Exercise of lecture 2

November 4, 2019

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We want to complement the notion of ker, ran, and dom for a linear relation T between X_0 and X_1 by

$$\operatorname{mul} T := \{y \in X_1 : (0, y) \in T\}.$$

Amongst other advantages defining this explicitely instead of using $T(\{0\})$ is that is perfectly fits known relations like

$$\ker T^* = (\operatorname{ran} T)^\perp \quad \text{and} \quad \operatorname{mul} T^* = (\operatorname{dom} T)^\perp.$$

Exercise 2.1. Let $A \subseteq X_0 \times X_1$ be an unbounded linear operator. Show that for every linear operator $B \subseteq X_0 \times X_1$ with $B \supseteq A$ and $\operatorname{dom} B = X_0$, we have B is not closed.

Solution. If B is closed, then B is a closed linear operator from X_0 to X_1. By the closed graph theorem B is bounded. Hence, every restriction of B is also bounded, which contradicts the assumption that A is unbounded. ❑

Exercise 2.2. Prove Proposition 2.1.4 and Corollary 2.1.5.

Solution. We will prove the statements separately.

Proof of Proposition 2.1.4. Note that $\overline{A} = A$ (closed graph theorem). Let $(x, y) \in A$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in D that converges to x. By the continuity of A, we even have $(x_n, Ax_n) \to (x, y)$.

Proof of Corollary 2.1.5. We define $B := \overline{A}$ and prove that $\operatorname{mul} B = \{0\}$ (B is single-valued). We assume $\operatorname{mul} B \neq \{0\}$. Then there is a sequence $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ in A that converges to $(0, y)$, where $y \neq 0$.

1. case: There exists a $n_0 \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq n_0$. Then $y_n = 0$ for all $n \geq n_0$, because A is a single-valued linear operator. This contradicts $y \neq 0$.
2. case: W.L.O.G. we assume that $x_n \neq 0$ for all $n \in \mathbb{N}$ (otherwise we regard the sequence without the 0 entries). Since $(x_n, y_n) \in A$ also $\big(\frac{x_n}{\|x_n\|}, \frac{y_n}{\|x_n\|}\big) \in A$. Clearly, $\big(\frac{x_n}{\|x_n\|}\big)$ is bounded, but $\big(\frac{y_n}{\|x_n\|}\big)$ is unbounded, which contradicts the assumption that A is bounded.

Hence, $\operatorname{mul} B = \{0\}$.

Since $B \supseteq A$, we have

$$\sup_{x\in X_0\setminus\{0\}} \frac{\|Bx\|}{\|x\|} \geq \sup_{x\in\operatorname{dom} A\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

On the other hand for every $x \in X_0$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ that converges to x such that also $(Ax_n)_{n\in\mathbb{N}}$ converges to Bx. By the continuity of the norm we have

$$\frac{\|Bx\|}{\|x\|} = \lim_{n\in\mathbb{N}} \frac{\|Ax_n\|}{\|x_n\|} \leq \sup_{z\in\operatorname{dom} A\setminus\{0\}} \frac{\|Az\|}{\|z\|},$$

which completes the proof. ❑

Exercise 2.3. Prove Lemma 2.2.2.

Solution. We will show that $-A^\perp = (-A)^\perp$, $(A^{-1})^\perp = (A^\perp)^{-1}$, and if A is linear, then $-A^{-1} = (-A)^{-1}$. The asseration is a consequence of these three equalities.

- $-A^\perp = (-A)^\perp$:

$$\begin{aligned}(x,y) \in -A^\perp &\Leftrightarrow (x,-y) \perp (a,b) \quad \text{for all} \quad (a,b) \in A \\ &\Leftrightarrow (x,y) \perp (a,-b) \quad \text{for all} \quad (a,b) \in A \\ &\Leftrightarrow (x,y) \in (-A)^\perp.\end{aligned}$$

