23st Internet Seminar – Evolutionary Equations

Lecture 5, solutions to exercises

Team: Ulm University (Patrick Brkic, Dennis Gallenmüller, Simon Hofmann, Nina Kliche, Manfred Sauter, Stefan Skondric, Stefan Wagner)

Exercise 5.1

Exercise 5.1. Let (Ω, Σ, μ) be a measure space, X a Banach space and $I \subset \mathbb{R}$ an open interval. Let $g\colon I \times \Omega \to X$ such that $g(t,\cdot) \in L_1(\mu; X)$ for each $t \in I$, and define

$$h\colon I \to X,\ t \mapsto \int_\Omega g(t,\omega) d\mu(\omega).$$

(a) Assume that $g(\cdot,\omega)$ is continuous for μ-almost every $\omega \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|g(t,\omega)\| \leq f(\omega) \qquad (t \in \Omega,\ \omega \in \Omega).$$

Prove that h is continuous.

(b) Assume that $g(\cdot,\omega)$ is differentiable for μ-almost every $w \in \Omega$ and let $f \in L_1(\mu)$ such that

$$\|\partial_t g(t,\omega)\| \leq f(\omega) \qquad (t \in I,\ \omega \in \Omega).$$

Prove that h is differentiable with

$$h'(t) = \int_\Omega \partial_t g(t,\omega) d\mu(\omega).$$

Proof. (a) Let $(t_n)_{n\in\mathbb{N}} \subset I, t_n \to t$. Since $g(\cdot,\omega)$ is continuous for almost every $\omega \in \Omega$, we have $g(t_n,\omega) \to g(t,\omega)$ for a.e. $\omega \in \Omega$. By assumption $g(t_n,\cdot) \in L_1(\mu, X)\ \forall n \in \mathbb{N}$ and $\|g(t_n,\omega)\|_X \leq f(w)$ for a.e. $w \in \Omega,\ \forall n \in \mathbb{N}$. Now Lebesgue's theorem gives us

$$\|h(t_n) - h(t)\|_X = \left\| \int_\Omega g(t_n,\omega) d\mu(\omega) - \int_\Omega g(t,\omega) d\mu(\omega) \right\|_X \leq \int_\Omega \|g(t_n,\omega) - g(t,\omega)\|_X d\mu(\omega) \to 0$$

as $n \to \infty$.

(b) By assumption we know that the mapping $g(\cdot,\omega)$ is differentiable for μ-a.e. $\omega \in \Omega$ and $g(t,\cdot) \in L_1(\mu, X)\ \forall t \in I$. Let $(t_n)_{n\in\mathbb{N}},\ t_n \to t$. It follows for μ-a.e. $\omega \in \Omega$ from the mean value inequality (or the fundamental theorem of calculus) that

$$\frac{\|g(t_n,\omega) - g(t,\omega)\|_X}{|t_n - t|} \leq \sup_{\xi \in I} \|\partial_t g(\xi,\omega)\| \leq f(\omega).$$

As $f \in L_1(\mu)$ the theorem of Lebesgue leads to

$$\left\| \int_\Omega \frac{g(t_n,\omega) - g(t,\omega)}{t_n - t} - \partial_t g(t,\omega) d\mu(\omega) \right\| \leq \int_\Omega \left\| \frac{g(t_n,\omega) - g(t,\omega)}{t_n - t} - \partial_t g(t,\omega) \right\| d\mu(\omega) \to 0,$$

as $n \to \infty$ and therefore

$$h'(t) = \int_\Omega \partial_t g(t,\omega) d\mu(\omega).$$

□

Exercise 5.2

Let H_0, H_1 be Hilbert spaces and $U\colon \operatorname{dom}(U) \subseteq H_0 \to H_1$ be linear such that

- $\operatorname{dom}(U)$ is dense in H_0 and $\operatorname{ran}(U)$ is dense in H_1,
- $\forall x \in \operatorname{dom}(U) : \|Ux\|_{H_1} = \|x\|_{H_0}$.

