ISEM 23: Solutions to exercises of Lecture 08
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Exercise 8.1. Let $\Lambda \subseteq \mathbb{R}_{>0}$ be a set with an accumulation point in $\mathbb{R}_{>0}$. Prove that $\left\{\left(x \mapsto \mathrm{e}^{-\lambda x}\right); \lambda \in \Lambda\right\}$ is a total set in $L_1(\mathbb{R}_{\geqslant 0})$.
Hint: Use that the set is total if and only if

$$\forall f \in L_\infty(\mathbb{R}_{\geqslant 0})\colon \left(\forall \lambda \in \Lambda\colon \int_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{-\lambda x} f(x) \mathrm{d}\, x = 0 \Rightarrow f = 0\right).$$

Solution. Let $f \in L_\infty(\mathbb{R}_{\geqslant 0})$ such that for all $\lambda \in \Lambda\colon \int_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{-\lambda x} f(x) \mathrm{d}\, x = 0$.
Let $\nu > 0$. Then $f \in L_{2,\nu}(\mathbb{R}_{\geqslant 0})$. By Corollary 8.1.3, the mapping

$$(\mathcal{L}f)(\lambda) = (\mathcal{L}_{\mathrm{Re}\,\lambda} f)(\mathrm{Im}\,\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\lambda x} f(x) \mathrm{d}\, x, \qquad \lambda \in \mathbb{C}_{\mathrm{Re}>\nu},$$

is holomorphic in $\mathbb{C}_{\mathrm{Re}>\nu}$. Let λ_0 be an accumulation point of Λ in $\mathbb{R}_{>0}$. By choosing ν such that $0 < \nu < \lambda_0$, $\mathbb{C}_{\mathrm{Re}>\nu} \cap \Lambda$ admits an accumulation point in $\mathbb{R}_{>0}$. Moreover, $(\mathcal{L}f)$ holomorphic in $\mathbb{C}_{\mathrm{Re}>\nu} \cap \Lambda$ and, $(\mathcal{L}f)(\lambda) = 0$ for all $\lambda \in \mathbb{C}_{\mathrm{Re}>\nu} \cap \Lambda$. Hence, by the identity principle we obtain that $(\mathcal{L}f)(\lambda) = 0$ for all $\lambda \in \mathbb{C}_{\mathrm{Re}>\nu}$. By injectivity of Laplace transform we infer that $f = 0$.

Exercise 8.2. Let $M\colon \mathrm{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Moreover, let $\nu > \mathrm{s_b}(M)$. Show that $\lim_{\rho \to \nu^+} M(\mathrm{im} + \rho) = M(\mathrm{im} + \nu)$ where the limit is meant in the strong operator topology on $L_2(\mathbb{R}; H)$.

Solution. Recall that the operator $M(\mathrm{im} + \nu) : L_2(\mathbb{R}; H) \to L_2(\mathbb{R}; H);\ f \mapsto (t \mapsto M(\mathrm{i}t + \nu)f)$ is a bounded operator on $L_2(\mathbb{R}; H)$ (see Proposition 5.3.2).
Let $f \in L_2(\mathbb{R}; H)$ and $\rho > \nu$. Then

$$\|M(\mathrm{im} + \rho)f - M(\mathrm{im} + \nu)f\|^2_{L_2(\mathbb{R};H)} = \int_{\mathbb{R}} \|M(\mathrm{i}t + \rho)f(t) - M(\mathrm{i}t + \nu)f(t)\|^2_H \mathrm{d}t.$$

By continuity of the map $\mathbb{C}_{\mathrm{Re}>\nu} \ni z \mapsto M(z)$, we have, for almost every $t \in \mathbb{R}$

$$\lim_{\rho \to \nu^+} \|M(\mathrm{i}t + \rho)f(t) - M(\mathrm{i}t + \nu)f(t)\|_H = 0,$$

and

$$\|M(\mathrm{i}t+\rho)f(t)\|_H \le M_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}\|f(t)\|_H.$$

Thus, by dominated convergence theorem we obtain that

$$\lim_{\rho\to\nu^+}\|M(\mathrm{im}+\rho)f - M(\mathrm{im}+\nu)f\|_{L_2(\mathbb{R};H)} = 0,$$

hence the assertion follows.

Exercise 8.3. Prove the uniqueness statement in Theorem 8.2.1.