- $(A^{-1})^\perp = (A^\perp)^{-1}$:

$$\begin{aligned}(x,y) \in (A^{-1})^\perp &\Leftrightarrow (x,y) \perp (b,a) \quad \text{for all} \quad (a,b) \in A \\ &\Leftrightarrow (y,x) \perp (a,b) \quad \text{for all} \quad (a,b) \in A \\ &\Leftrightarrow (y,x) \in A^\perp \\ &\Leftrightarrow (x,y) \in (A^\perp)^{-1}.\end{aligned}$$

- $-A^{-1} = (-A)^{-1}$, if A is linear:

$$\begin{aligned}(x,y) \in -A^{-1} &\Leftrightarrow (x,-y) \in A^{-1} \Leftrightarrow (-y,x) \in A \Leftrightarrow (y,x) \in -A \\ &\Leftrightarrow (x,y) \in (-A)^{-1}.\end{aligned}$$

Note that $A^\perp$ is always linear. ❑

Exercise 2.4. Let $A : \operatorname{dom} A \subseteq H_0 \to H_0$ be a closed and densely defined linear operator. Show that for all $\lambda \in \mathbb{K}$ we have

$$\lambda \in \rho(A) \Leftrightarrow \lambda^* \in \rho(A^*).$$

Solution. Note that by the previous exercise we have for a linear relation B on H_0

$$(B^{-1})^* = -((B^{-1})^{-1})^\perp = (-(B^{-1})^\perp)^{-1} = (B^*)^{-1}.$$

Hence, we have $\big((A-\lambda)^{-1}\big)^* = \big((A-\lambda)^*\big)^{-1} = (A^* - \lambda^*)^{-1}$ (in the sense of linear relations).

Let $\lambda \in \rho(A)$. Then $(A-\lambda)^{-1} \in \mathcal{L}_\mathrm{b}(H_0)$, which implies that also its adjoint $(A^* - \lambda^*)^{-1}$ is in $\mathcal{L}_\mathrm{b}(H_0)$. Consequently, $\lambda^* \in \rho(A^*)$.

Since A is closed we have that $A^{**} = A$ and therefore the reverse implication can be derived by the same argument. ❑

Exercise 2.5. Let $U \subseteq H_0 \times H_1$ satisfy $U^{-1} = U^*$. Show that $U \in \mathcal{L}_\mathrm{b}(H_0, H_1)$ and that U is *unitary*, that is, U is onto and for all $x \in H_0$ we have $\|Ux\|_{H_1} = \|x\|_{H_0}$.

Solution. By $U^{-1} = U^*$ and Exercise 2.3 we have

$$U = (U^*)^{-1} = \big((-U^\perp)^{-1}\big)^{-1} = -U^\perp.$$

Furthermore,

$$(x,y) \in U \Rightarrow (x,-y) \in U^\perp \quad \text{implies} \quad \langle x,x\rangle_{H_0} + \langle y,-y\rangle_{H_1} = 0 \quad \forall (x,y) \in U.$$

Hence, $\|x\|_{H_0} = \|y\|_{H_1}$ for all $(x,y) \in U$. This also yields $\ker U = \operatorname{mul} U = \{0\}$ (U is injective and single-valued) and U is isometric (in paticular bounded). By

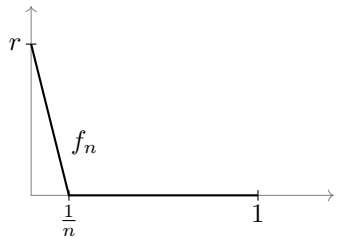
$$U \oplus (-U) = U \oplus U^\perp = H_0 \times H_1,$$

we have $\operatorname{dom} U = H_0$ and $\operatorname{ran} U = H_1$. ❑

Exercise 2.6. Let $\delta : C[0,1] \subseteq L^2(0,1) \to \mathbb{K}, f \mapsto f(0)$ where $C[0,1]$ denotes the set of $\mathbb{K}$-valued continuous functions on $[0,1]$. Show that δ is not closable and compute $\overline{\delta}$.

