note that $(\phi w_n)_n$ weakly converges to $\phi w$ in $H_\sharp^1(Y)$. In particular, $\operatorname{grad}_\sharp(\phi w_n) \to \operatorname{grad}_\sharp(\phi w)$ weakly in $L_2(Y)^d$. For $n \in \mathbb{N}$, we compute

$$\begin{aligned}\int_Y \langle r_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x &= \langle r_n, q_n\phi\rangle_{L(Y)^d} = \left\langle r_n, (\operatorname{grad}_\sharp w_n)\phi\right\rangle_{L(Y)^d} \\ &= \left\langle r_n, \operatorname{grad}_\sharp(\phi w_n)\right\rangle_{L(Y)^d} - \left\langle r_n, w_n \operatorname{grad}_\sharp \phi\right\rangle_{L_2(Y)^d} .\end{aligned}$$

Now, the first term on the right-hand side of this equality tends to $\langle r, \operatorname{grad}_\sharp(\phi w)\rangle_{L_2(Y)^d}$ by Lemma 14.4.6 applied to $X = \operatorname{ran}(\operatorname{grad}_\sharp)$, which is closed by Proposition 14.3.7. The second term tends to $\langle r, w \operatorname{grad}_\sharp \phi\rangle_{L_2(Y)^d}$ by strong convergence of $(w_n)_n$ and weak convergence of $(r_n)_n$ in $L_2(Y)^d$. Thus, we obtain

$$\begin{aligned}\int_Y \langle r_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x &\to \langle r, \operatorname{grad}_\sharp(\phi w)\rangle_{L_2(Y)^d} - \langle r, w \operatorname{grad}_\sharp \phi\rangle_{L_2(Y)^d} \\ &= \int_Y \langle r(x), q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x \quad (n \to \infty). \qquad \square\end{aligned}$$

We will apply the latter theorem to the concrete case when $r_n = a(n\mathrm{m})q_n$ in order to determine the weak limit of $(a(n\mathrm{m})q_n)_n$.

**Lemma 14.4.8.** *Let $(q_n)_n$ and $(a(n\mathrm{m})q_n)_n$ be weakly convergent in $L_2(Y)^d$ to some $q$ and $r$, respectively. Assume that*

$q_n \in \operatorname{ran}(\operatorname{grad}_\sharp)$ *for each $n \in \mathbb{N}$ and $\left(\iota_\sharp^* a(n\mathrm{m})q_n\right)_n$ is strongly convergent in* $\operatorname{ran}(\operatorname{grad}_\sharp)$.

*Then $r = a_{\mathrm{hom}} q$.*

*Proof.* Let $\xi \in \mathbb{K}^d$ and choose $v := v_\xi \in L_2(Y)^d$ according to Lemma 14.4.1 for $a^*$ instead of $a$; that is, $v - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and $a^*(\mathrm{m})v \in \ker(\operatorname{div}_\sharp)$. For $n \in \mathbb{N}$, we define $v_n := v_{\mathrm{pe}}(n\cdot) \in L_2(Y)^d$. Next, let $g \in C_\sharp^\infty(Y)$. Then we compute

$$\begin{aligned}\langle a^*(n\mathrm{m})v_n, \operatorname{grad}_\sharp g\rangle_{L_2(Y)^d} &= \int_Y \langle a^*(nx)v_{\mathrm{pe}}(nx), \operatorname{grad}_\sharp g(x)\rangle_{\mathbb{K}^d}\,\mathrm{d}x \\ &= \frac{1}{n^d}\int_{nY} \langle a^*(y)v_{\mathrm{pe}}(y), \left(\operatorname{grad}_\sharp g\right)(y/n)\rangle_{\mathbb{K}^d}\,\mathrm{d}y \\ &= \frac{1}{n^{d-1}}\int_{nY} \langle a^*(y)v_{\mathrm{pe}}(y), (\operatorname{grad} g(\cdot/n))(y)\rangle_{\mathbb{K}^d}\,\mathrm{d}y.\end{aligned}$$