Solution. Let $M_1, M_2\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ holomorphic and bounded such that

$$T_\nu = M_1(\partial_{t,\nu}) = M_2(\partial_{t,\nu}),\ \forall\,\nu > \nu_0.$$

Let us show that $M_1 = M_2$. Since $M_1(\partial_{t,\nu}) = M_2(\partial_{t,\nu})$ for all $\nu > \nu_0$, by using the definition of $M_k(\partial_{t,\nu})$, $k = 1, 2$, and the fact that $\mathcal{L}_\nu$ is unitary, we get

$$M_1(\mathrm{im}+\nu) = M_2(\mathrm{im}+\nu), \qquad \forall\,\nu > \nu_0.$$

The last equality means that for $f \in L_2(\mathbb{R}; H)$ we have

$$M_1(\mathrm{i}t+\nu)f(t) = M_2(\mathrm{i}t+\nu)f(t), \qquad \forall\,\nu > \nu_0 \text{ and a.e } t \text{ in } \mathbb{R}.$$

In particular, for $f = \mathbb{1}_{(a,b)}(\mathrm{m})\, y$, $a < b$ and $y \in H$ arbitrarily chosen, we get

$$M_1(\mathrm{i}t+\nu)y = M_2(\mathrm{i}t+\nu)y, \qquad \forall\,\nu > \nu_0, \text{ and a.e } t \text{ in } (a,b).$$

By continuity, we deduce that the equality holds for all $t \in (a, b)$. That is

$$M_1(\mathrm{i}t+\nu)y = M_2(\mathrm{i}t+\nu)y, \qquad \forall\,\nu > \nu_0,\ \forall\, t \in (a,b).$$

Since a and b are arbitrary in $\mathbb{R}$, we obtain

$$M_1(z)y = M_2(z)y, \qquad \forall\, z \in \mathbb{C}_{\mathrm{Re}>\nu_0},\ \forall\, y \in H.$$

Therefore, $M_1(z) = M_2(z)$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. i.e., $M_1 = M_2$.

Exercise 8.4. Give an example of a continuous and bounded function $\mathbb{C}_{\mathrm{Re}>0} \to L(H)$ such that the corresponding operator $M(\partial_{t,\nu})$ is not causal for any $\nu > 0$.

Solution. Let $\nu > 0$ and $g : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C},\ z \mapsto g(z) := \frac{\mathrm{Re}\, z}{\bar{z}}$.
Now consider $M\colon \mathbb{C}_{\mathrm{Re}>0} \to L(H); z \mapsto M(z) := g(z)I_H$. Note that g is a continuous (not holomorphic) and bounded function. Then, M is continuous and bounded.
As in Exercise 5.7., one can prove that $M(\partial_{t,\nu})$ is autonomous. Then, by the same exercise: $M(\partial_{t,\nu})$ is not causal if and only if, there exists $f_0 \in L_{2,\nu}(\mathbb{R}, H)$ such that

$\operatorname{spt} f_0 \subseteq [0,+\infty)$ but $\operatorname{spt} M(\partial_{t,\nu})f \not\subseteq [0,+\infty)$.
Let $x \in H$, we define $f_0(t) := \mathbb{1}_{\mathbb{R}_{\geqslant 0}}(t)x$. By definition we have

$$M(\partial_{t,\nu})f_0(t) := \mathcal{L}_\nu^* M(\mathrm{i}t+\nu)\mathcal{L}_\nu f_0(t),\ \forall t \in \mathbb{R}.$$

Next, we calculate $\mathcal{L}_\nu f_0$.

$$\begin{aligned}(\mathcal{L}_\nu f_0)(t) &:= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_0(s) e^{-(\mathrm{i}t+\nu)s} x \,\mathrm{d}s \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-(\mathrm{i}t+\nu)s} x \,\mathrm{d}s \\ &= \frac{1}{\sqrt{2\pi}} \frac{x}{\mathrm{i}t+\nu}.\end{aligned}$$

Thus, $M(\mathrm{i}t+\nu)(\mathcal{L}_\nu f_0)(t) = \dfrac{1}{\sqrt{2\pi}} \dfrac{\nu}{t^2+\nu^2} x$. Finally,

$$\begin{aligned}M(\partial_{t,\nu})f_0(t) &= \mathcal{L}_\nu^* \left(\frac{1}{\sqrt{2\pi}} \frac{\nu}{t^2+\nu^2} \right) x \\ &= \frac{1}{2}\, \mathrm{e}^{\nu(t-|t|)} x.\end{aligned}$$

Hence, $\operatorname{spt} M(\partial_{t,\nu})f_0 \not\subseteq [0,+\infty)$.