Show that U can be uniquely extended to a unitary operator between H_0 and H_1.

Let $x \in H_0$. Since $\operatorname{dom}(U) \subset H_0$ is dense, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \operatorname{dom}(U)$ such that $x_n \to x$ in H_0. Let $\epsilon > 0$. Since H_0 is a Hilbert space, $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. Hence $\exists$ $N \in \mathbb{N}$ such that

$$\|Ux_n - Ux_m\|_{H_1} = \|U(x_n - x_m)\|_{H_1} = \|x_n - x_m\|_{H_0} \leq \epsilon \qquad \forall n, m \geq N.$$

Hence, the limit $\tilde{U}x := \lim_{n\to\infty} Ux_n \in H_1$ exists and is independent of the approximating sequence. One sees that, by that definition, $\tilde{U}\colon H_0 \to H_1$ is indeed a linear operator extending U. That extension is unique as a continuous extension.

Finally, we need to check, that $\tilde{U}$ is unitary, that is, $\tilde{U}$ is onto and for all $x \in H_0$ we have $\|\tilde{U}x\|_{H_1} = \|x\|_{H_0}$.

Let $x \in H_0$ and $(x_n)_{n\in\mathbb{N}} \subset H_0$ such that $x_n \to x$ in H_0. It holds that

$$\|\tilde{U}x\| = \|\lim_{n\to\infty} Ux_n\| = \lim_{n\to\infty} \|Ux_n\| = \lim_{n\to\infty} \|x_n\| = \|x\|.$$

It remains to show that $\tilde{U}$ is onto. Let $y \in H_1$. Since $\operatorname{ran}(U) \subset H_1$ is dense, there exists $(Ux_n)_{n\in\mathbb{N}} \subset \operatorname{ran}(U)$ such that $Ux_n \to y$ in H_1. H_1 being a Hilbert space implies that $(Ux_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H_1 and since U is isometric by assumption, we get that $(x_n)_{n\in\mathbb{N}} \subset \operatorname{dom}(U)$ is a Cauchy sequence in H_0. Hence, there exists $x \in H_0$ such that $x_n \to x$ in H_0. Therefore $\lim_{n\to\infty} Ux_n = \tilde{U}x$ in H_1 as before and thus $\tilde{U}x = y$.

Exercise 5.3

(i) $\Rightarrow$ (ii) Use linearity and continuity of x'.

(ii) $\Rightarrow$ (iii) X' is a norming set (like in the hint).

(iii) $\Rightarrow$ (i) Let $z_0 \in \Omega$. W.l.o.g $z_0 = 0$. Define

$$u(h,k) := \frac{f(h) - f(0)}{h} - \frac{f(k) - f(0)}{k}$$

for small $h, k \in \mathbb{C} \setminus \{0\}$. Choose $r > 0$ s.t. $\overline{B_{2r}(0)} \subset \Omega$ and write

$$M := \sup_{z \in \overline{B_{2r}(0)}} \|f(z)\|.$$

Let $|h|, |k| \leq r$ with $h, k \neq 0$ and $x' \in D$. Use Cauchy's integral formula to obtain

$$\begin{aligned}\langle u(h,k), x' \rangle &= \frac{1}{h}\left(\langle f(h), x'\rangle - \langle f(0), x'\rangle\right) - \frac{1}{k}\left(\langle f(k), x'\rangle - \langle f(0), x'\rangle\right) = \\ &= \frac{1}{2\pi i}\int_{|z|=2r} \langle f(z), x'\rangle \left(\frac{1}{h}\left(\frac{1}{z-h} - \frac{1}{z}\right) - \frac{1}{k}\left(\frac{1}{z-k} - \frac{1}{z}\right)\right) \mathrm{d}z \\ &= \frac{h-k}{2\pi i}\int_{|z|=2r} \frac{\langle f(z), x'\rangle}{z(z-h)(z-k)}\,\mathrm{d}z.\end{aligned}$$

