QUANTITATIVE HOMOGENISATION THEORY

SHANE COOPER

In ISem23 Lecture 14, amongst other things, homogenisation for the wave equation was studied; in particular the solution $u_n \in L_{2,\nu}(\mathbb{R}; L^2(Y))$ to

$$\partial_{t,\nu}^2 u_n - \mathrm{div}_\# a(nm)\mathrm{grad}_\# u_n = f \in L_{2,\nu}(\mathbb{R}; L^2(Y)),$$

was shown to converge strongly in L^2 to the solution u_0 of the homogenised equation:

$$\partial_{t,\nu}^2 u - \mathrm{div}_\# a_{\mathrm{hom}}\mathrm{grad}_\# u = f \in L_{2,\nu}(\mathbb{R}; L^2(Y)).$$

Much more can be said; for example the L^2-norm of the difference $r_n := u_n - u$ is known to be bounded from above by $C\frac{1}{n}\|f\|_{L_{2,\nu}(\mathbb{R};L^2(Y))}$, where the constant C depends only on a and Y. Such quantitative statements are called error estimates in homogenisation. Additionally, so-called corrector estimates exist: here one seeks to determine corrections to u that allow for improvements to the error estimate. For example one can aim to control the difference in a stronger norm (for example the $L_{2,\nu}(\mathbb{R}; H^1_\#(Y))$-norm). Or, seek corrections to improve the above rate of convergence in L^2-norm from $\frac{1}{n}$ to $\left(\frac{1}{n}\right)^2$. Such types of quantitative homogenisation results began to appear in the literature as early as 2003/4, in [2], for L^2 type estimates, and 2005/6 in [3], for H^1 corrector-type estimates.

Error estimates in homogenisation have since been proved, by various means, in various contexts. However, even now it is still an important research activity to revisit these results and develop further our understanding on the underlying fundamental concepts of their proof.

In [4] a novel approach to proving error estimates by combining quantitative homogenisation theory with evolutionary equations appeared. In this project we will study and learn this approach in the context of the wave equation. If time permits, we shall see how this approach allows one to determine corrector-type estimates such as those mentioned above.

This project is suited for 3 to 4 students.

References

[1] ISem 23 Lecture Notes, 2020.

[2] M. Sh. Birman and T.A. Suslina, "Second order periodic differential operators. Threshold properties and homogenization", *St Petersburg Math. J.* **15**:5 (2004).

[3] V.V. Zhikov, "Some estimates from homogenization theory", *Dok. Math.* **73**:1 (2006), 534-538.

[4] S. Cooper and M. Waurick, "Fibre homogenisation". *Hournal of Functional Analysis.* **276** (2019), 3363-3405.