Exercise 8.5. Prove the following distributional variant of the Paley-Wiener theorem: Let $\nu_0 > 0$, $k \in \mathbb{N}$, $f\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathbb{C}$, and set $h(z) := \frac{1}{z^k} f(z)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. We assume that $h \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; \mathbb{C})$. For $\nu > \nu_0$ we define the distribution $u\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$u(\psi) := \left\langle \mathcal{L}_\nu^* h(\mathrm{i}\cdot + \nu), (\partial_{t,\nu}^*)^k \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} \qquad (\psi \in C_{\mathrm{c}}^\infty(\mathbb{R};\mathbb{C}))\,.$$

Prove that $\operatorname{spt} u \subseteq \mathbb{R}_{\geqslant 0}$, where

$$\operatorname{spt} u := \mathbb{R} \setminus \bigcup \{U \subseteq \mathbb{R} \text{ open } ;\ \forall \psi \in C_{\mathrm{c}}^\infty(U;\mathbb{C}) : u(\psi) = 0\}.$$

What is u if $f = \mathbb{1}_{\mathbb{C}_{\mathrm{Re}>\nu_0}}$?

Solution. Let $\nu > \nu_0$. It suffices to prove that for all $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}_{<0};\mathbb{C})$ we have $u(\psi) = 0$. Indeed, let $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}_{<0};\mathbb{C})$, by Corollary 3.2.6. we have that

$$\operatorname{dom}(\partial_{t,\nu}^*) = \operatorname{dom}(\partial_{t,\nu}) \text{ and } \partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu.$$

Using Newton's binomial formula we obtain

$$\begin{aligned}(\partial_{t,\nu}^*)^k \psi = (-\partial_{t,\nu} + 2\nu)^k \psi &= \sum_{j=0}^k \binom{k}{j} (-1)^j \partial_{t,\nu}^j(\psi) (2\nu)^{k-j} \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^j \psi^{(j)} (2\nu)^{k-j}.\end{aligned}$$

Then

$$u(\psi) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j (2\nu)^{k-j} \left\langle \mathcal{L}_\nu^* h(\mathrm{i}\cdot + \nu), \psi^{(j)} \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})}.$$

Since $h \in \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>\nu_0}; \mathbb{C})$, by Corollary 8.1.3, there exists $g \in L_{2,\nu_0}(\mathbb{R}_{\geqslant 0}; \mathbb{C})$ such that $\mathcal{L}_\nu g(\cdot) = h(\mathrm{i}\cdot + \nu)$ for all $\nu > \nu_0$. Since $\operatorname{spt} g \subseteq \mathbb{R}_{\geq 0}$ and $\operatorname{spt} \psi^{(j)} \subseteq \mathbb{R}_{<0}$, for $j = 0, 1, \ldots, k$, then

$$u(\psi) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j (2\nu)^{k-j} \left\langle g, \psi^{(j)} \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} = 0.$$

Hence the assertion follows. Now, for $f = \mathbb{1}_{\mathbb{C}_{\mathrm{Re}>\nu_0}}$, we have

$$\begin{aligned} u(\psi) &= \left\langle \mathcal{L}_\nu^* \left(\frac{1}{(\mathrm{i}\cdot + \nu)^k} \right), (\partial_{t,\nu}^*)^k \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} \\ &= \left\langle g_k, (\partial_{t,\nu}^*)^k \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})}, \end{aligned}$$

where $g_k(t) = \sqrt{2\pi} \dfrac{t^{k-1}}{(k-1)!} \mathbb{1}_{\mathbb{R}_{\geqslant 0}}$. Note that by Corollary 3.2.6 and Theorem 5.2.3 we have

$$(\partial_{t,\nu}^*)^k = \mathcal{L}_\nu^* (-\mathrm{im} + \nu)^k \mathcal{L}_\nu = (\partial_{t,\nu}^k)^*.$$

By Proposition 4.1.1, we can easily check that $g_k \in \operatorname{dom}(\partial_{t,\nu}^{k-1})$ (but $g_k \notin \operatorname{dom}(\partial_{t,\nu}^k)$) and $\partial_{t,\nu}^{k-1} g_k = \sqrt{2\pi} \mathbb{1}_{\mathbb{R}_{\geqslant 0}}$. Then we deduce that

$$u(\psi) = \left\langle \partial_{t,\nu}^{k-1} g_k, \partial_{t,\nu}^* \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} = \sqrt{2\pi} \left\langle \mathbb{1}_{\mathbb{R}_{\geqslant 0}}, \partial_{t,\nu}^* \psi \right\rangle_{L_{2,\nu}(\mathbb{R};\mathbb{C})} = \sqrt{2\pi} \psi(0).$$

Hence $u \equiv \sqrt{2\pi} \delta_0$.

