Exercise 2.1

Let $A \subseteq X_0 \times X_1$ be an unbounded linear operator. Show that for every linear operator $B \subseteq X_0 \times X_1$ with $A \subseteq B$ and $\mathrm{dom}(B) = X_0$, we have that B is not closed.

Proof. We assume that B is closed. Since B is a linear operator and $\mathrm{dom}(B) = X_0$ is closed in X_0 then it follows by Lemma 2.1.3 that B is a bounded operator. However, since $A \subseteq B$ and A is unbounded which lead to a contradiction. Therefore, B is not closed. □

Exercise 2.2

Prove Proposition 2.1.4 and Corollary 2.1.5.

Proposition 2.1.4. Let $A \in L(X_0, X_1)$, and $D \subseteq X_0$ a dense linear subspace. Then D is a core for A.

Corollary 2.1.5. Let $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ be a densely defined, bounded linear operator. Then there exists a unique $B \in L(X_0, X_1)$ with $B \supseteq A$. In particular, we have $B = \bar{A}$ and

$$\|B\| = \sup_{\substack{x \in \mathrm{dom}(A) \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}.$$

Proof of Proposition 2.1.4.

1. Since $A \cap (D \times X_1) \subset A$, we have
$$\overline{A \cap (D \times X_1)} \subset \bar{A}.$$

2. We only need to show
$$\overline{A \cap (D \times X_1)} \supset \bar{A}$$
which is equivalent to
$$\overline{A \cap (D \times X_1)} \supset A.$$

3. Take $(x, y) \in A$. For $n \in \mathbb{N}$, we have $x_n \in D$ such that $\|x - x_n\| < \frac{1}{n}$. Therefore,
$$\begin{aligned} \|(x, y) - (x_n, Ax_n)\| &= \|x - x_n\| + \|y - Ax_n\| \\ &= \|x - x_n\| + \|Ax - Ax_n\| \\ &\leq \|x - x_n\| + \|A\| \, \|x - x_n\| \\ &\leq \frac{1}{n} (1 + \|A\|) \xrightarrow{n \to \infty} 0. \end{aligned}$$

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Since $(x_n, Ax_n) \in A \cap D \times X_1$, we get $(x, y) \in \overline{A \cap D \times X_1}$. □

Proof of Corollary 2.1.5. 1. *Existence.* Put $B = \bar{A}$ (the closure of A in $X_0 \times X_1$ equipped with the product topology), we have $B \supseteq A$. We now prove $B \in L(X_0, X_1)$.

First, we claim B is a linear relation. Indeed, for $(x, y), (u, v) \in B$ and $\lambda \in \mathbb{K}$, we have

$$A \ni (x_n, y_n) \longrightarrow (x, y)$$
$$A \ni (u_n, v_n) \longrightarrow (u, v)$$

and

$$\begin{aligned}\lim_{n\to\infty}(x_n + \lambda u_n, y_n + \lambda v_n) &= (\lim_{n\to\infty}(x_n + \lambda u_n), \lim_{n\to\infty}(y_n + \lambda v_n)) \\ &= (x + \lambda u, y + \lambda v).\end{aligned}$$

Since $(x_n + \lambda u_n, y_n + \lambda v_n) \in A$, we have $(x + \lambda u, y + \lambda v) \in B$ so B is a linear relation.

Second, we claim B is a linear operator. Assume $(0, y) \in B$ then there exists $(x_n, y_n) \in A$ such that $(x_n, y_n) \to (0, y)$. Thus

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_n = A \lim_{n\to\infty} x_n = A\ 0 = 0$$

so $B[\{0\}] = \{0\}$.

Third, we claim $\mathrm{dom}(B) = X_0$. Take $x \in X_0$ then we have $x_n \in \mathrm{dom}(A)$, $x_n \to x$ since A is densely defined. Because A is bounded, $C \coloneqq \sup_{x\in\mathrm{dom}(A)\setminus\{0\}} \frac{\|Ax\|}{\|x\|} < \infty$. Observe

$$\|Ax_n - Ax_m\| \le C \|x_n - x_m\| \xrightarrow{n,m\to\infty} 0$$

so (Ax_n) is a Cauchy sequence in X_1, hence $Ax_n \to y$ for some $y \in X_1$. Now, $(x_n, Ax_n) \to (x, y)$ implies $(x, y) \in B$, which concludes that $\mathrm{dom}(B) = X_0$.