In order to compute the last integral, we employ Lemma 14.3.3 and Remark 14.3.4 to find a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C_\sharp^\infty(Y)^d \cap \ker(\operatorname{div}_\sharp)$ such that $\phi_k \to a^*(\mathrm{m})v$ as $k \to \infty$ in $L_2(Y)^d$. The latter implies $(\phi_k)_{\mathrm{pe}} \to a^*(\mathrm{m})v_{\mathrm{pe}}$ as $k \to \infty$ in $L_2(nY)^d$ for each $n \in \mathbb{N}$ and $\operatorname{div}(\phi_k)_{\mathrm{pe}} = 0$ for all $k \in \mathbb{N}$ by Proposition 14.3.5. Thus, we obtain with integration by parts (note that the boundary terms vanish due to the periodicity of $\phi_k$ and $g$)

$$\begin{aligned}\langle a^*(n\mathrm{m})v_n, \operatorname{grad}_\sharp g\rangle_{L_2(Y)^d} &= \frac{1}{n^{d-1}} \langle a^*(\mathrm{m})v_{\mathrm{pe}}, (\operatorname{grad} g(\cdot/n))\rangle_{L_2(nY)^d} \\ &= \frac{1}{n^{d-1}} \lim_{k\to\infty} \langle (\phi_k)_{\mathrm{pe}}, (\operatorname{grad} g(\cdot/n))\rangle_{L_2(nY)^d} = 0.\end{aligned}$$

Since $C_\sharp^\infty(Y)$ is a core for $\operatorname{grad}_\sharp$, we infer that $a^*(n\mathrm{m})v_n \in \operatorname{ran}(\operatorname{grad}_\sharp)^\perp$ and hence,

$$\iota_\sharp^* a^*(n\mathrm{m})v_n = 0 \quad (n \in \mathbb{N}).$$

Moreover, we have $a^*(n\mathrm{m})v_n \to \int_Y a^* v = (a^*)_{\mathrm{hom}}\xi$ weakly in $L_2(Y)$ as $n \to \infty$ by Lemma 14.4.5. Thus, by Theorem 14.4.7 applied to $q_n$ and $r_n \coloneqq a^*(n\mathrm{m})v_n$, we deduce that for all $\phi \in C_{\mathrm{c}}^\infty(Y)$

$$\lim_{n\to\infty} \int_Y \langle a^*(nx)v_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x = \int_Y \langle (a^*)_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x.$$

On the other hand, $v_n \to \left(\int_Y v\right)\mathbb{1}_Y = \xi\mathbb{1}_Y$ weakly in $L_2(Y)$ as $n \to \infty$ by Lemma 14.4.5, where $\int_Y v = \xi$ follows from $v - \xi \in \operatorname{ran}(\operatorname{grad}_\sharp)$. Thus, we can apply Theorem 14.4.7 to $q_n \coloneqq v_n$ and $r_n \coloneqq a(n\mathrm{m})q_n$ and obtain for all $\phi \in C_{\mathrm{c}}^\infty(Y)$

$$\begin{aligned}\lim_{n\to\infty} \int_Y \langle a^*(nx)v_n(x), q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_Y \langle v_n(x), a(nx)q_n(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x \\ &= \int_Y \langle \xi, r(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\int_Y \langle (a^*)_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x = \int_Y \langle \xi, r(x)\rangle_{\mathbb{K}^d}\, \phi(x)\,\mathrm{d}x$$

for each $\phi \in C_{\mathrm{c}}^\infty(Y)$. Hence, we infer

$$\langle \xi, r(x)\rangle_{\mathbb{K}^d} = \langle (a^*)_{\mathrm{hom}}\xi, q(x)\rangle_{\mathbb{K}^d} = \langle \xi, a_{\mathrm{hom}}q(x)\rangle_{\mathbb{K}^d}$$

for almost every $x \in Y$, where we have used Remark 14.4.2(a). Since the latter holds for each $\xi \in \mathbb{K}^d$, we deduce $r = a_{\mathrm{hom}}q$. □

