Exercise 14.1

Let V, H be separable Hilbert spaces such that $V \overset{d}{\hookrightarrow} H$, $\dim H = \infty$. Assume that the injection of V into H is compact. Let $a\colon V \times V \to \mathbb{K}$ be continuous, coercive and symmetric.
(a) Show that there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$ such that

$$V = \Big\{u \in H;\ \sum_{n=1}^{\infty} \lambda_n |(u \mid e_n)_H|^2 < \infty\Big\}$$

and $a(u,v) = \sum_{n=1}^{\infty} \lambda_n (u \mid e_n)_H (e_n \mid v)_H$ for all $u, v \in V$.
Hint: Use Theorem 6.17 and Corollary 6.18.
(b) Show that V^* can be identified with the space $\{(y_n) \in \mathbb{K}^{\mathbb{N}};\ \sum_{n=1}^{\infty} \lambda_n^{-1} |y_n|^2 < \infty\}$, with the canonical mapping $H \hookrightarrow V^*$ given by $u \mapsto ((u \mid e_n)_H)_{n\in\mathbb{N}}$.
(c) Show that the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ from Section 14.1, considered as an operator in V^*, is given by

$$\operatorname{dom}(\mathcal{A}) = \{y \in V^*;\ (\lambda_n y_n)_{n\in\mathbb{N}} \in V^*\},$$
$$\mathcal{A}y = (\lambda_n y_n)_{n\in\mathbb{N}} \quad (y \in \operatorname{dom}(\mathcal{A})).$$

Show that $-\mathcal{A}$ generates a C_0-semigroup on V^*, given by

$$e^{-t\mathcal{A}} y = (e^{-t\lambda_n} y_n)_{n\in\mathbb{N}}.$$

(a) Let A be the operator associated with a. Then by Corollary 6.18, A is a positive self-adjoint operator with compact resolvent. Hence Theorem 6.17 applies and thus there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\lim\limits_{n\to\infty} \lambda_n = \infty$ such that

$$\operatorname{dom}(A) = \left\{u \in H; \sum_{n=1}^{\infty} \lambda_n^2 |(u|e_n)_H|^2 < \infty\right\},$$
$$Au = \sum_{n=1}^{\infty} \lambda_n (u|e_n)_H e_n \quad (u \in \operatorname{dom}(A)).$$

Let

$$W := \left\{u \in H; \sum_{n=1}^{\infty} \lambda_n |(u|e_n)_H|^2 < \infty\right\}$$

and $b(u,v) = \sum\limits_{n=1}^{\infty} \lambda_n (u|e_n)_H (e_n|v)_H$ for all $u, v \in W$ as in Example 6.16. Clearly, both a and b are densely defined embedded closed sectorial forms and A is associated with both a and b. But Corollary 12.6 implies that this form is unique, hence $a = b$ and $V = W$.

(b) We can identify H with $\ell^2(\mathbb{N})$ using $u \mapsto ((u|e_n)_H)_{n\in\mathbb{N}}$. Similarly we can identify V with an ℓ^2-space weighted by $(\lambda_n)_{n\in\mathbb{N}}$. Taking the counting measure and $m = (\lambda_n)_{n\in\mathbb{N}}$, we can apply Example 14.1 to get the assertion.
(c) Clearly, $(\lambda_n y_n)_{n\in\mathbb{N}} \in V^*$ if and only if $(y_n)_{n\in\mathbb{N}} \in V$. Hence, applying Example 14.3, we get

$$\begin{aligned} \operatorname{dom}(\mathcal{A}) = V = \{y \in V^*; (\lambda_n y_n)_{n\in\mathbb{N}} \in V^*\}\,, \\ \mathcal{A}y = (\lambda_n y_n)_{n\in\mathbb{N}} \quad (y \in \operatorname{dom}(\mathcal{A})). \end{aligned}$$

For the second assertion we can apply Exercise 1.3 with $a = (-\lambda_n)_{n\in\mathbb{N}}$.