Moreover we have for all $z \in \mathbb{C}$ with $|z| = 2r$

$$|z|\,|z-h|\,|z-k| \geq |z|\,(|z| - |h|)(|z| - |k|) \geq 2r(2r-r)(2r-r) = 2r^3.$$

So we have

$$\left|\langle u(h,k), x'\rangle\right| \leq \frac{|h-k|}{2\pi}\int_{|z|=2r} \frac{M\|x'\|}{2r^3}\,\mathrm{d}z = \frac{|h-k|\,M\|x'\|}{r^2}.$$

If we use the norming property of D we get

$$\|u(h,k)\| \leq \frac{|h-k|\,M}{r^2} \to 0$$

as $h, k \to 0$. As a result, f is holomorphic.

(i) $\Rightarrow$ (iv) Let $z_0 \in \Omega$. Choose $r > 0$ s.t. $\overline{B_r(z_0)} \subset \Omega$ and consider again

$$M := \sup_{z \in \overline{B_r(0)}} \|f(z)\|.$$

Put

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

for all $n \in \mathbb{N}_0$. Note that

$$\|a_n\| \leq \frac{1}{2\pi}\int_{|z-z_0|=r} \frac{M}{r^{n+1}}\,\mathrm{d}z = Mr^{-n}$$

implies that the series $\sum_{k=0}^{n} a_n(z-z_0)^n$ converges absolutely. Since $z \mapsto \langle f(z), x'\rangle$ is holomorphic ($x' \in X'$ fixed) we know

$$\langle f(z), x'\rangle = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

where

$$b_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\langle f(z), x'\rangle}{(z-z_0)^{n+1}} = \langle a_n, x'\rangle.$$

By linearity and continuity we obtain

$$\langle f(z), x'\rangle = \langle \sum_{n=0}^{\infty} a_n (z-z_0)^n, x'\rangle.$$

Since X' separated the points, we get the series representation in X.

(iv) $\Rightarrow$ (ii) Let $z_0 \in \mathbb{C}$. We have

$$\langle f(z), x' \rangle = \sum_{n=0}^{\infty} \langle a_n, x' \rangle (z - z_0)^n.$$

Hence $z \mapsto \langle f(z), x' \rangle$ is holomorphic for all $x' \in X'$.

(i) $\Rightarrow$ (v) If f is holomorphic we obtain by standard computations that

$$z \mapsto \langle f(z) x_1, x_2' \rangle$$

is holomorphic.

(v) $\Rightarrow$ (iii) When proceeding similarly to the proof of implication (iii)$\Rightarrow$(i) we obtain the inequality

$$\left| \langle u(h,k) x_1, x_2' \rangle \right| \leq \frac{|h-k| \, M \|x_1\| \|x_2'\|}{r^2}.$$

By using the norming property of D_2 we get

$$\|u(h,k) x_1\| \leq \frac{|h-k| \, M \|x_1\|}{r^2}$$

and therefore by density of D_1 and continuity of $u(h,k)$

$$\|u(h,k)\| \leq \frac{|h-k| \, M}{r^2},$$

which yields that f is holomorphic.

Exercise 5.4

Let $\nu \in \mathbb{R}$ and $k \in \mathcal{L}_{1,\nu}(\mathbb{R}; H)$ First, we note that it is enough to show the equality for each $f \in \{\varphi(\cdot) x : x \in H, \varphi \in C_c^\infty(\mathbb{R})\}$. Therefore we have that $k, f, k * f \in \mathcal{L}_{1,\nu}(\mathbb{R}; H)$ and we can use Remark 5.2.1 to get that for each $t \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{L}_\nu(k)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)s} k(s) \, \mathrm{d}s, \\
\mathcal{L}_\nu(f)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)s} f(s) \, \mathrm{d}s \text{ and} \\
\mathcal{L}_\nu(k * f)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)s} \left(\int_{\mathbb{R}} k(x) f(s-x) \, \mathrm{d}x \right) \mathrm{d}s.
\end{aligned}$$

