Exercise 1.1

Let $\phi \in C(\mathbb{R}, \mathbb{R})$. Assume that $\phi(t+s) = \phi(t)\phi(s)$ for all $s, t \in \mathbb{R}$, $\phi(0) = 1$. Show that $\phi(t) = \mathrm{e}^{\alpha t}$ $(t \in \mathbb{R})$ for some $\alpha \in \mathbb{R}$.

Solution. 1. We claim $\phi > 0$. First, $\phi(s) = \phi\left(\frac{s}{2} + \frac{s}{2}\right) = \phi\left(\frac{s}{2}\right)^2 \geq 0$ for $s \in \mathbb{R}$. Now for $s, t \in \mathbb{R}$,

$$\phi(2s) = \phi(s)^2 \quad \text{so} \quad \phi(t)^{\frac{1}{2}} = \phi\left(\frac{t}{2}\right).$$

Assume that there exists $s \in \mathbb{R}$ such that $\phi(s) = 0$, then

$$0 = \phi(s)^{\frac{1}{2^n}} = \phi\left(\frac{s}{2^n}\right) \xrightarrow{n \to \infty} \phi(0) = 1,$$

which is a contradiction.

2. Now put $u(s) = \ln \phi(s)$ then u is continuous and

$$u(s+t) = u(s) + u(t)$$

Thus for any $n \in \mathbb{Z}$, this equation implies $u(ns) = nu(s)$ or $u(\frac{t}{n}) = \frac{1}{n}u(t)$. Note that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, we always have a strictly monotone sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ such that $r = \sum_{k=1}^{\infty} \frac{1}{n_k}$ for any $r \in \mathbb{R}$. Therefore,

$$u(rs) = u\left(\lim_{N\to\infty} s \sum_{k=1}^{N} \frac{1}{n_k}\right) = \lim_{N\to\infty} u\left(\sum_{k=1}^{N} \frac{s}{n_k}\right) = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{n_k} u(s) = ru(s).$$

Choose $s = 1$, we get $u(r) = ru(1)$. Since r is arbitrary, we conclude that $u(t) = u(1)t$ for all $t \in \mathbb{R}$. Put $\alpha = u(1)$, we get $\phi(t) = \mathrm{e}^{\alpha t}$ for all $t \in \mathbb{R}$. □

Exercise 1.2

Let $n \in \mathbb{N}$, $T\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ continuously differentiable such that $T(s+t) = T(s)T(t)$ for all $t, s \in \mathbb{R}$, $T(0) = I$. Show that there exists $A \in \mathbb{R}^{n\times n}$ with the property that $T(t) = \mathrm{e}^{tA}$, $(t \in \mathbb{R})$.

Solution. 1. Fix $t \in \mathbb{R}$. Put $A = T'(0)$, then

$$\begin{aligned} T(t+h) - T(t) &= T(t)T(h) - T(t)T(0) \quad \text{(by semigroup law)} \\ &= T(t)\left(T(h) - T(0)\right) \\ &= T(t)\left(Ah + \varphi(h)\right) \quad (T \text{ is differentiable}) \end{aligned}$$

where $\frac{\varphi(h)}{h} \to 0$ as $h \to 0$.

2. Divide both sides by h and let $h \to 0$, we get

$$T'(t) = AT(t)$$

Together with $T(0) = I$, we conclude that $T(t) = \mathrm{e}^{tA}$ for all $t \in \mathbb{R}$. □

Exercise 1.3

Show that $x \mapsto u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\ dy$ satisfies Poisson's equation, given $f \in C_c^\infty(\mathbb{R}^3)$.

Solution. 1. Write $u(x) = \int_{\mathbb{R}^3} \Gamma(x-y)f(y)\ dy = \int_{\mathbb{R}^3} \Gamma(y)f(x-y)$, where $\Gamma(x) := \frac{1}{4\pi|x|}$. Note that u is well-defined since f is bounded, compact support and $\int_{B_R(0)} \Gamma(x)\ dx < \infty$ for any $R > 0$. Fix $x \in \mathbb{R}^3$, by Lebesgue Dominated Convergence Theorem

$$\frac{u(x+te_i) - u(x)}{t} = \int_{\mathbb{R}^3} \Gamma(y) \frac{f(x-y+te_i) - f(x-y)}{t}\ dy \xrightarrow{t\to 0} \int_{\mathbb{R}^3} \Gamma(x) f_{x_i}(x-y)\ dy$$

Thus

$$u_{x_i} = \int_{\mathbb{R}^3} \Gamma(y) f_{x_i}(x-y)\ dy$$

Since $f \in C_c^\infty(\mathbb{R}^3)$, we can proceed similarly to get $u \in C^\infty(\mathbb{R}^3)$, in particular,

$$u_{x_i x_i} = \int_{\mathbb{R}^3} \Gamma(y) f_{x_i x_i}(x-y)\ dy.$$

Therefore,

$$\Delta u(x) = \int_{\mathbb{R}^3} \Gamma(y) \Delta_x f(x-y)\ dy.$$

Thus the problem is solved if we have

$$\int_{\mathbb{R}^3} \Gamma(y) \Delta_x f(x-y)\ dy = -f(x). \tag{1}$$

2. The idea is to isolate the singularity of Γ. Fix $\varepsilon > 0$,

$$\left| \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\ dy \right| \le C \left\| D^2 f \right\|_\infty \int_{B_\varepsilon(0)} \Gamma(y)\ dy \le C\varepsilon^2. \tag{2}$$

