Internet Seminar 23

Solutions - Exercise 12

Karlsruher Institut für Technologie

Exercise 12.1. *Let $\phi \in C_c^\infty(\mathbb{R}^d), f \in L^2(\mathbb{R}^d)$. Show that $\phi * f : x \mapsto \int_{\mathbb{R}^d} \phi(x-y)f(y)dy$ belongs to $H^1(\mathbb{R}^d)$ and that $\operatorname{grad}(\phi * f) = (\operatorname{grad}\phi) * f$. If, in addition, $f \in H^1(\mathbb{R}^d) = dom(\operatorname{grad})$, then $\operatorname{grad}(\phi * f) = \phi * \operatorname{grad} f$, where the convolution is always taken component wise.*

Proof. Firstly we show that $\phi * f$ belongs to $C^\infty(\mathbb{R}^d)$. Let $h \in \mathbb{R}, i \in \{1, \dots, n\}, x \in \mathbb{R}^d$ and $\operatorname{supp}(\phi) \subseteq B(0,R)$, then

$$\frac{\phi * f(x + e_i h) - \phi * f(x)}{h} = \int_{\mathbb{R}^d} \frac{\phi(x + e_i h - y) - \phi(x-y)}{h} f(y)dy$$

By the mean value theorem the left hand side of the integrand can be estimated by $\|\partial_i \phi\|_\infty$. Also note that it is compactly supported on $B(x, R+1)$ for $|h| \leq 1$. Since $f \in L^2(\mathbb{R}^d)$ the integrand can be dominated by an integrable function. Letting $h \to 0$ and applying Lebesgue yields

$$\frac{\phi * f(x + e_i h) - \phi * f(x)}{h} \to \int_{\mathbb{R}^d} \partial_i \phi(x-y) f(y)dy =: \partial_i \phi * f.$$

Therefore, by iteration, we see that $\phi * f \in C^\infty(\mathbb{R}^d)$ with $\partial_i(\phi * f) = (\partial_i \phi) * f$. Applying Young to the convolutions shows that furthermore $\phi * f \in H^1(\mathbb{R}^d)$. In case $f \in H^1(\mathbb{R}^d)$ it follows from the definition of the weak derivative that

$$\begin{aligned} \partial_i \phi * f(x) &= \int_{\mathbb{R}^d} (\partial_i \phi)(x-y) f(y)dy \\ &= \int_{\mathbb{R}^d} -\partial_i(\phi(x-y)) f(y)dy = \int_{\mathbb{R}^d} \phi(x-y)\partial_i f(y)dy = \phi * \partial_i f(x). \end{aligned}$$

□

Exercise 12.2. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in L^2(\Omega)$ and denote by $\tilde{f} \in L^2(\mathbb{R}^d)$ the extension of f by zero. Let $v \in \mathbb{R}^d, \tau > 0$ and define $f_\tau := \tilde{f}(\cdot + \tau v)\big|_\Omega$.*

a) Show that $f_\tau \to f$ in $L^2(\Omega)$ as $\tau \to 0$.

b) Let now $f \in H^1(\Omega)$ and $\Omega + \tau v \subseteq \Omega$ for all $\tau > 0$. Show that $f_\tau \to f$ in $H^1(\Omega)$ as $\tau \to 0$.

Proof. a) Let $\epsilon > 0$. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ there exist a $g \in C_c^\infty(\Omega)$ such that $\|f - g\|_2 < \epsilon$. Notice that $\|f_\tau - g_\tau\|_2 \leq \|f - g\|_2$. Furthermore, since $g \in C_c^\infty(\Omega)$, Lebesgue implies

$$\|g_\tau - g\|_2^2 = \int_\Omega |g(x + \tau v) - g(x)|^2 \, dx \xrightarrow{\tau \to 0} 0$$

Thus, for τ sufficiently small,

$$\begin{aligned}\|f(\cdot+\tau v)-f(\cdot)\|_2 \le \|f(\cdot+\tau v)-g(\cdot+\tau v)\|_2 + \|g(\cdot+\tau v)-g(\cdot)\|_2 \\ + \|f-g\|_2 \le 3\epsilon.\end{aligned}$$

b) Since the derivative of $f \in H^1(\Omega)$ is an $L^2(\Omega)$ function and $f_\tau \in H^1(\Omega)$ with derivative $\partial_i(f_\tau) = (\partial_i f)_\tau$, the assertion follows from a).