From this we can now infer for every $t \in \mathbb{R}$, that

$$\begin{aligned}
\mathcal{L}_\nu(k * f)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)s} \left(\int_{\mathbb{R}} k(x) f(s-x) \, \mathrm{d}x \right) \mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(it+\nu)s} k(x) f(s-x) \, \mathrm{d}x \, \mathrm{d}s \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(it+\nu)s} k(x) f(s-x) \, \mathrm{d}s \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)x} k(x) \int_{\mathbb{R}} e^{-(it+\nu)(s-x)} f(s-x) \, \mathrm{d}s \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)x} k(x) \int_{\mathbb{R}} e^{-(it+\nu)(s)} f(s) \, \mathrm{d}s \, \mathrm{d}x \\
&= \mathcal{L}_\nu(k)(t) \, \mathcal{L}_\nu(k)(t) \sqrt{2\pi},
\end{aligned}$$

which completes this proof.

Exercise 5.5

Let $\alpha, \nu > 0$. Obviously g_α is measurable, hence to prove that $g_\alpha \in \mathcal{L}_{1,\nu}(\mathbb{R})$ it suffices to show that

$$\int_{\mathbb{R}} |g_\alpha(s)| e^{-\nu s} \, \mathrm{d}s = \int_0^\infty s^{\alpha-1} e^{-\nu s} \, \mathrm{d}s < \infty,$$

which can be seen by integration by parts.

Since $g_\alpha \in \mathcal{L}_{1,\nu}(\mathbb{R})$, we have by Remark 5.2.1 that

$$h(t) := (\mathcal{L}_\nu g_\alpha)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(it+\nu)s} g_\alpha(s) \, \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(it+\nu)s} s^{\alpha-1} \, \mathrm{d}s,$$

where $h \in C_{\mathrm{b}}(\mathbb{R}; \mathbb{C})$. An easy calculation reveals that h is differentiable with

$$h'(t) = -\frac{i\alpha}{it+\nu} h(t)$$

and $h(0) = \frac{1}{\sqrt{2\pi}} \Gamma(\alpha) \nu^{-\alpha}$. One checks that the given function is the unique solution to this differential equation. So $(\mathcal{L}_\nu g_\alpha)(t) = \frac{1}{\sqrt{2\pi}} \Gamma(\alpha)(it+\nu)^{-\alpha}$. (Alternatively, write $\frac{h'(s)}{h(s)} = \frac{-\alpha}{s-i\nu}$ and integrate from 0 to t in order to obtain the solution to the differential equation.)

To prove formula (5.8), let $M_0 \in L(H), \alpha, \nu > 0$. Since $M(\partial_{t,\nu})$ is bounded, it again suffices to prove the equation for $f \in \{\varphi(\cdot) x : x \in H, \varphi \in C_c^\infty(\mathbb{R})\}$. We then have for almost every $t \in \mathbb{R}$

$$\begin{aligned}
(M(\partial_{t,\nu}) f)(t) &= (\mathcal{L}_\nu^* M(it+\nu) \mathcal{L}_\nu f)(t) \\
&= \mathcal{L}_\nu^* \left(x \mapsto M_0 (ix+\nu)^{-\alpha} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(ix+\nu)s} f(s) \, \mathrm{d}s \right)(t) \\
&= \mathcal{L}_\nu^* \left(M_0 (\mathcal{L}_\nu g_\alpha)(\mathcal{L}_\nu f) \sqrt{2\pi} \frac{1}{\Gamma(\alpha)} \right)(t) \\
&= \frac{1}{\Gamma(\alpha)} M_0 (\mathcal{L}_\nu^* \mathcal{L}_\nu (f * g_\alpha))(t) = \frac{1}{\Gamma(\alpha)} M_0 (f * g_\alpha)(t) \\
&= M_0 \int_{\mathbb{R}} \frac{f(s)}{\Gamma(\alpha)} \mathbb{1}_{[0,\infty)}(t-s) (t-s)^{\alpha-1} \, \mathrm{d}s \\
&= M_0 \int_{-\infty}^t \frac{f(s)}{\Gamma(\alpha)} (t-s)^{\alpha-1} \, \mathrm{d}s.
\end{aligned}$$