Finally, for $(x, y) \in B$, we have $(x_n, y_n) \in A$ that converges to (x, y), so

$$\|Bx\| = \|y\| = \left\|\lim_{n\to\infty} y_n\right\| = \left\|\lim_{n\to\infty} Ax_n\right\| = \lim_{n\to\infty} \|Ax_n\| \le \lim_{n\to\infty} C \|x_n\| = C \|x\|$$

so $\|B\| \le C$. Therefore, $B \in L(X_0, X_1)$.

Moreover, by construction, $B \supseteq A$, so $\|B\| \ge C$. Therefore, we actually have

$$\|B\| = C = \sup_{x\in\mathrm{dom}(A)\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

2. *Uniqueness.* Assume we have $B' \in L(X_0, X_1)$ such that $B' \supseteq A$. We only need to show $B' \equiv \bar{A}$. Indeed, since $\mathrm{dom}(A)$ is dense in X_0, it is a core for B' by Proposition 2.1.4. Therefore,

$$\begin{aligned}B' &= \overline{B'} \quad (B' \text{ is closed by Lemma 2.1.2}) \\ &= \overline{B' \cap (\mathrm{dom}(A) \times X_1)} \quad (\text{definition of core}) \\ &= \bar{A} \quad (\text{since } B' \supseteq A).\end{aligned}$$

□

Exercise 2.3

Let $A \subseteq H_0 \times H_1$ be a relation. Then A^* is a linear relation. Moreover, we have

$$A^* = -\left(\left(A^\perp\right)^{-1}\right) = \left((-A)^{-1}\right)^\perp = \left(-\left(A^{-1}\right)\right)^\perp = \left((-A)^\perp\right)^{-1} = \left(-\left(A\right)^\perp\right)^{-1}.$$

Proof. We first show that

$$A^* = \{(u,v) \in H_1 \times H_0;\, \forall (x,y) \in A : \langle u, y\rangle = \langle v, x\rangle\}$$

as in Remark 2.2.1. From the definition we have $A^* := -\left(\left(A^{-1}\right)^\perp\right) \subseteq H_1 \times H_0$. Moreover, one has

$$\begin{aligned}
A^{-1} &= \{(y,x) \in H_1 \times H_0; \forall (x,y) \in A\}, \\
\left(A^{-1}\right)^\perp &= \{(h_1,h_0) \in H_1 \times H_0; ((h_1,h_0),(y,x)) = \langle h_1, y\rangle + \langle h_0, x\rangle = 0,\ \forall (x,y) \in A\}, \\
-\left(\left(A^{-1}\right)^\perp\right) &= \{(h_1,-h_0) \in H_1 \times H_0; ((h_1,h_0),(y,x)) = \langle h_1, y\rangle + \langle h_0, x\rangle = 0,\ \forall (x,y) \in A\} \\
&= \{(h_1,h_0) \in H_1 \times H_0; \langle h_1, y\rangle = \langle h_0, x\rangle,\ \forall (x,y) \in A\},
\end{aligned}$$

which yields the above statement.

We then prove that A^* is a linear relation. Of course, one has A^* is a relation in $H_1 \times H_0$ by its definition. In order to prove that in fact A^* is a linear relation, let (h_1,h_0), (g_1,g_0) in A^* and $\lambda \in \mathbb{K}$, we aim to show that $(h_1 + \lambda g_1, h_0 + \lambda g_0) \in A^*$. That means we need to prove

$$\langle h_1 + \lambda g_1, y\rangle = \langle h_0 + \lambda g_0, x\rangle \quad \forall (x,y) \in A.$$

In fact, the last equality follows directly from (h_1,h_0), (g_1,g_0) in A^*. Thus A^* is a linear relation.

Finally, the result follows by step by step:

1. Proof of $A^* = -\left(\left(A^\perp\right)^{-1}\right)$.

We have

$$\begin{aligned}
A^\perp &= \{(h_0,h_1) \in H_0 \times H_1; \langle h_0, x\rangle + \langle h_1, y\rangle = 0,\ \forall (x,y) \in A\}, \\
\left(A^\perp\right)^{-1} &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle + \langle h_1, y\rangle = 0,\ \forall (x,y) \in A\}, \\
-\left(\left(A^\perp\right)^{-1}\right) &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle,\ \forall (x,y) \in A\} = A^*.
\end{aligned}$$