Exercise 8.6. Let $g \in L_2(\mathbb{R}), a > 0$ such that $\operatorname{spt} g \subseteq [-a, a]$. Show that $f := \mathcal{F} g$ extends to a holomorphic function $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ with $\widetilde{f}(\mathrm{i}t) = f(t)$ for each $t \in \mathbb{R}$ such that

$$\exists C \geqslant 0 \; \forall z \in \mathbb{C}\colon \; \left| \widetilde{f}(z) \right| \leqslant C \mathrm{e}^{a |\operatorname{Re} z|}.$$

Solution. Consider the function

$$\begin{aligned} \widetilde{f}(z) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{e}^{-zs} g(s) \mathrm{d}s \\ &= (\mathcal{L}_{\operatorname{Re} z} g)(\operatorname{Im} z), \qquad \text{for all } z \in \mathbb{C}. \end{aligned}$$

By Corollary 8.1.3, the mapping $\mathbb{C}_{\mathrm{Re}>\nu} \ni z \mapsto \widetilde{f}(z)$ is holomorphic for all $\nu \in \mathbb{R}$, since $g \in L_{2,\nu}(\mathbb{R})$ for all $\nu \in \mathbb{R}$. Hence $\mathbb{C} \ni z \mapsto \widetilde{f}(z)$ is holomorphic and we have

$$\left| \widetilde{f}(z) \right| \leq \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{e}^{|\operatorname{Re} z \, s|} |g(t)| \, \mathrm{d}s \leq \frac{1}{\sqrt{2\pi}} \mathrm{e}^{|\operatorname{Re} z| a} \int_{-a}^{a} |g(s)| \, \mathrm{d}s = C \mathrm{e}^{|\operatorname{Re} z| a},$$

where the constant C is equal to $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-a}^{a}|g(s)|\mathrm{d}s$. By definition of Fourier transform we have $\widetilde{f}(\mathrm{i}t)=f(t)$ for all $t\in\mathbb{R}$.

Exercise 8.7. Let $f:\mathbb{C}\to\mathbb{C}$ be holomorphic such that
(a) $\exists C\geqslant 0, a>0\ \forall z\in\mathbb{C}:|f(z)|\leqslant C\mathrm{e}^{a|\operatorname{Re} z|}$,
(b) $f(\mathrm{i}\cdot)\in L_2(\mathbb{R})$.
Prove that $g:=\mathcal{F}^* f(\mathrm{i}\cdot)$ satisfies $\operatorname{spt} g\subseteq[-a,a]$.
Hint: Apply Theorem 8.1.2 to the function $h:\mathbb{C}_{\mathrm{Re}>0}\to\mathbb{C}$ given by

$$h(z):=\mathrm{e}^{-za}\frac{f(z)}{z+1}\quad(z\in\mathbb{C}_{\mathrm{Re}>0}),\tag{1}$$

to derive that $\operatorname{spt} g\subseteq\mathbb{R}_{\geqslant -a}$.
Note: The assertion even holds true if one replaces condition (a) by

$$\exists C\geqslant 0, a>0\ \forall z\in\mathbb{C}:|f(z)|\leqslant C\mathrm{e}^{a|z|}.$$

Solution. We will apply Theorem 8.1.2 to the function h given by (1). Next, we verify that $h\in\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};\mathbb{C})$. Let $\rho>0$, we have

$$\int_{\mathbb{R}}|h(\mathrm{i}t+\rho)|^2\,\mathrm{d}t=\int_{\mathbb{R}}\mathrm{e}^{-2a\rho}\frac{|f(\mathrm{i}t+\rho)|^2}{t^2+\rho^2+1}\,\mathrm{d}t\leq C^2\int_{\mathbb{R}}\frac{1}{t^2+\rho^2+1}\,\mathrm{d}t$$

where the last inequality holds from the condition (a), then

$$\|h\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};\mathbb{C})}:=\sup_{\rho>0}\int_{\mathbb{R}}|h(\mathrm{i}t+\rho)|^2\,\mathrm{d}t\leq C^2\int_{\mathbb{R}}\frac{1}{t^2+1}\,\mathrm{d}t<\infty,$$

