Solutions to the Exercises of Lecture 13

Team Dresden

Exercise 13.1. Let $(V_n)_n$ in $L_\infty(\mathbb{R}^d)$ and $V \in L_\infty(\mathbb{R}^d)$. Characterise convergence of $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$ in terms of convergence of $(V_n)_n$ similar to as was done in Proposition 13.2.1.

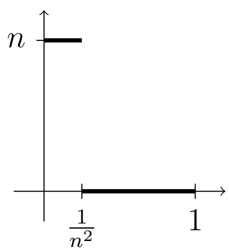
Solution 13.1. $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$ if and only if (V_n) is bounded in $L_\infty(\mathbb{R}^d)$ and $V_n \to V$ in $L_{1,loc}(\mathbb{R}^d)$.

Proof. Let $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$. Then $(V_n(m))_n$ is bounded in $L(L_{2,loc}(\mathbb{R}^d))$ by the uniform boundedness principle and as in the proof of Proposition 13.2.1 (V_n) is bounded in $L_\infty(\mathbb{R}^d)$. So let $K \subset \mathbb{R}^d)$ compact, then $\mathbb{1}_K \in L_2(\mathbb{R}^d)$ and $V_n(m)\mathbb{1}_K \to V(m)\mathbb{1}_K$ in $L_2(K)$ and in $L_1(K)$. Hence $V_n \to V$ in $L_{1,loc}(\mathbb{R}^d)$, since K is arbitrary.

Assume $V_n \to V$ in $L_{1,loc}(\mathbb{R}^d)$ and V_n, V bounded. Let $f \in L_2(\mathbb{R}^d)$ with compact support. Then by the compactness of $\operatorname{supp} f$ we can interpret V_n, V as operators on some finite measure space; so we can apply Proposition 13.2.1 and derive $\|V_n(m)f - V(m)f\|_{L_2(\mathbb{R}^d)} \to 0$. By density we obtain $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$. □

Exercise 13.2. Show that there exists an unbounded sequence $(V_n)_n$ in $L_\infty((0,1))$ and $V \in L_\infty((0,1))$ with $V_n \to V$ in $L_1((0,1))$.

Solution 13.2. Let $V_n = n \cdot \mathbb{1}_{[1,n^{-2}]}$ and $V = 0$.



Then

$$\|V_n\|_{L_\infty((0,1))} = n,$$

so the sequence is unbounded in $L_\infty((0,1))$ and

$$\|V_n - V\|_{L_1((0,1))} = \frac{1}{n} \to 0,$$

so $V_n \to V$ in $L_1((0,1))$.

Exercise 13.3. Let (Ω, Σ, μ) be a finite measure space, $(V_n)_n$ a bounded sequence in $L_\infty(\mu)$ and assume that $V_n \to V$ in $L_1(\mu)$ for some $V \in L_\infty(\mu)$. Show that there exists $\nu > 0$ such that

$$(\partial_{t,\nu} + V_n(\mathrm{m}))^{-1} \to (\partial_{t,\nu} + V(\mathrm{m}))^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu)), H^1_\nu(\mathbb{R}; L_2(\mu))\big)$.

Solution 13.3. First, we rewrite

$$\begin{aligned}(\partial_{t,\nu} + V_n(\mathrm{m}))^{-1} &= \partial_{t,\nu}^{-1}\partial_{t,\nu}(\partial_{t,\nu} + V_n(\mathrm{m}))^{-1} \\ &= \partial_{t,\nu}^{-1} M_n(\partial_{t,\nu})\end{aligned}$$

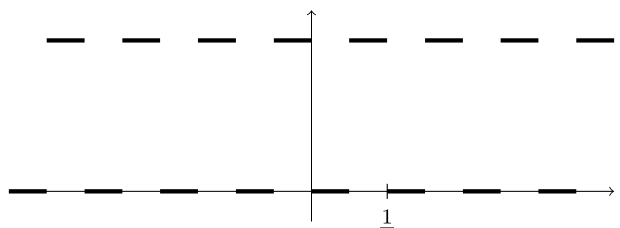
where the functions $M_n : \mathrm{dom}(M_n) \to L(L_2(\mu))$ are defined by $M_n(z) = (1+z^{-1}V_n(m))^{-1}$ for $n \in \mathbb{N}$. Now, if $\nu > \sup_{n\in\mathbb{N}} ||V_n||_{L_\infty(\mu)}$ we see that $(M_n)_n$ is a bounded sequence of material laws. By proposition 13.2.1(b) we obtain $V_n(m) \to V(m)$ in the strong operator topology of $L(L_2(\mu))$, which implies that $(M_n(z))_n$ converges to $M(z) := (1 + z^{-1}V(m))$ in the strong operator topology of $L(L_2(\mu))$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$. Then one has $M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$ strongly by Thereom 13.1.1. and the claim follows.