□

Exercise 12.3. *Prove Theorem 12.2.1: $\mathcal{D}^d$ is dense in $H(\operatorname{div},\Omega)$.*

Proof. Let $f \in H(\operatorname{div},\Omega)$ and $\tilde f \in L^2(\mathbb{R}^d)^d$ be the extension of f by zero. For all $\tau > 0$, we have $\Omega + \tau e_d \subseteq \Omega$ and, thus by Exercise 12.2, we deduce $f_\tau := \tilde f(\cdot+\tau e_d)|_\Omega \to f$ in $L^2(\Omega)^d$ and $\operatorname{div}(f_\tau) = \operatorname{div}\tilde f(\cdot+\tau e_d)|_\Omega \to \operatorname{div} f$ in $L^2(\Omega)$ as $\tau \to 0$. Hence, $f_\tau \to f$ in $H(\operatorname{div},\Omega)$ as $\tau \to 0$. Thus, it suffices to approximate f_τ for $\tau > 0$. Let $\tau > 0$ and let $(\psi_k)_k$ be a δ-sequence. Then $\psi_k * \tilde f(\cdot+\tau e_d) \in H^1(\mathbb{R}^d)^d \subseteq H(\operatorname{div},\Omega)$ by Exercise 12.1. Define $f_{k,\tau} := (\psi_k * \tilde f(\cdot+\tau e_d))|_\Omega$. Then we obtain by Exercise 3.2 that $f_{k,\tau} \to f_\tau$ in $L^2(\Omega)^d$ as $k \to \infty$. Indeed, the only thing left to prove is that $\operatorname{div} f_{k,\tau} \to \operatorname{div} f_\tau$ in $L^2(\Omega)$ as $k \to \infty$. For this, we denote by g the extension of $\operatorname{div} f$ by zero. Since $g \in L^2(\Omega)$, it suffices to show that $\operatorname{div} f_{k,\tau} = \psi_k * g_\tau$ on Ω for all large enough $k \in \mathbb{N}$, where $g_\tau := g(\cdot+\tau e_d)$. Let $k > 1/\tau$. Then for all $x \in \Omega$ and $y \in \operatorname{spt}\psi_k \subseteq [-1/k,1/k]^d$ we infer $x - y + \tau e_d \in \Omega$. In particular, $f(\cdot - y + \tau e_d) \in H(\operatorname{div},\Omega)$ and $\operatorname{div} f(\cdot - y + \tau e_d) = g(\cdot - y + \tau e_d)$. Take $\eta \in C_c^\infty(\Omega)$ and compute

$$\begin{aligned}-\langle f_{k,\tau}, \operatorname{grad}\eta\rangle_{L^2(\Omega)^d} &= -\Big\langle \int_{\mathbb{R}^d} \psi_k(y)\tilde f(\cdot - y + \tau e_d)\,\mathrm{d}y, \operatorname{grad}\eta\Big\rangle_{L^2(\Omega)^d} \\ &= -\int_{[-1/k,1/k]^d} \psi_k(y)\langle f(\cdot - y + \tau e_d), \operatorname{grad}\eta\rangle_{L^2(\Omega)^d}\,\mathrm{d}y \\ &= \int_{[-1/k,1/k]^d} \psi_k(y)\langle g(\cdot - y + \tau e_d), \eta\rangle_{L^2(\Omega)}\,\mathrm{d}y \\ &= \Big\langle \int_{[-1/k,1/k]^d} \psi_k(y) g(\cdot - y + \tau e_d)\,\mathrm{d}y, \eta\Big\rangle_{L^2(\Omega)} \\ &= \langle \psi_k * g_\tau, \eta\rangle_{L^2(\Omega)^d}.\end{aligned}$$

As $\psi_k * \tilde f(\cdot+\tau e_d) \in H^1(\mathbb{R}^d)^d \subseteq H(\operatorname{div},\Omega)$, we conclude the proof using Lemma 12.1.2. □

Exercise 12.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open, $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(L_2(\Omega)\times L_2(\Omega))$ with $s_b(M) < \nu_0$ for some $\nu_0 \in \mathbb{R}$, $c > 0$ such that for all $z \in \mathbb{C}_{\operatorname{Re}\ge\nu_0}$ we have $\operatorname{Re} zM(z) \ge c$, $\nu \ge \max\{\nu_0, 0\}$ and $\nu \ne 0$. Show that there exists a unique*

$$\begin{pmatrix} v \\ q \end{pmatrix} \in H^1_\nu\Big(\mathbb{R}; \operatorname{dom}\Big(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\Big)\Big)$$

satisfying

$$\begin{cases} \Big(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\Big)\begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \Omega \\ \pi_{\mathrm{BD}(\operatorname{grad})} v(t) = \phi(t) f & \text{for all } t \in \mathbb{R}, \end{cases}$$

for some bounded $\phi \in C^\infty(\mathbb{R})$ *with* $\inf \operatorname{spt}(\phi) > -\infty$ *and* $f \in \operatorname{BD}(\operatorname{grad})$.