2. Proof of $A^* = \left((-A)^{-1}\right)^\perp$.

We have

$$\begin{aligned} -A &= \{(x,-y) \in H_0 \times H_1; \forall (x,y) \in A\}, \\ (-A)^{-1} &= \{(-y,x) \in H_1 \times H_0; \forall (x,y) \in A\}, \\ \left((-A)^{-1}\right)^{\perp} &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in A\} = A^*. \end{aligned}$$

3. Proof of $A^* = \left(-\left(A^{-1}\right)\right)^{\perp}$.

We have

$$\begin{aligned} A^{-1} &= \{(y,x) \in H_1 \times H_0; \forall (x,y) \in A\}, \\ -A^{-1} &= \{(y,-x) \in H_1 \times H_0; \forall (x,y) \in A\}, \\ \left(-\left(A^{-1}\right)\right)^{\perp} &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in A\} = A^*. \end{aligned}$$

4. Proof of $A^* = \left((-A)^{\perp}\right)^{-1}$.

We have

$$\begin{aligned} -A &= \{(x,-y) \in H_0 \times H_1; \forall (x,y) \in A\}, \\ (-A)^{\perp} &= \{(h_0,h_1) \in H_0 \times H_1; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in A\}, \\ \left((-A)^{\perp}\right)^{-1} &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in A\} = A^*. \end{aligned}$$

5. Proof of $A^* = \left(-\left(A^{\perp}\right)\right)^{-1}$.

We have

$$\begin{aligned} A^{\perp} &= \{(h_0,h_1) \in H_0 \times H_1; \langle h_0, x\rangle + \langle h_1, y\rangle = 0, \forall (x,y) \in A\}, \\ -(A)^{\perp} &= \{(h_0,-h_1) \in H_0 \times H_1; \langle h_0, x\rangle + \langle h_1, y\rangle = 0, \forall (x,y) \in A\}, \\ \left(-\left(A^{\perp}\right)\right)^{-1} &= \{(-h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle + \langle h_1, y\rangle = 0, \forall (x,y) \in A\} \\ &= \{(h_1,h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in A\} = A^*. \end{aligned}$$

□

Exercise 2.4

Let $A\colon \operatorname{dom}(A) \subseteq H_0 \to H_0$ be a closed and densely defined linear operator. Show that for all $\lambda \in \mathbb{K}$ we have

$$\lambda \in \rho(A) \Longleftrightarrow \lambda^* \in \rho(A^*).$$

Proof. 1. Since A is densely defined, A^* is a linear operator by Lemma 2.2.7. Moreover, A^* is also closed by definition of $*$. Thus we can talk about the spectrum of A^*.[1]

2. For any $\lambda \in \mathbb{K}$, we use Lemma 2.2.2 to compute

$$\begin{aligned}\left((\lambda - A)^{-1}\right)^* &= \left(-\left(\left((\lambda - A)^{-1}\right)^{-1}\right)\right)^{\perp} \\ &= (-(\lambda - A))^{\perp} \\ &= \left(\left((-(\lambda - A))^{\perp}\right)^{-1}\right)^{-1} \\ &= ((\lambda - A)^*)^{-1} \\ &= (\lambda^* - A^*)^{-1}, \end{aligned} \tag{1}$$

where the last identity follows from Theorem 2.3.2.

3. Now, assume $\lambda \in \rho(A)$, then $(\lambda - A)^{-1} \in L(H_0)$. By Lemma 2.1.2, $(\lambda - A)^{-1}$ is closed. Moreover, $(\lambda - A)^{-1}$ is densely defined on H_0 (in fact, its domain is H_0), so by Lemma 2.2.7 and (1), $(\lambda^* - A^*)^{-1}$ is a linear operator. Therefore, its operator norm is well-defined and

$$\begin{aligned}\left\|(\lambda^* - A^*)^{-1}\right\| &= \left\|\left((\lambda - A)^{-1}\right)^*\right\| && \text{(by (1))} \\ &= \left\|\overline{(\lambda - A)^{-1}}\right\| && \text{(by Lemma 2.2.8)} \\ &= \left\|(\lambda - A)^{-1}\right\| && \text{(since } (\lambda - A)^{-1} \text{ is closed)} \\ &< \infty. \end{aligned}$$

Hence, $(\lambda^* - A^*)^{-1} \in L(H_0)$, or $\lambda^* \in \rho(A^*)$.