Exercise 13.4. Let $D := \bigcup_{n\in\mathbb{Z}}[n + 1/2, n + 1]$ and $V_n := \mathbb{1}_D(n\cdot)$. For suitable $\nu > 0$ compute the limit of

$$\big((\partial_{t,\nu} + V_n(\mathrm{m}))^{-1}\big)_n$$

in the weak operator topolgy of $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$.

Solution 13.4. Here is the graph of V_n.



By Theorem 13.2.3, $V_n \to \frac{1}{2}$ in the weak* topology of $L_\infty(\mathbb{R}^n)$. Note that $V_n^k = V_n$ for every $k \in \mathbb{N}$. Hence, $V_n^k \to \frac{1}{2}$ in the weak* topology of $L_\infty(\mathbb{R}^n)$ for every $k \in \mathbb{N}$.
By Proposition 13.2.1, this implies that, $V_n^k(m) \to \frac{1}{2}Id$ for every $k \in \mathbb{N}$ in the weak operator topology of $L_2((0,1))$.
Hence for $\nu \geqslant 1$, Theorem 13.3.1 yields

$$\begin{aligned}(\partial_{t,\nu} + V_n(m))^{-1} \to &\sum_{k=0}^{\infty}(-\partial_{t,\nu}^{-1})^k \frac{1}{2}\partial_{t,\nu}^{-1} \\ &= \frac{1}{2}\partial_{t,\nu}^{-1}\sum_{k=0}^{\infty}(-\partial_{t,\nu}^{-1})^k \\ &= \frac{1}{2}(\partial_{t,\nu} + 1)^{-1}.\end{aligned}$$

Exercise 13.5. Let H be a Hilbert space, $c > 0$ and $c \leqslant B_n = B_n^* \in L(H)$ for all $n \in \mathbb{N}$. Characterise, in terms of convergence of $(B_n)_n$ in a suitable sense, that

$$\left((\partial_{t,\nu} B_n)^{-1}\right)_n$$

converges in the weak operator topology. In the case of convergence, find its limit and a sufficient condition for which there exists a $B \in L(H)$ such that

$$(\partial_{t,\nu} B_n)^{-1} \to (\partial_{t,\nu} B)^{-1}$$

in the weak operator topology.

Solution 13.5. We first proof the following lemma.

Lemma. *Let H be a Hilbert space, $(B_n)_n$ be a sequence in $L(H)$ such that $\operatorname{Re} B_n \geqslant c$ $(n \in \mathbb{N}), \sup_n \|B_n\| \leqslant d$ for some $c, d > 0$ and $B_n^{-1} \to C$ in $L(H)$ in the weak operator topology. Then C is invertible in $L(H)$.*

Proof. We know that B_n is invertible in $L(H)$ and $\|B_n^{-1}\| \leqslant 1/c$. Let $y \in H, x := B_n^{-1} y$ (then $\|y\| = \|B_n x\| \leqslant d\|x\|$). We have

$$\operatorname{Re}\langle B_n^{-1} y, y\rangle = \operatorname{Re}\langle x, B_n x\rangle \geqslant c\|x\|^2 \geqslant \frac{c}{d^2}\|y\|^2$$

for all $n \in \mathbb{N}$. This implies that $\operatorname{Re}\langle Cy, y\rangle \geqslant \frac{c}{d^2}\|y\|^2$ (for all $y \in H$), and therefore C is invertible (with $\|C^{-1}\| \leqslant d^2/c$). $\square$

Let H be a Hilbert space, $\nu > 0$ and $(B_n)_n$ be a sequence in $L(H)$ such that $\operatorname{Re} B_n \geqslant c$ for some $c > 0$. Then $\left((\partial_{t,\nu} B_n)^{-1}\right)_n$ converges in the weak operator topology if and only if $(B_n^{-1})_n$ converges in the weak operator topology. Moreover, if $\sup \|B_n\| \leqslant d$ for some $d > 0$ and $B_n^{-1} \to C \in L(H)$ in the weak operator topology, then C is invertible and $(\partial_{t,\nu} B_n)^{-1} \to (\partial_{t,\nu} B)^{-1}$ (where $B := C^{-1}$).