*Proof of Theorem 14.4.3.* Let $n \in \mathbb{N}$ and for $u \in \operatorname{ran}(\operatorname{grad}_\sharp)$ we put $q_n \coloneqq \mathfrak{a}_n u$. We need to show that $(q_n)_n$ weakly converges to $\mathfrak{a}_{\mathrm{hom}}u$. For this, we choose subsequences (without relabeling) such that both $(q_n)_n$ and $(a(n\mathrm{m})q_n)_n$ weakly converge to some $q$ and $r$, respectively. By definition, we have $q_n \in \operatorname{ran}(\operatorname{grad}_\sharp)$ and $\iota_\sharp^* a(n\mathrm{m})q_n = u$ for each $n \in \mathbb{N}$. Hence, by Lemma 14.4.8, we deduce $a_{\mathrm{hom}}q = r$. As $\operatorname{ran}(\operatorname{grad}_\sharp)$ is closed, it is also weakly closed, and hence, $q \in \operatorname{ran}(\operatorname{grad}_\sharp)$. Thus, we have

$$\iota_\sharp^* a_{\mathrm{hom}}\iota_\sharp q = \iota_\sharp^* r,$$

or equivalently

$$q = \mathfrak{a}_{\mathrm{hom}}\iota_\sharp^* r.$$

Now, since $u = \iota_\sharp^* a(n\mathrm{m})q_n \to \iota_\sharp^* r$ weakly, we infer

$$q = \mathfrak{a}_{\mathrm{hom}}u.$$

A subsequence argument now yields the claim. □

## 14.5 Comments

The theory of finding partial differential equations as appropriate limit problems of partial differential equations with highly oscillatory coefficients is commonly referred to as 'homogenisation'. The mathematical theory of homogenisation goes back to the late 1960s and early 70s. We refer to [BLP78] as an early monograph wrapping up the available theory to that date.

The usual way of addressing homogenisation problems is to look at static (i.e., time-independent) problems first. The corresponding elliptic equation is then intensively studied. Even though it might be hidden in the derivations above, the 'study of the elliptic problem' essentially boils down to addressing the limit behaviour of $\mathfrak{a}_n$ as $n \to \infty$; see [EGW18; Wau16a]. Consequently, generalisations of the periodic case have been introduced. The periodic case (and beyond) is covered in [BLP78; CD99]; non-periodic cases and corresponding notions have been introduced in [Spa67; Spa68] and, independently, in [MT97; Mur78].

An important technical tool to obtain results in this direction is the div-curl lemma or the notion of 'compensated compactness'. In the above presented material, this is Theorem 14.4.7; the main difficulty to overcome is that of finding a limit of a product $(\langle q_n, r_n \rangle)_n$ of weakly convergent sequences $(q_n)_n, (r_n)_n$ in $L_2(\Omega)^3$ for some open $\Omega \subseteq \mathbb{R}^3$. It turns out that if $(\operatorname{curl} q_n)_n$ and $(\operatorname{div} r_n)_n$ converge strongly in an appropriate sense, then $\int_\Omega \langle q_n, r_n \rangle \phi$ converges to the desired limit for all $\phi \in C_c^\infty(\Omega)$. In Theorem 14.4.7 the curl-condition is strengthened in as much as we ask $q_n$ to be a gradient, which results in $\operatorname{curl} q_n = 0$. The div-condition is replaced by the condition involving $\iota_\sharp^*$, which can in fact be shown to be equivalent, see [Wau18a]. The restriction to periodic boundary value problems is a mere convenience. It can be shown that the arguments work similarly for non-periodic boundary conditions, and even with the same limit, see [Tar09, Lemma 10.3].

There are many generalisations of the div-curl lemma. For this, we refer to [BCM09] (and the references given there) and to the rather recently found operator-theoretic perspective, with plenty of applications not solely restricted to the operators div and curl, see [Wau18a; Pau19].

The way of deriving the homogenised equation (i.e., the limit of $\mathfrak{a}_n$) is akin to some derivations in [VSO94; CD99]. Further reading on homogenisation problems can also be found in these references. The first step of combining homogenisation processes and evolutionary equations has been made in [Wau11] and has had some profound developments for both quantitative and qualitative results; see [Wau16b; FW18; CW19; Wau18b].