Exercise 5.6

Let $\mu, \nu \in \mathbb{R}$ with $\mu < \nu$ and $f \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H)$.

1. Let $\rho \in (\mu, \nu)$. We claim that $f \in L_{2,\rho}(\mathbb{R}; H) \cap L_{1,\rho}(\mathbb{R}; H)$.

Indeed, one has

$$\begin{aligned}
\|f\|_{L_{2,\rho}}^2 &= \int_{\mathbb{R}} \|f(t)\|^2 e^{-2\rho t} \, \mathrm{d}t \\
&\le \int_0^\infty \|f(t)\|^2 e^{-2\mu t} \, \mathrm{d}t + \int_{-\infty}^0 \|f(t)\|^2 e^{-2\nu t} \, \mathrm{d}t \\
&\le \|f\|_{L_{2,\mu}}^2 + \|f\|_{L_{2,\nu}}^2.
\end{aligned}$$

So $f \in L_{2,\rho}(\mathbb{R};H)$. Moreover, by using Hölder's inequality one obtains

$$\begin{aligned}\|f\|_{L_{1,\rho}} &= \int_{\mathbb{R}} \|f(t)\| e^{-\rho t}\,\mathrm{d}t \\ &= \int_0^\infty \|f(t)\| e^{-\mu t} e^{-(\rho-\mu)t}\,\mathrm{d}t + \int_{-\infty}^0 \|f(t)\| e^{-\nu t} e^{-(\rho-\nu)t}\,\mathrm{d}t \\ &\le \|f\|_{L_{2,\mu}} \Big(\int_0^\infty e^{-2(\rho-\mu)t}\,\mathrm{d}t \Big)^{1/2} + \|f\|_{L_{2,\nu}} \Big(\int_{-\infty}^0 e^{-2(\rho-\nu)t}\,\mathrm{d}t \Big)^{1/2} \\ &= \frac{1}{\sqrt{2(\rho-\mu)}} \|f\|_{L_{2,\mu}} + \frac{1}{\sqrt{2(\nu-\rho)}} \|f\|_{L_{2,\nu}} < \infty.\end{aligned}$$

So $f \in L_{1,\rho}(\mathbb{R};H)$.

2. We prove that $g\colon U \to H$ defined by $g(z) = (\mathcal{L}_{\operatorname{Re} z} f)(\operatorname{Im} z)$ is holomorphic. To this end, first note that since $f \in L_{1,\nu}(\mathbb{R};H)$ one has

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-zs} f(s)\,\mathrm{d}s.$$

Fix $z \in U$ and let $\rho := \operatorname{Re} z$. For all $h \in \mathbb{C} \setminus \{0\}$ in a small neighbourhood of 0 one has

$$\frac{g(z+h)-g(z)}{h} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-hs}-1}{h} e^{-zs} f(s)\,\mathrm{d}s, \tag{1}$$

Obviously

$$\lim_{h\to 0} \frac{e^{-hs}-1}{h} = -s$$

and

$$\left| \frac{e^{-hs}-1}{h} \right| \le |s| + e^{|h|s}.$$

Suppose $|h| < \delta := \frac{1}{2}\min\{\rho-\mu, \nu-\rho\}$. Then

$$\int_{\mathbb{R}} (|s| + e^{\delta s}) e^{-\rho s} \|f(s)\|\,\mathrm{d}s < \infty.$$

Therefore we can apply Lebesgue's dominated convergence theorem in (1) to obtain complex differentiability of g at $z \in U$. Hence g is holomorphic. (Note that by Exercise 5.3 it would suffice to test with functionals, which would allow to use the scalar-valued dominated convergence theorem.)