Proof. First, we want to show existence of solutions of the system above. We define $G \in H^3_\nu(\mathbb{R}, H^1(\Omega))$ by $t \mapsto \phi(t)f$. To see that this is well-defined we note that $\operatorname{supp}\phi \subseteq [-N, \infty)$ for some $N \in \mathbb{N}$ by assumption.

Now we consider the equation

$$\begin{aligned}\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right) \begin{pmatrix} r \\ q \end{pmatrix} \\ = \underbrace{\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix}}_{\in H^2_\nu(\mathbb{R}, H^1(\Omega)^2)} + \underbrace{\begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix}}_{\in H^3_\nu(\mathbb{R}, L^2(\Omega)\times L^2(\Omega))}.\end{aligned} \tag{12.1}$$

The right-hand side of (12.1) belongs to $H^2_\nu(\mathbb{R}, L^2(\Omega) \times L^2(\Omega))$. Theorem 6.2.1 (Picard) then yields a solution

$$\begin{pmatrix} r \\ q \end{pmatrix} = \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix}\right)$$

of (12.1) in $H^1_\nu(\mathbb{R}, L^2(\Omega) \times L^2(\Omega)) \cap L^2_\nu(\mathbb{R}, \operatorname{dom}(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}))$. Moreover, it holds that

$$\begin{aligned}\partial_{t,\nu} \begin{pmatrix} r \\ q \end{pmatrix} &= \partial_{t,\nu} \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix}\right) \\ &= \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \underbrace{\partial_{t,\nu} \underbrace{\left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix}\right)}_{H^2_\nu(\mathbb{R}, L^2(\Omega)\times L^2(\Omega))}}_{H^1_\nu(\mathbb{R}, L^2(\Omega)\times L^2(\Omega))}\end{aligned}$$

Therefore we obtain $\partial_{t,\nu} \begin{pmatrix} r \\ q \end{pmatrix} \in H^1_\nu(\mathbb{R}, L^2(\Omega) \times L^2(\Omega))$ and $\begin{pmatrix} r \\ q \end{pmatrix} \in H^2_\nu(\mathbb{R}, L^2(\Omega) \times L^2(\Omega))$ which allows us to drop the closure of the operator in (12.1). Thus

$$\begin{aligned}\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} &= \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix} - \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} r \\ q \end{pmatrix} \\ &\in H^1_\nu(\mathbb{R}, L^2(\Omega) \times L^2(\Omega))\end{aligned}$$

which also yields that $\begin{pmatrix} r \\ q \end{pmatrix} \in H^1_\nu(\mathbb{R}, \operatorname{dom}(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}))$. By our construction we

then get

$$\begin{aligned}
&\left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right)\begin{pmatrix} r+G \\ q \end{pmatrix} \\
&\quad = \left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right)\left(\begin{pmatrix} r \\ q \end{pmatrix} + \begin{pmatrix} G \\ 0 \end{pmatrix}\right) \\
&\quad = \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix} + \left(\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right)\begin{pmatrix} G \\ 0 \end{pmatrix} \\
&\quad = -\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} G \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \operatorname{grad} G \end{pmatrix} + \partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \operatorname{grad} G \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Moreover, we have $G(t) \in \mathrm{BD}(\mathrm{grad})$ for all $t \in \mathbb{R}$ (since $f \in \mathrm{BD}(\mathrm{grad})$ by assumption) and $r \in H^1_0(\Omega)$. Since $\mathrm{BD}(\mathrm{grad}) = H^1_0(\Omega)^{\perp_{H^1(\Omega)}}$ (cf. p. 153) we obtain

$$\begin{aligned}
\pi_{\mathrm{BD}(\mathrm{grad})}(r+G)(t) &= \pi_{\mathrm{BD}(\mathrm{grad})} r(t) + \pi_{\mathrm{BD}(\mathrm{grad})} G(t) \\
&= 0 + \phi(t) f
\end{aligned}$$

Thus, $\begin{pmatrix} v \\ q \end{pmatrix} := \begin{pmatrix} r+G \\ q \end{pmatrix} \in H^1_\nu(\mathbb{R}, \operatorname{dom}(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}))$ solves (12.1).