which implies that $h\in\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};\mathbb{C})$. Then, by Theorem 8.1.2, there exists a unique $\varphi\in L_2(\mathbb{R}_{\geqslant 0};\mathbb{C})$ such that $h(\mathrm{i}\cdot+\nu)=\mathcal{L}_\nu\varphi(\cdot)$ for all $\nu>0$. Therefore, by definition of $\mathcal{L}_\nu$ and (1) we have

$$\mathrm{e}^{-(\mathrm{i}\cdot+\nu)a}\frac{f(\mathrm{i}\cdot+\nu)}{\mathrm{i}\cdot+\nu+1}=\mathcal{F}(\mathrm{e}^{-\mathrm{m}\nu}\varphi)(\cdot),$$

and then

$$\mathrm{e}^{-\nu a}\frac{f(\mathrm{i}\cdot+\nu)}{\mathrm{i}\cdot+\nu+1}=\mathcal{F}\left(\tau_a(\mathrm{e}^{-\mathrm{m}\nu}\varphi)\right)(\cdot),$$

since $\mathrm{e}^{\mathrm{i}\cdot a}\mathcal{F}(g)(\cdot)=\mathcal{F}(\tau_a g)(\cdot)$. By dominated convergence theorem we have $\tau_a(\mathrm{e}^{-\mathrm{m}\nu}\varphi)\to\tau_a\varphi$ in $L^2(\mathbb{R})$ with $\nu\downarrow 0$. On the other hand, the continuity of Fourier transform on $L^2(\mathbb{R})$ yields that $\mathcal{F}(\tau_a(\mathrm{e}^{-\mathrm{m}\nu}\varphi))\to\mathcal{F}(\tau_a\varphi)$ in $L^2(\mathbb{R})$, and then

$$\frac{f(\mathrm{i}\cdot)}{\mathrm{i}\cdot+1}=\mathcal{F}(\tau_a\varphi)(\cdot)\qquad(\text{equality in } L^2(\mathbb{R})).$$

By definition of g we have $f(\mathrm{i}\cdot)=\mathcal{F}(g)$, and by the fact that $\dfrac{1}{\mathrm{i}\cdot+1}=\mathcal{F}(\mathrm{e}^{-t}\mathbb{1}_{[0,\infty)}(t))(\cdot)$ we deduce that

$$\mathcal{F}(\tau_a\varphi)=\mathcal{F}(g*\mathrm{e}^{-\cdot}\mathbb{1}_{[0,\infty)}(\cdot)).$$

Then, by the injectivity of Fourier transform we have

$$\varphi(a+t) = \mathrm{e}^{-t} \int_{-\infty}^{t} g(s)\mathrm{e}^{s}\,\mathrm{d}s \quad \text{for almost all } t \in \mathbb{R}.$$

Using the the fact that $\operatorname{spt}\varphi \subseteq \mathbb{R}_{\geqslant 0}$ and the continuity of the map $t \mapsto \displaystyle\int_{-\infty}^{t} g(s)e^{s}\,\mathrm{d}s$, we get

$$\int_{-\infty}^{t} g(s)\mathrm{e}^{s}\,\mathrm{d}s = 0 \quad \text{for all } t < -a.$$

Then

$$\int_{A} g(s)\mathrm{e}^{s}\,\mathrm{d}s = 0 \quad \text{for all Borel set } A \subset (-\infty, -a),$$

which implies that $g = 0$ on $(-\infty, -a)$. Hence, $\operatorname{spt} g \subset \mathbb{R}_{\geqslant -a}$. It remains to prove that $\operatorname{spt} g \subset \mathbb{R}_{\leqslant a}$, for this aim we apply Theorem 8.1.2 to the function $\widetilde{h} : \mathbb{C}_{\mathrm{Re}>0} \to \mathbb{C}$ given by

$$\widetilde{h}(z) := \mathrm{e}^{-za}\frac{\widetilde{f}(z)}{z+1} \quad (z \in \mathbb{C}_{\mathrm{Re}>0}),$$

where $\widetilde{f}(z) = f(-z)$. It is easy to see that $\widetilde{f}$ satisfies (a) and (b). By the same arguments above we obtain that $\operatorname{spt}\widetilde{g} \subseteq \mathbb{R}_{\geqslant -a}$, where $\widetilde{g} := \mathcal{F}^{*}(f(-\mathrm{i}\cdot))$, and the definition of $\mathcal{F}^{*}$ allows to deduce that $\widetilde{g}(\cdot) = g(-\cdot)$. Finally, $\operatorname{spt} g \subseteq \mathbb{R}_{\leqslant a}$.