Proof. We know that B_n is invertible in $L(H)$ and $\|B_n^{-1}\| \leqslant 1/c$. Let $M_n(z) := (zB_n)^{-1}$ for $z \in \mathbb{C}_{\operatorname{Re}>\nu}$. Then $(M_n)_n$ is a bounded sequence in $\mathcal{M}(H, \nu)$. The equivalence of convergence of $\left((\partial_{t,\nu} B_n)^{-1}\right)_n$ and $(B_n^{-1})_n$ in the weak operator topology now follows from Theorem 13.1.1.

Now assume $\sup \|B_n\| \leqslant d$ for some $d > 0$ and $B_n^{-1} \to C \in L(H)$ in the weak operator topology. Then by the above Lemma we obtain that C is invertible. Let $B = C^{-1}$. Applying Theorem 13.1.1 gives $(\partial_{t,\nu} B_n)^{-1} \to (\partial_{t,\nu} B)^{-1}$ in the weak operator topology. $\square$

Exercise 13.6. Let H be a Hilbert space. Show that $B_{L(H)} := \{T \in L(H); \|T\| \leqslant 1\}$ is a compact subset under the weak operator topology. If, in addition, H is separable, show that $B_{L(H)}$ is also metrisable under the weak operator topology.

Solution 13.6. The proof will be given in two parts: (a) for the compactness, (b) for the metrisability. The proof of (a) is remindful of the proof of the Banach–Alaoglu theorem.

(a) The weak operator topology τ_{w} is defined as the initial topology with respect to the family of mappings $(L(H) \ni A \mapsto \langle x, Ay\rangle \in \mathbb{K})_{x,y \in H}$, i.e., τ_{w} is the coarsest topology on $L(H)$ making all the mappings $A \mapsto \langle x, Ay\rangle \in \mathbb{K}$ continuous. Equivalently, τ_{w} is defined by the set $\{p_{x,y};\, x, y \in H\}$ of seminorms, where

$$p_{x,y}(A) := |\langle x, Ay\rangle| \qquad (A \in L(H),\ x, y \in H).$$

Let S denote the set of sesquilinear mappings $b\colon H \times H \to \mathbb{K}$. Then S is a closed subset of $\mathbb{K}^{H\times H}$ (with the product topology). Indeed, the mappings

$$\begin{aligned} \varphi_{\lambda,x,y,z}\colon \mathbb{K}^{H\times H} &\to \mathbb{K},\ b \mapsto b(\lambda x + y, z) - \lambda^* b(x,z) - b(y,z), \\ \psi_{\lambda,x,y,z}\colon \mathbb{K}^{H\times H} &\to \mathbb{K},\ b \mapsto b(x, \lambda y + z) - \lambda b(x,y) - b(x,z) \end{aligned}$$

are continuous. (The reason is, that the canonical projections $\mathbb{K}^{H\times H} \ni b \mapsto b(x,y) \in \mathbb{K}$ are continuous, for all $(x,y) \in H \times H$.) Therefore

$$S = \bigcap_{\lambda\in\mathbb{K},x,y,z\in H} \varphi_{\lambda,x,y,z}^{-1}(0) \cap \bigcap_{\lambda\in\mathbb{K},x,y,z\in H} \psi_{\lambda,x,y,z}^{-1}(0)$$

is a closed subset of $\mathbb{K}^{H\times H}$.

We define the mapping $\eta\colon L(H) \to \mathbb{K}^{H\times H}$ by $\eta(A) := (\langle x, Ay\rangle)_{(x,y)\in H\times H}$; then obviously $\eta(L(H)) \subseteq S$. (Note that τ_{w} is the initial topology with respect to the mapping η.)