To show uniqueness assume there exists another solution $\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}$ of (12.1). Then it follows that

$$\begin{aligned}
\begin{pmatrix} \tilde{v} - G \\ \tilde{q} \end{pmatrix} &= \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}}\right)^{-1} \left(\partial_{t,\nu} M(\partial_{t,\nu}) \begin{pmatrix} -G \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\operatorname{grad} G \end{pmatrix}\right) \\
&= \begin{pmatrix} v - G \\ q \end{pmatrix}.
\end{aligned}$$

Thus, we have $\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} v \\ q \end{pmatrix}$ and the solution of is unique.

□

Exercise 12.5. *Let $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$. Show that there exists a continuous linear operator $E\colon H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that $E(\phi)\big|_\Omega = \phi$.*

Proof. Firstly we define the linear operator E on $\mathcal{D} \subseteq H^1(\Omega)$ as

$$Ef(\hat{x}, x_d) = \begin{cases} f(\hat{x}, x_d) & \text{for } (\hat{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_{>0} \\ 4f(\hat{x}, -\frac{x_d}{2}) - 3f(\hat{x}, -x_d) & \text{for } (\hat{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_{\leq 0} \end{cases}.$$

Then Ef is in $C^1(\mathbb{R}^d)$ and $\|Ef\|_{H^1(\mathbb{R}^d)} \leq 12 \|f\|_{H^1(\Omega)}$. By Theorem 12.1.1 the set $\mathcal{D}$ is dense in $H^1(\Omega)$ which therefore yields a continuous extension of E to the whole space $H^1(\Omega)$ with $\|E\| \leq 12$. Since $\mathcal{D} \subseteq \mathcal{S} \coloneqq N(E\big|_\Omega - I) = \{\varphi \in H^1(\Omega) \mid E(\varphi)\big|_\Omega = \varphi\}$ is closed, as kernel of a continuous operator, the assertion follows. □

Exercise 12.7. *Let* $\Omega \subseteq \mathbb{R}^3$ *be open. Compute* $\mathrm{BD}(\mathrm{curl}) \coloneqq H_0(\mathrm{curl}\,,\Omega)^{\perp_{H(\mathrm{curl}\,,\Omega)}}$ *and show that* $\mathrm{curl}\colon \mathrm{BD}(\mathrm{curl}) \to \mathrm{BD}(\mathrm{curl})$ *is well-defined, unitary and skew-selfadjoint. Regarding the representation of* $\mathrm{BD}(\mathrm{curl})$ *we will show that*

$$\mathrm{BD}(\mathrm{curl}) = \{q \in H(\mathrm{curl}\,,\Omega) |\, \mathrm{curl}\; q \in H(\mathrm{curl}\,,\Omega), -\mathrm{curl}\;\mathrm{curl}\; q = q\}.$$

Proof. Let $q \in H(\mathrm{curl}\,,\Omega)$. By definition we have $q \in \mathrm{BD}(\mathrm{curl}) \Leftrightarrow \forall\, r \in H_0(\mathrm{curl}\,,\Omega) : 0 = \langle r, q\rangle_{H(\mathrm{curl}\,,\Omega)}$. Using $\mathrm{curl}\; r = \mathrm{curl}_{\,0}\, r$ as $r \in H_0(\mathrm{curl}\,,\Omega)$, we can rewrite this as

$$\begin{aligned}
0 &= \langle r, q\rangle_{H(\mathrm{curl}\,,\Omega)} \\
&= \langle r, q\rangle_{L^2(\Omega)} + \langle \mathrm{curl}\; r, \mathrm{curl}\; q\rangle_{L^2(\Omega)} \\
&= \langle \mathrm{curl}_{\,0}\, r, \mathrm{curl}\; q\rangle_{L^2(\Omega)} + \langle r, q\rangle_{L^2(\Omega)} \\
&= \langle \mathrm{curl}^{\,*}\, r, \mathrm{curl}\; q\rangle_{L^2(\Omega)} + \langle r, q\rangle_{L^2(\Omega)} \\
&= \langle r, \mathrm{curl}^{\,**}\, \mathrm{curl}\; q\rangle_{L^2(\Omega)} + \langle r, q\rangle_{L^2(\Omega)} \\
&= \langle r, \mathrm{curl}\;\mathrm{curl}\; q\rangle_{L^2(\Omega)} + \langle r, q\rangle_{L^2(\Omega)} \\
&= \langle r, \mathrm{curl}\;\mathrm{curl}\; q + q\rangle_{L^2(\Omega)}
\end{aligned}$$