Exercise 5.7

Exercise 5.7. Let H_0, H_1 be Hilbert spaces and $T\colon L_{2,\nu}(\mathbb{R};H_0) \to L_{2,\nu}(\mathbb{R};H_1)$ linear and bounded. We call T *autonomous* if $T\tau_h = \tau_h T$ for each $h \in \mathbb{R}$ (τ_h denotes the translation operator defined in Example 5.3.3). Prove that for autonomous T the following statements are equivalent:

(i) T is causal.

(ii) For all $f \in L_{2,\nu}(\mathbb{R};H_0)$ with $\operatorname{spt} f \subseteq [0,\infty)$ one has $\operatorname{spt} Tf \subseteq [0,\infty)$.

Moreover, prove that for a material law M, the operator $M(\partial_{t,\nu})$ is autonomous for each $\nu > \mathrm{s_b}(M)$.

Proof. Note that T is clearly Lipschitz continuous, since it is a linear and bounded operator.
‘ $\Rightarrow$ ‘ Assume that T is causal and let $f \in L_{2,\nu}(\mathbb{R}; H_0)$ with $\operatorname{spt} f \subseteq [0,\infty)$. By the causality of T and $f = 0$ on $(-\infty, 0)$ we obtain that $Tf = 0$ on $(-\infty, 0)$. Hence $\operatorname{spt} Tf \subseteq [0,\infty)$.
‘ $\Leftarrow$ ‘ Let $a \in \mathbb{R}$ and let $f, g \in L_{2,\nu}(\mathbb{R}, H_0)$ with $f = g$ on $(-\infty, a]$. Then we have $\operatorname{spt} \tau_a(f-g) \subseteq [0,\infty)$. So it holds by the assumptions that $\operatorname{spt} \tau_a T(f-g) = \operatorname{spt} T(\tau_a(f-g)) \subset [0,\infty)$. This is equivalent to $Tf = Tg$ on $(-\infty, a]$.
Let H be a Hilbert space and M be a material law and let $\nu > \mathrm{s_b}(M)$. From Example 5.3.3 (c) we know that

$$\mathcal{L}_\nu(\tau_h \varphi)(t) = e^{it+\nu} \mathcal{L}_\nu(\varphi)(t)$$

for every $t \in \mathbb{R}$ and every $\varphi \in C_c^\infty(\mathbb{R}, H)$. Hence

$$\begin{aligned}
& M(it+\nu) e^{it+\nu} \mathcal{L}_\nu(\varphi)(t) = e^{it+\nu} M(it+\nu) \mathcal{L}_\nu(\varphi)(t) \\
\iff & M(it+\nu) e^{it+\nu} \mathcal{L}_\nu(\varphi)(t) = e^{it+\nu} \mathcal{L}_\nu \mathcal{L}_\nu^* M(it+\nu) \mathcal{L}_\nu(\varphi)(t) \\
\iff & M(it+\nu) \mathcal{L}_\nu(\tau_h \varphi)(t) = \mathcal{L}_\nu \tau_h M(\partial_{t,\nu})(\varphi)(t) \\
\iff & M(\partial_{t,\nu})(\tau_h \varphi)(t) = \tau_h M(\partial_{t,\nu})(\varphi)(t)
\end{aligned}$$

for every $t \in \mathbb{R}$ and every $\varphi \in C_c^\infty(\mathbb{R}, H)$. Since $C_c^\infty(\mathbb{R}, H)$ is dense in $L_{2,\nu}(\mathbb{R}, H)$, we see that $M(\partial_{t,\nu})$ is autonomous for each $\nu > \mathrm{s_b}(M)$. □