## Exercises

**Exercise 14.1.** Under the same assumptions of Theorem 14.1.1 show

$$\left\| \left( \left(\overline{\partial_{t,\nu} M_n(\partial_{t,\nu}) + A}\right)^{-1} - \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} \right) \partial_{t,\nu}^{-1} \right\|_{L(L_{2,\nu}(\mathbb{R};H))} \to 0.$$

**Exercise 14.2.** Let $\Omega \subseteq \mathbb{R}^d$ be open and $\varepsilon > 0$. We define the set

$$\Omega_\varepsilon := \{x \in \Omega\,;\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\}\,.$$

(a) Let $(\phi_k)_{k\in\mathbb{N}}$ in $C_\mathrm{c}^\infty(\mathbb{R}^d)$ be a $\delta$-sequence (cf. Exercise 3.1) and $u \in H^1(\Omega)$. We identify each function on $\Omega$ by its extension to $\mathbb{R}^d$ by 0. Prove that for $k \in \mathbb{N}$ large enough, $\phi_k * u \in H^1(\Omega_\varepsilon)$ with

$$\mathrm{grad}(\phi_k * u) = \phi_k * \mathrm{grad}\, u \text{ on } \Omega_\varepsilon.$$

(b) Use (a) to prove Lemma 14.2.2.

**Exercise 14.3.** Prove the ‘subsequence argument’: Let $X$ be a topological space and $(x_n)_n$ a sequence in $X$. Assume that there exists $x \in X$ such that each subsequence of $(x_n)_n$ has a subsequence converging to $x$. Show that $x_n \to x$ as $n \to \infty$.

**Exercise 14.4.** Let $H_0, H_1$ be Hilbert spaces and $C\colon \mathrm{dom}(C) \subseteq H_0 \to H_1$ be a closed linear operator such that $\mathrm{dom}(C) \hookrightarrow H_0$ compactly. Let $P_{\ker(C)^\perp}\colon H_0 \to H_0$ denote the orthogonal projection onto the closed subspace $\ker(C)^\perp$. Prove that there exists $c > 0$ such that

$$\forall u \in \mathrm{dom}(C) : \left\| P_{\ker(C)^\perp} u \right\|_{H_0} \leqslant c \|Cu\|_{H_1}.$$

Apply this result to prove Proposition 14.3.7.

**Exercise 14.5.** Let $H_0, H_1$ be Hilbert spaces. Let $C\colon \mathrm{dom}(C) \subseteq H_0 \to H_1$ be closed and densely defined. Assume that $\mathrm{dom}(C) \cap \ker(C)^\perp \hookrightarrow H_0$ compactly. Show that, then, $\mathrm{dom}(C^*) \cap \ker(C^*)^\perp \hookrightarrow H_1$ compactly.

**Exercise 14.6.** Let $\nu > 0$, $\Omega = [0,1)^d$, $s \in L_\infty(\mathbb{R})$ be 1-periodic, $0 \leqslant s \leqslant 1$, and $a$ as in Example 14.3.8. Show that $(u_n)_n$ in $L_{2,\nu}(\mathbb{R}; L_2(Y))$ satisfying

$$\partial_{t,\nu}^2 s(n\mathrm{m})u_n + \partial_{t,\nu}(1 - s(n\mathrm{m}))u_n - \mathrm{div}_\sharp\, a(n\mathrm{m})\, \mathrm{grad}_\sharp\, u_n = f$$

for some $f \in L_{2,\nu}(\mathbb{R}; L_2(Y))$ is convergent to some $u \in L_{2,\nu}(\mathbb{R}; L_2(Y))$. Which limit equation is satisfied by $u$?

**Exercise 14.7.** Let $(\alpha_n)_n$ be a null-sequence in $[0,1]$ and let $a$ be as in Example 14.3.8. Show

$$\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \partial_{t,\nu}^{\alpha_n} \mathfrak{a}_n \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\sharp\, \iota_\sharp \\ \iota_\sharp^* \,\mathrm{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1}$$
$$\to \left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\sharp\, \iota_\sharp \\ \iota_\sharp^* \,\mathrm{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1}$$

in the strong operator topology. Show that if $f \in L_{2,-\mu}(\mathbb{R}; L_2(Y)_\perp)$, where $L_2(Y)_\perp := \left\{\phi \in L_2(Y)\,;\, \int_Y \phi = 0\right\}$ for some small enough $\mu > 0$, we have

$$\left(\overline{\begin{pmatrix} \partial_{t,\nu} & 0 \\ 0 & \mathfrak{a}_{\mathrm{hom}} \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{div}_\sharp\, \iota_\sharp \\ \iota_\sharp^* \,\mathrm{grad}_\sharp & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} \in L_{2,-\mu}\big(\mathbb{R}; L_2(Y) \times \mathrm{ran}(\mathrm{grad}_\sharp)\big).$$

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