4. Now, assume $\lambda^* \in \rho(A^*)$. Apply above result

$$\lambda = \lambda^{**} \in \rho(A^{**}) = \rho(\bar{A}) = \rho(A)$$

since A is closed. □

Exercise 2.5

Let $U \subseteq H_0 \times H_1$ satisfy $U^{-1} = U^*$. Show that $U \in L(H_0, H_1)$ and that U is *unitary*, that is, U is onto and for all $x \in H_0$ we have $\|Ux\|_{H_1} = \|x\|_{H_0}$.

Proof. 1. Let $(x, y_1), (x, y_2) \in U$, we prove $y_1 = y_2$ (i.e, U is a "function"). We have $(y_1, x), (y_2, x) \in U^{-1} \equiv U^*$, so by Remark 2.2.1,

$$\langle y_1, y_1\rangle_{H_1} = \langle y_1, y_2\rangle_{H_1} = \langle y_2, y_1\rangle_{H_1} = \langle y_2, y_2\rangle_{H_1} = \langle x, x\rangle_{H_0}. \tag{2}$$

[1] In fact, if a operator is not closed, its "spectrum" is trivial, i.e, equal to $\mathbb{K}$.

Therefore,

$$\begin{aligned}\|y_1 - y_2\|_{H_1}^2 &= \langle y_1 - y_2, y_1 - y_2\rangle_{H_1} \\ &= \langle y_1, y_1\rangle_{H_1} - \langle y_1, y_2\rangle_{H_1} - \langle y_2, y_1\rangle_{H_1} + \langle y_2, y_2\rangle_{H_1} \\ &= 0 \quad \text{(by (2))}.\end{aligned}$$

So $y_1 \equiv y_2$. Hence, Ux is well-defined (single value) for $x \in H_0$. Moreover, (2) also gives

$$\|Ux\|_{H_1}^2 = \|y_1\|^2 = \|x\|_{H_0}^2$$

or

$$\|Ux\|_{H_1} = \|x\|_{H_0}. \tag{3}$$

2. We now prove U is linear. For $x, y \in H_0$, $\lambda \in \mathbb{K}$, put $a = U(x + \lambda y),\ b = Ux + \lambda Uy$. Observe that

$$\langle Uw, Uz\rangle = \langle U^*Uw, z\rangle = \left\langle U^{-1}Uw, z\right\rangle = \langle w, z\rangle \quad \forall w, z \in H_0,$$

we have

$$\langle a, a\rangle = \langle a, b\rangle = \langle b, a\rangle = \langle b, b\rangle = \|x + \lambda y\|^2.$$

Thus

$$\|a - b\|^2 = \|a\|^2 + \|b\|^2 - \langle a, b\rangle - \langle b, a\rangle = 0.$$

Hence U is linear. From (3), we get $U \in L(H_0, H_1)$.

3. Now U is onto by scaling. Indeed, (3) implies that U maps closed set to closed set. Hence, U is open map, so we have $r > 0$ such that $B_{H_1}(0, r) \subset U\left(B_{H_0}(0, 1)\right)$. If $y = 0$, then $U(0) = 0$ by linearity. For $y \in H_1 \setminus \{0\}$, we have $\frac{y}{2\|y\|} r \in B_{H_1}(0, r)$, so there exists $x \in B_{H_0}(0, 1)$ such that

$$Ux = \frac{y}{2\|y\|} r \qquad \text{or} \qquad U\left(\frac{2\|y\| x}{r}\right) = y.$$

□

Exercise 2.6

Let $\delta : C([0, 1]) \subseteq L_2((0, 1)) \to \mathbb{K}$, $f \mapsto \delta(f) = f(0)$, where $C([0, 1])$ denotes the set of $\mathbb{K}$-valued functions on $[0, 1]$. Show that δ is not closable. Compute $\bar{\delta}$.

First proof. Let $H_0 = L_2((0, 1))$ and $H_1 = \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), which are Hilbert spaces with the usual inner products given (g, h) and $a \cdot b = ab$, respectively, for all $g, h \in H_0$ and $a, b \in \mathbb{K}$.