Solution. We will show that $\overline{\delta} = L^2(0,1) \times \mathbb{K}$, which already implies that δ is not closable in the sense of linear operators. For arbitrary $r \in \mathbb{K}$ we define

$$f_n(x) := \begin{cases} 0, & \text{if } x \geq \frac{1}{n}, \\ r(1-nx), & \text{if } x < \frac{1}{n}. \end{cases}$$



This sequence converges to 0 in $L^2(0,1)$ as the following equation shows

$$\|f_n\|_{L^2(0,1)} = \int_0^1 |f_n(x)|^2 \,\mathrm{d}x = |r|^2 \int_0^{\frac{1}{n}} |1-nx|^2 \,\mathrm{d}x = \frac{|r|^2}{3n} \to 0.$$

Hence, $\big((f_n, f_n(0))\big)_{n\in\mathbb{N}}$ is a sequence in δ that converges to $(0,r)$. Since r was arbitrarily chosen in $\mathbb{K}$, we conclude that $\operatorname{mul}\overline{\delta} = \mathbb{K}$ (this already implies that δ is not closable in the sense of linear operators). For every $g \in L^2(0,1)$ there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of $C_c^\infty(0,1)$ functions that converges to g. Since $(g_n, 0)$ is in δ and converges to $(g,0)$, we have $\operatorname{dom}\overline{\delta} = L^2(0,1)$. By linearity, $\overline{\delta} = L^2(0,1) \times \mathbb{K}$.

For a more constructive proof one could combine the sequences $(g_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$. ❑

Exercise 2.7. Let $C \subseteq \mathbb{C}$ be closed. Provide a Hilbert space H and a densely defined closed linear operator A on H such that $\sigma(A) = C$.

Solution. For convenient we choose $H := \ell^2(\mathbb{N})$. Since $\ell^2(\mathbb{N})$ is a space of sequences and we will have sequences of sequences, we will denote sequences of $\ell^2(\mathbb{N})$ elements by a superscript and the entries of a $\ell^2(\mathbb{N})$ by a subscript. Let e^n denote the sequence

$$e_k^n := \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{else.} \end{cases}$$

For a closed $C \subseteq \mathbb{C}$ there exists a countable $D \subseteq C$ such that $\overline{D} = C$. Since D is countable we can write it as $D = \{\lambda_k : k \in \mathbb{N}\}$. we define the operator A via

$$Ax = (\lambda_k x_k)_{k\in\mathbb{N}} \quad \text{with} \quad \operatorname{dom} A = \Big\{x \in \ell^2(\mathbb{N}) : \sum_{k\in\mathbb{N}} |\lambda_k x_k|^2 < +\infty\Big\}.$$

Clearly, the dense set $\operatorname{span}\{e^n : n \in \mathbb{N}\}$ is contained in $\operatorname{dom} A$, which implies the density of $\operatorname{dom} A$.

Show that A is closed: Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} A$ such that (x^n, Ax^n) converges to $(x,y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Since $\ell^2(\mathbb{N})$ convergence implies pointwise convergence, we have $x_k^n \to x_k$ and $\lambda_k x_k^n \to y_k$. Hence, we have $\lambda_k x_k = y_k$ and consequently $x \in \operatorname{dom} A$ and $Ax = y$. Therefore, A is closed.

Show that $\sigma(A) = C$: It is easy to see that $e^n \in \ker(A - \lambda_n)$, which yields that $\lambda_n \in \sigma(A)$ for every $n \in \mathbb{N}$. Therefore, $D \subseteq \sigma(A)$ and because the spectrum is always closed $C \subseteq \sigma(A)$.

For $\lambda \notin C$ we have $|\lambda_k - \lambda| > \epsilon$ for an $\epsilon > 0$. Therefore, $(\frac{1}{\lambda_k - \lambda})_{k\in\mathbb{N}}$ is a bounded sequence and $x \mapsto (\frac{x_k}{\lambda_k - \lambda})_{k\in\mathbb{N}}$ is a bounded and everywhere-defined linear

operator. It is ease to check that this mapping is indeed $(A-\lambda)^{-1}$. So $\lambda \notin \sigma(A)$ and $\sigma(A) = C$. ❑