The set $C := \prod_{(x,y)\in H\times H} B_{\mathbb{K}}[0, \|x\|\|y\|]$ is a compact subset of $\mathbb{K}^{H\times H}$, by Tikhonov's theorem; hence the closed subset $C \cap S$ of C is compact as well.

We show that $\eta(B_{L(H)}) = C \cap S$. '$\subseteq$' follows from $|\langle x, Ay\rangle| \leqslant \|A\|\|x\|\|y\| \leqslant \|x\|\|y\|$ $(A \in B_{L(H)},\ x, y \in H)$. '$\supseteq$': $b \in C \cap S$ means that b is sesquilinear and $|b(x,y)| \leqslant \|x\|\|y\|$ $(x, y \in H)$. This implies that there exists $A \in L(H)$ with $\|A\| \leqslant 1$ such that $b(x,y) = \langle x, Ay\rangle$ $(x, y \in H)$, i.e., $b = \eta(A)$.

As both topologies, τ_{w} on $B_{L(H)}$ and the topology of $C\cap S$, are induced by the product topology on $\mathbb{K}^{H\times H}$, it follows that $\eta\colon (B_{L(H)}, \tau_{\mathrm{w}}) \to C \cap S$ is a homeomorphism; hence $(B_{L(H)}, \tau_{\mathrm{w}})$ is compact.

(b) Now let H be separable; let $M \subseteq H$ be a countable dense subset. Let τ be the topology on $L(H)$ defined by the countable (!) set $P_M := \{p_{x,y};\ x, y \in M\}$ of seminorms. Then $\mathrm{id}\colon (L(H), \tau_{\mathrm{w}}) \to (L(H), \tau)$ is continuous. As the set P_M is countable, the topology is induced by a semi-metric, which in fact is a metric, because $A \in L(H)$, $p_{x,y}(A) = 0$ for all $x, y \in M$ implies that $A = 0$.

Now it follows that the identity maps the compact set $(B_{L(H)}, \tau_{\mathrm{w}})$ continuously to the metrisable (hence Hausdorff) set $(B_{L(H)}, \tau)$. A well-known theorem from elementary topology then implies that the topologies τ_{w} and τ coincide on $B_{L(H)}$, and so $(B_{L(H)}, \tau_{\mathrm{w}})$ is metrisable. (The theorem referred to above says: If C and D are topological spaces, C compact, D Hausdorff, and $\varphi\colon C \to D$ is bijective and continuous, then φ is a homeomorphism. The proof is by noting that the image of a closed (hence compact) subset of C is a compact (hence closed) subset of D. By taking complements one concludes that φ maps open sets to open sets, i.e., φ^{-1} is continuous.)

Exercise 13.7. Let H be a separable Hilbert space, $(B_n)_n$ in $L(H)$ bounded. Show that there exist a subsequence $(B_{n_j})_j$ of $(B_n)_n$, a material law $M : \mathrm{dom}(M) \to L(H)$ and $\nu > 0$ such that given $f \in L_{2,\nu}(\mathbb{R}; H)$ and $(u_k)_k$ in $L_{2,\nu}(\mathbb{R}; H)$ with

$$\partial_{t,\nu} u_j + B_{n_j} u_j = f \quad (j \in \mathbb{N}),$$

one can deduce that $(u_j)_j$ converges weakly to some $u \in L_{2,\nu}(\mathbb{R}; H)$ with the property that

$$\partial_{t,\nu} M(\partial_{t,\nu}) u = f.$$

Solution 13.7. From Exercise 13.6 we know that the balls $rB_{L(H)}$ are sequentially compact for the weak operator topology, for all $r > 0$. As the sequence $(B_n^k)_n$ is bounded for all $k \in \mathbb{N}$, a standard diagonal procedure shows that there exists a subsequence $(B_{n_j})_j$ such that $(B_{n_j}^k)_j$ is convergent in the weak operator topology, for all $k \in \mathbb{N}$. For simplicity of notation we now suppose that the sequence $(B_n)_n$ has already the property that the sequences $(B_n^k)_n$, for all $k \in \mathbb{N}$, are convergent to some operator $C_k \in L(H)$ in the weak operator topology. Put $C_0 := I_H$.
Let $\nu > \sup_{n\in\mathbb{N}} \|B_n\|$, and let $f \in L_{2,\nu}(\mathbb{R}; H)$. Theorem 13.3.1 shows that then