3. Integrating by parts twice,

$$\begin{aligned}
\int_{B_\varepsilon(0)^c} \Gamma(y)\Delta_x f(x-y)\ dy &= \int_{B_\varepsilon(0)^c} \Gamma(y)\Delta_y f(x-y)\ dy \\
&= -\int_{B_\varepsilon(0)^c} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\ dy + \int_{\partial B_\varepsilon(0)} \Gamma(y)\frac{\partial f}{\partial \nu}(x-y)\ d\sigma(y) \\
&= \left(\int_{B_\varepsilon(0)^c} \Delta\Gamma(y) f(x-y)\ dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x-y)\ d\sigma(y) \right) \\
&\qquad + \int_{\partial B_\varepsilon(0)} \Gamma(y)\frac{\partial f}{\partial \nu}(x-y)\ d\sigma(y) \\
&= -\int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x-y)\ d\sigma(y) + \int_{\partial B_\varepsilon(0)} \Gamma(y)\frac{\partial f}{\partial \nu}(x-y)\ d\sigma(y) \\
&\qquad \text{(since } \Delta\Gamma = 0 \text{ on } B_\varepsilon(0)^c\text{)} \\
&=: I_\varepsilon + J_\varepsilon
\end{aligned} \tag{3}$$

where ν is the inward unit vector on $\partial B_\varepsilon(0)$, σ is the surface measure.

4. Now, for $y \in \partial B_\varepsilon(0)$,

$$\frac{\partial \Gamma}{\partial \nu}(y) = \nabla\Gamma(y)\cdot\nu = \left(-\frac{1}{4\pi}\frac{y}{|y|^3}\right)\left(-\frac{y}{|y|}\right) = \frac{1}{4\pi|y|^2} = \frac{1}{4\pi\varepsilon^2} = \frac{1}{\sigma(\partial B_\varepsilon(0))}$$

Thus,

$$I_\varepsilon = -\frac{1}{\sigma(\partial B_\varepsilon(0))}\int_{\partial B_\varepsilon(0)} f(x-y)\ d\sigma(y) \xrightarrow{\varepsilon\to 0} -f(x) \tag{4}$$

5. Moreover,

$$|J_\varepsilon| \le \|\nabla f\|_\infty \int_{\partial B_\varepsilon(0)} |\Gamma(y)|\ \ d\sigma(y) \le C\varepsilon \tag{5}$$

From (2)–(5), (1) is proved by letting $\varepsilon \to 0$. □

Exercise 1.4

Let $f \in C_c^\infty(\mathbb{R})$. Define $u(t,x) \coloneqq f(x+t)$ for $x, t \in \mathbb{R}$. Show that u satisfies the differential equation $\partial_t u = \partial_x u$ and $u(0,x) = f(x)$ for all $x \in \mathbb{R}$.

Solution. Let $g(x,t) \coloneqq x + t$ then $u = f \circ g$. By chain rule

$$\partial_t u(x_0,t_0) = f'|_{g(x_0,t_0)}\partial_t g|_{(x_0,t_0)} = f'|_{g(x_0,t_0)} = f'|_{g(x_0,t_0)}\partial_x g|_{(x_0,t_0)} = \partial_x u(x_0,t_0).$$

□

Exercise 1.5

Let X, Y be Banach spaces, $(T_n)_{n\in\mathbb{N}}$ be a sequence in $L(X,Y)$, the set of bounded linear operators. If $\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty$, show that there is $x \in X$ and a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\|T_{n_k}(x)\| \to \infty$.

Solution. Suppose by contradiction that for all $x \in X$ and for all strictly increasing sequences $(n_k)_k$ in $\mathbb{N}$, we have $\|T_{n_k}x\|$ doesn't converge to ∞. That implies $\sup_{k\in\mathbb{N}}\|T_{n_k}(x)\| < M$, for some $0 < M < \infty$ (if not, one can extract a subsequence n_{k_j} such that $\left\|T_{n_{k_j}}(x)\right\| \to \infty$). In particular, for $n_k = k$, we have $\sup_{k\in\mathbb{N}}\|T_k(x)\| < \infty$. By Uniformly Boundedness Principle, we get $\sup_{k\in\mathbb{N}}\|T_k\| < \infty$ (contradiction).

□

Exercise 1.6

Let $n \in \mathbb{N}$. Denote by $\mathrm{GL}(n;\mathbb{K})$ the set of continuously invertible $n \times n$ matrices. Show that $\mathrm{GL}(n;\mathbb{K}) \subset \mathbb{K}^{n\times n}$ is open.