Firstly, we shall prove that δ is a linear operator from H_0 to H_1. Of course, δ is a relation in $H_0 \times H_1$. In order to show that δ is linear, let (x, y), (u, v) in δ and $\lambda \in \mathbb{K}$, we need to

prove that $(x+\lambda u, y+\lambda v) \in \delta$. That means does $\delta(x+\lambda u) = y + \lambda v$? In fact, the result follows by $\delta(x+\lambda u) = (x+\lambda u)(0) = x(0) + \lambda u(0) = y + \lambda v$ as both (x,y) and (u,v) in δ. It implies that δ is a linear in $H_0 \times H_1$. Let $M = \{\mathbf{0}\} \subset H_0$, we consider

$$\delta[M] = \{h_1 \in H_1; (\mathbf{0}, h_1) \in \delta\} = \{h_1 \in H_1; \delta(\mathbf{0}) = \mathbf{0}(0) = 0 = h_1) \in \delta\} = \{0\}.$$

Thus, δ is a linear operator from H_0 to H_1.

Secondly, we will show that δ is not closable. Since δ is linear relation in $H_0 \times H_1$ as above argument. Therefore, by Lemma 2.2.7, it is enough to prove that $\delta^* \subseteq H_1 \times H_0$ is not densely defined. As in Exercise 2.3 above, one has

$$\begin{aligned}
\delta^* &= \{(h_1, h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, y\rangle, \forall (x,y) \in \delta\} \\
&= \{(h_1, h_0) \in H_1 \times H_0; \langle h_0, x\rangle = \langle h_1, \delta(x)\rangle, \forall x \in C([0,1])\} \\
&= \{(h_1, h_0) \in H_1 \times H_0; (h_0, x) = x(0)h_1, \forall x \in C([0,1])\}.
\end{aligned}$$

In particular, since $C([0,1])$ is dense in H_0 and δ is linear then δ^* is a linear operator, see Lemma 2.2.7 again. In fact, it can be seen also by Lemma 2.2.2 (Exercise 2.3 above), δ^* is a linear relation. Moreover, the domain of δ^*, $\mathrm{dom}(\delta^*)$, is given by:

$$\begin{aligned}
\mathrm{dom}(\delta^*) &= \{k \in H_1; \exists \ell \in H_0 : (k, \ell) \in \delta^*\} \quad (\text{or } \ell = \delta^*(k)) \\
&= \{k \in H_1; \exists \ell \in H_0 : kx(0) = (\ell, x), \forall x \in C([0,1])\}.
\end{aligned}$$

So if $k \in \mathrm{dom}(\delta^*)$, we have $\ell \in H_0$ such that

$$kx(0) = (\ell, x), \forall x \in C([0,1]). \tag{4}$$

Choose $x \in C_c^\infty([0,1])$, (4) becomes

$$0 = (\ell, x), \forall x \in C_c^\infty([0,1]) \tag{5}$$

so $\ell \equiv 0$ by du Bois-Reymond lemma. Take into account with (4), we have

$$kx(0) = (0, x) = 0, \forall x \in C([0,1])$$

and hence $k = 0$. Therefore, $\mathrm{dom}(\delta^*) = \{0\}$, so it is not densely defined in $H_1 = \mathbb{K}$.

Thirdly, we will compute $\bar{\delta}$. We already showed that $\mathrm{dom}(\delta^*) = 0$ and δ^* is a linear operator, so $\delta^* = \{(0,0)\}$. Since δ is a linear relation, by using Lemma 2.2.4 and Lemma 2.2.2, we know that $\bar{\delta} = \delta^{**} = (\delta^*)^* = \{(0,0)\}^* = \left(\left(-\{(0,0)\}\right)^{-1}\right)^\perp = H_0 \times H_1$. □

Second proof. 1. We claim that $\bar{\delta} = H_0 \times H_1 \equiv L_2[0,1] \times \mathbb{K}$, where the product space is equipped with the usual norm

$$\|(f,k)\|^2 := \|f\|_{L_2}^2 + |k|^2, \qquad f \in L_2[0,1], k \in \mathbb{K}.$$

To prove this claim, take $(f,k) \in L_2((0,1)) \times \mathbb{K}$ and we show that it can be approximated by $(f_n, f_n(0))_{n=1}^\infty$ where $f_n \in C([0,1])$.

2. For each $n \in \mathbb{N}$, put $I_n = \left[\frac{1}{n}, 1\right]$ and $J_n = \left[0, \frac{1}{n}\right]$. Since $C_c^\infty(I_n)$ is dense in $L_2(I_n)$, we have $\phi_n \in C_c^\infty(I_n)$ so that

$$\|f\chi_{I_n} - \phi_n\|_{L_2(I_n)} \leq \frac{1}{n}. \tag{6}$$

Define

$$f_n(x) = \begin{cases} \phi_n(x), & \text{on } I_n \\ -nkx + k, & \text{on } J_n \end{cases}$$

then $f_n \in C([0,1])$ (since $\phi_n\left(\frac{1}{n}\right) = 0 = -nk\frac{1}{n} + k$) and $f_n(0) = k$.