$$u_n = (\partial_{t,\nu} + B_n)^{-1} f \to u := \partial_{t,\nu}^{-1} \sum_{k=0}^{\infty} (\partial_{t,\nu}^{-1})^k C_k f = \partial_{t,\nu}^{-1} N(\partial_{t,\nu}) f$$

weakly in $L_{2,\nu}(\mathbb{R}; H)$, with the material law

$$N(z) := \sum_{k=0}^{\infty} \Big(-\frac{1}{z}\Big)^k C_k.$$

(Note that $\frac{1}{z} N(z) = M(z)$, with $M(z)$ as in the proof of Theorem 13.3.1.) This shows that $u \in \operatorname{dom}(\partial_{t,\nu})$ and $\partial_{t,\nu} u = N(\partial_{t,\nu}) f$.
Now let $\nu > 2 \sup_{n\in\mathbb{N}} \|B_n\|$. It was shown in Remark 13.3.2 that then the last equality can be transformed to

$$f = \partial_{t,\nu} M(\partial_{t,\nu}) u,$$

with the material law

$$M(z) := \sum_{\ell=0}^{\infty} \Big(- \sum_{k=1}^{\infty} \Big(-\frac{1}{z}\Big)^k C_k \Big)^{\ell} = I_H + \sum_{\ell=1}^{\infty} \Big(- \sum_{k=1}^{\infty} \Big(-\frac{1}{z}\Big)^k C_k \Big)^{\ell}.$$

And here is an **additional exercise**, suggested in the forum, concerning the weak operator topology convergence used in the proof of Theorem 13.3.1, and its solution.

Exercise 13.8. Let H be a Hilbert space. For $k \in \mathbb{N}$ let $(A_{k,n})_n$ be a sequence in $L(H)$ converging to $B_k \in L(H)$ in the weak operator topology. Let $(A_n)_n$ be a sequence in $L(H)$, and assume that the sequence $\big((A_{k,n})_n\big)_k$ converges to $(A_n)_n$ 'uniformly in operator norm', i.e., $\sup_n \|A_n - A_{k,n}\| \to 0$ as $k \to \infty$.
Show that $(B_k)_k$ is convergent in operator norm to some $B \in L(H)$ and that $(A_n)_n$ converges to B in the weak operator topology.

Solution 13.8. Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \|A_n - A_{k,n}\| \leqslant \varepsilon$ for all $k \geqslant k_0$. Then $\sup_{n\in\mathbb{N}} \|A_{k',n} - A_{k'',n}\| \leqslant 2\varepsilon$ for all $k', k'' \geqslant k_0$. From $A_{k',n} - A_{k'',n} \to B_{k'} - B_{k''}$ $(n \to \infty)$ in the weak operator topology we conclude that $\|B_{k'} - B_{k''}\| \leqslant 2\varepsilon$. Hence (B_k) is a Cauchy sequence in $L(H)$, and therefore convergent in operator norm, to some $B \in L(H)$.
Let $x, y \in H$, $\|x\|, \|y\| \leqslant 1$. Let again $\varepsilon > 0$, and choose k_0 as above. Then there exists $n_0 \in \mathbb{N}$ such that $|\langle x, (A_{k_0,n} - B_{k_0})y\rangle| \leqslant \varepsilon$ for all $n \geqslant n_0$. This implies that

$$\begin{aligned}
&|\langle x, (A_n - B)y\rangle| \\
&\leqslant |\langle x, (A_n - A_{k_0,n})y\rangle| + |\langle x, (A_{k_0,n} - B_{k_0})y\rangle| + |\langle x, (B_{k_0} - B)y\rangle| \\
&\leqslant \|A_n - A_{k_0,n}\| + |\langle x, (A_{k_0,n} - B_{k_0})y\rangle| + \|B_{k_0} - B\| \\
&\leqslant \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon
\end{aligned}$$

for all $n \geqslant n_0$. This shows that $A_n \to B$ $(n \to \infty)$ in the weak operator topology.