for all $r \in H_0(\mathrm{curl}\,,\Omega)$, where we used the definition $\mathrm{curl}_{\,0} \coloneqq \mathrm{curl}^{\,*}$ and $\mathrm{curl}^{\,**} = \mathrm{curl}\,$, since curl is a closed operator. Thus, we obtain the representation

$$\mathrm{BD}(\mathrm{curl}) = \{q \in H(\mathrm{curl}\,,\Omega) |\, \mathrm{curl}\; q \in H(\mathrm{curl}\,,\Omega), -\mathrm{curl}\;\mathrm{curl}\; q = q\}.$$

Let $\phi \in \mathrm{BD}(\mathrm{curl})$. From our representation of $\mathrm{BD}(\mathrm{curl})$ above we obtain $\mathrm{curl}\;\phi \in H(\mathrm{curl}\,,\Omega)$ and $-\mathrm{curl}\;\mathrm{curl}\;\phi = \phi \in H(\mathrm{curl}\,,\Omega)$. Moreover, we get $\mathrm{curl}\;\phi = -\mathrm{curl}\;\mathrm{curl}\;\mathrm{curl}\;\phi$ and therefore obtain $\mathrm{curl}\;\phi \in \mathrm{BD}(\mathrm{curl})$. Thus, the operator curl maps $\mathrm{BD}(\mathrm{curl})$ to itself and $-\mathrm{curl}\colon \mathrm{BD}(\mathrm{curl}) \to \mathrm{BD}(\mathrm{curl})$ is the inverse.

Further, let $\phi \in \mathrm{BD}(\mathrm{curl})$ and consider

$$\begin{aligned}
\langle \mathrm{curl}\;\phi, \mathrm{curl}\;\phi\rangle_{\mathrm{BD}(\mathrm{curl})} &= \langle \mathrm{curl}\;\phi, \mathrm{curl}\;\phi\rangle_{L^2(\Omega)} + \langle \mathrm{curl}\;\mathrm{curl}\;\phi, \mathrm{curl}\;\mathrm{curl}\;\phi\rangle_{L^2(\Omega)} \\
&= \langle \mathrm{curl}\;\phi, \mathrm{curl}\;\phi\rangle_{L^2(\Omega)} + \langle -\phi, -\phi\rangle_{L^2(\Omega)} \\
&= \langle \phi, \phi\rangle_{L^2(\Omega)} + \langle \mathrm{curl}\;\phi, \mathrm{curl}\;\phi\rangle_{L^2(\Omega)} \\
&= \langle \phi, \phi\rangle_{\mathrm{BD}(\mathrm{curl})}
\end{aligned}$$

which yields the unitarity of curl.

To show that curl is skew-selfadjoint let $\phi, \psi \in \mathrm{BD}(\mathrm{curl})$. We then obtain

$$\begin{aligned}
\langle \mathrm{curl}\;\phi, \psi\rangle_{\mathrm{BD}(\mathrm{curl})} &= \langle \mathrm{curl}\;\phi, \psi\rangle_{L^2(\Omega)} + \langle \mathrm{curl}\;\mathrm{curl}\;\phi, \mathrm{curl}\;\psi\rangle_{L^2(\Omega)} \\
&= \langle \mathrm{curl}\;\phi, -\mathrm{curl}\;\mathrm{curl}\;\psi\rangle_{L^2(\Omega)} + \langle -\phi, \mathrm{curl}\;\psi\rangle_{L^2(\Omega)} \\
&= \langle \phi, -\mathrm{curl}\;\psi\rangle_{L^2(\Omega)} + \langle \mathrm{curl}\;\phi, \mathrm{curl}\,(-\mathrm{curl}\;\psi)\rangle_{L^2(\Omega)} \\
&= \langle \phi, -\mathrm{curl}\;\psi\rangle_{\mathrm{BD}(\mathrm{curl})}.
\end{aligned}$$

□