3. We prove $f_n \to f$ in $L_2((0,1))$. First, observe that

$$(f_n - f)\chi_{J_n} \xrightarrow{n\to\infty} 0 \text{ a.e on } [0,1] \qquad \text{and} \tag{7}$$

$$|f_n\chi_{J_n}| \leq \begin{cases} |f\chi_{J_n}| + |k| & \text{a.e on } J_n \\ 0 & \text{on } I_n \end{cases} \leq |f| + |k| \text{ a.e on } [0,1]. \tag{8}$$

Then

$$\begin{aligned} \|f_n - f\|^2_{L_2((0,1))} &= \int_0^1 |f_n - f|^2\,dx = \int_{I_n} |f_n - f|^2\,dx + \int_{J_n} |f_n - f|^2\,dx \\ &\leq \frac{1}{n^2} + \int_{J_n} |f_n - f|^2\,dx \qquad \text{(by (6))} \\ &= \frac{1}{n^2} + \int_0^1 |(f_n - f)\chi_{J_n}|^2\,dx \\ &\xrightarrow{n\to\infty} 0. \qquad \text{(by (7) - (8) and Lebesgue Dominated Convergence Theorem)} \end{aligned}$$

Therefore, $\bar{\delta} = L_2[0,1] \times \mathbb{K}$.

4. From definition, δ is not closable if $\bar{\delta}$ is not a linear operator. Since $(0,1) \in \bar{\delta} = L_2((0,1)) \times \mathbb{K}$, we conclude that δ is not closable. □

Exercise 2.7

Let $C \subseteq \mathbb{C}$ be closed. Provide a Hilbert space H and a densely defined closed linear operator A on H such that $\sigma(A) = C$.

Proof. 1. Since each Hilbert space is isomorphic to a ℓ_2 space, we find our example among those ℓ_2's. Moreover, by Proposition 2.4.1, $\sigma(A)$ is closed, so we only need to construct A whose eigenvalues form a dense subset of C. Choose $H = \ell_2(\mathbb{N})$, let $(\lambda_n)_{n=1}^\infty$ be a dense subset of C. Put $A\colon \operatorname{dom}(A) \subset \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$, $(x_n) \mapsto (\lambda_n x_n)$, where

$$\operatorname{dom}(A) := \{(x_n) \in \ell_2(\mathbb{N})\colon (\lambda_n x_n) \in \ell_2(\mathbb{N})\}.$$

Of course, $\operatorname{dom}(A) \neq \varnothing$ since it contains the standard basis of $\ell_2(\mathbb{N})$. Moreover, $\operatorname{dom}(A)$ is a subspace of $\ell_2(\mathbb{N})$ thanks to Minkowski inequality. Then it's obvious to see that A is linear.

2. Compute directly

$$\langle (x_n), A(y_n)\rangle = \langle (x_n), (\lambda_n y_n)\rangle = \langle (\lambda_n^* x_n), (y_n)\rangle$$

so A^* is a linear operator with $\mathrm{dom}(A^*) = \mathrm{dom}(A)$, $A^*(x_n) = (\lambda_n^* x_n)$. By Lemma 2.2.7, A is densely defined. Similarly, A^{**} is also a linear operator with $\mathrm{dom}(A^{**}) = \mathrm{dom}(A^*) = \mathrm{dom}(A)$, moreover,

$$A^{**}(x_n) = A^*(\lambda_n^* x_n) = (\lambda_n^{**} x_n) = (\lambda_n x_n) = A(x_n).$$

thus $A = A^{**}$ is closed by definition of $*$.

3. Now, each λ_n is an eigenvalue of A since $(\lambda_n - A)e_n = 0$. Therefore, $\sigma(A) \supset \overline{(\lambda_n)_{n=1}^{\infty}} = C$ since $\sigma(A)$ is closed. Finally, if $\lambda \notin C$, then

$$(\lambda - A)^{-1}(y_n) = \left(\frac{y_n}{\lambda - \lambda_n}\right) \in \ell_2(\mathbb{N})$$

since $\mathrm{dist}(\lambda, C) > 0$. Therefore, $\sigma(A) = C$. □