Solution. Note that determinant is a continuous function, $\mathrm{GL}(n;\mathbb{K}) = \det^{-1}(\mathbb{K} \setminus \{0\})$ is an open set, since $\mathbb{K} \setminus \{0\}$ is open. □

Exercise 1.7

Let $n \in \mathbb{N}$. Show that $\Phi\colon \mathrm{GL}(n;\mathbb{K}) \ni A \mapsto A^{-1} \in \mathbb{K}^{n\times n}$ is continuously differentiable. Compute Φ'.

Solution. 1. Fix $A \in \mathrm{GL}(n;\mathbb{K})$. For any $H \in \mathrm{GL}(n;\mathbb{K})$, Exercise 1.6 ensures that if $|t| > 0$ small enough, we have $A + tH \in \mathrm{GL}(n;\mathbb{K})$.

2. Now,

$$\begin{aligned}
\Phi(A+tH) - \Phi(A) &= (A+tH)^{-1} - A^{-1} \\
&= \left(A(I + tA^{-1}H)\right)^{-1} - A^{-1} \\
&= (I - tB)^{-1}A^{-1} - A^{-1} \quad (\text{where } B := -A^{-1}H) \\
&= \left[(I-tB)^{-1} - I\right] A^{-1}
\end{aligned} \tag{6}$$

3. Choose $0 \le |t| \le \frac{1}{2}\min\left\{1, \|B\|^{-1}\right\}$ then $\|tB\| < 1$, and we have

$$(I - tB)^{-1} = I + tB + (tB)^2 + \cdots + (tB)^n + \cdots$$

where the right hand side converges uniformly in t. Thanks to the uniform convergence, we can rearrange

$$\begin{aligned}
(I-tB)^{-1} - I &= tB + (tB)^2 + \cdots + (tB)^n + \cdots \\
&= tB\left[I + tB + (tB)^2 + \cdots + (tB)^n + \cdots\right] \\
&= tB(I-tB)^{-1}
\end{aligned} \tag{7}$$

Note that $\|I - tB\| \ge \|I\| - \|tB\| \ge 1 - \frac{1}{2} = \frac{1}{2}$, (7) gives

$$\left\|(I-tB)^{-1} - I\right\| = |t|\left\|B(I-tB)^{-1}\right\| \le \frac{1}{2}t\|B\| \quad \xrightarrow{t\to 0} \quad 0 \tag{8}$$

4. Substitute (7) into (6), and divide both sides by t,

$$\frac{\Phi(A+tH) - \Phi(A)}{t} = B(I-tB)^{-1}A^{-1} \quad \xrightarrow{t\to 0} \quad BA^{-1} = -A^{-1}HA^{-1} \tag{9}$$

here we use the limit (8).

5. Put

$$\begin{aligned} D\Phi \colon \mathrm{GL}(n;\mathbb{K}) &\longrightarrow L(\mathrm{GL}(n;\mathbb{K})) \\ A &\longmapsto \begin{pmatrix} D\Phi(A)\colon & \mathrm{GL}(n;\mathbb{K}) & \longrightarrow & \mathrm{GL}(n;\mathbb{K}) \\ & H & \longmapsto & -A^{-1}HA^{-1} \end{pmatrix} \end{aligned}$$

The fact that $D\Phi(A) \in L(\mathrm{GL}(n;\mathbb{K}))$ (i.e, it is a bounded linear operator on $\mathrm{GL}(n;\mathbb{K})$) follows directly from $D\Phi(A)(H) = -A^{-1}HA^{-1}$.

Now we prove $D\Phi \colon \mathrm{GL}(n;\mathbb{K}) \to L(\mathrm{GL}(n;\mathbb{K}))$ is continuous. For $A, B, H \in \mathrm{GL}(n;\mathbb{K})$,

$$\begin{aligned} \left\|-A^{-1}HA + B^{-1}HB^{-1}\right\| &\leq \left\|(A^{-1} - B^{-1})H(-A^{-1} - B^{-1})\right\| + \left\|A^{-1}H(B^{-1} - A^{-1})\right\| \\ &\quad + \left\|(A^{-1} - B^{-1})HA^{-1}\right\| \\ &\leq \|H\| \left\|A^{-1} - B^{-1}\right\| \left(\left\|A^{-1} + B^{-1}\right\| + 2\left\|A^{-1}\right\|\right) \end{aligned}$$

Thus,

$$\begin{aligned} \|D\Phi(A) - D\Phi(B)\| &= \sup_{\|H\|=1} \|D\Phi(A)(H) - D\Phi(B)(H)\| \\ &= \sup_{\|H\|=1} \left\|-A^{-1}HA + B^{-1}HB^{-1}\right\| \\ &\leq \left\|A^{-1} - B^{-1}\right\| \left(\left\|A^{-1} + B^{-1}\right\| + 2\left\|A^{-1}\right\|\right) \quad \xrightarrow{B \to A} 0 \end{aligned}$$

Therefore, Φ is continuous differentiable and $\Phi'(A)(H) = -A^{-1}HA^{-1}$ for $A, H \in \mathrm{GL}(n;\mathbb{K})$.

$\square$