Exercise 14.2

Prove Lemma 14.4.
Let $f \in L_1\,(a, b; H)$ and define

$$T_f \colon H \to \mathbb{K},\ T_f(v) := \int_a^b (f(t)|v)_H\, dt.$$

The mapping T_f is antilinear and continuous since

$$|T_f(v)| \leq \int_a^b |(f(t)|v)_H| dt \leq \|v\| \int_a^b \|f(t)\| dt = \|v\|\|f\|_{L_1(a,b;H)} \tag{1}$$

holds for every $v \in H$ by Cauchy-Schwarz, implying $T_f \in H^*$. Applying the theorem of Riesz-Fréchet to the mapping $\overline{T_f} \in H'$ defined by $\overline{T_f}(v) := \overline{T_f(v)}$, we obtain that there is an unique $w_f \in H$ such that

$$\int_a^b (f(t)|v)_H\, dt = T_f(v) = \overline{\overline{T_f(v)}} = \overline{(v|w_f)_H} = (w_f|v)_H \quad \text{and} \quad \|T_f\|_{H^*} = \|w_f\| \tag{2}$$

holds for every $v \in H$. Now we define the mapping

$$T \colon L_1\,(a, b; H) \to H,\ T(f) := \int_a^b f(t) dt := w_f.$$

This mapping is linear and continuous since

$$\|T(f)\| = \|w_f\| \underset{(2)}{=} \|T_f\|_{H^*} = \sup_{\|v\|\leq 1} |T_f(v)| \underset{(1)}{\leq} \|f\|_{L_1(a,b;H)}$$

is valid for every $f \in L_1\,(a, b; H)$.

Exercise 14.3

(a) Let $w \in H$, $\phi \in C_c^\infty(a,b)$. Then Lemma 14.4 gives:

$$0 = \left(\int_a^b \phi(t)u(t)\,\mathrm{d}t \,\middle|\, w \right)_H = \int_a^b (\phi(t)u(t) \,|\, w)_H \,\mathrm{d}t = \int_a^b \phi(t)(u(t) \,|\, w)_H \,\mathrm{d}t,$$

where $(u(\cdot) \,|\, w)_H \in L_1(a,b)$ is a scalar function. Using Lemma 4.5 we get that

$$(u(\cdot) \,|\, w)_H = 0 \text{ a.e. on } (a,b). \tag{1}$$

Thus, there exists a measurable set $\Omega_w \subseteq (a,b)$, such that

$$\int_{\Omega_w} 1\,\mathrm{d}t = 0 \text{ and } (u(\cdot) \,|\, w)_H|_{(a,b)\setminus\Omega_w} = 0.$$

Since H is separable, there exists a countable set $\{x_n \mid n \in \mathbb{N}\} \subseteq H$ such that for any $x \in H$ we have

$$\|x\|_H = \sup_{n\in\mathbb{N}} |(x \,|\, x_n)_H|.$$

The countable union of measurable subsets is measurable again. And if the measure of each of those sets equals zero, then the same holds for the countable union. Therefore, $\Omega := \bigcup_{n\in\mathbb{N}} \Omega_{x_n}$ is a measurable set such that

$$\int_{\Omega} 1\,\mathrm{d}t = 0 \text{ and } (u(\cdot) \,|\, x_n)_H|_{(a,b)\setminus\Omega} = 0 \text{ for all } n \in \mathbb{N}.$$

This in turn means that $\|u(\cdot)\|_H = 0$ a.e. on (a,b), hence $u = 0$ a.e. on (a,b).

In particular, this shows, that the weak derivative defined as in section 14.3, is unique, if it exists:
Let $u \in L_1(a,b;H)$ and suppose that there exist weak derivatives $v_1, v_2 \in L_1(a,b;H)$ of u. Then the definition yields that for any $\phi \in C_c^\infty(a,b)$ we have

$$\int_a^b \Big(v_1(t) - v_2(t)\Big)\phi(t)\,\mathrm{d}t = \int_a^b \Big(u(t) - u(t)\Big)\phi'(t)\,\mathrm{d}t = 0,$$

and thusm $v_1 = v_2$ a.e.

(b) Consider the non-separable Hilbert space $\ell_2(0,1)$. Recall that $f\colon (0,1) \to \mathbb{K}$ is in $\ell_2(0,1)$ by definition if and only if

$$\exists\ S_f \subseteq (0,1) \text{ countable such that } f|_{(0,1)\setminus S_f} = 0, \text{ and } \sum_{s\in S_f} |f(s)|^2 < \infty.$$

The scalar product is given by

$$(f \,|\, g) = \sum_{s\in S_f\cup S_g} f(s)\overline{g(s)}.$$

For $t \in (0,1)$ we set $u(t) := \mathbf{1}_{\{t\}}$, where $\mathbf{1}_C$ denotes the indicator function for a set $C \subseteq (0,1)$:

$$\mathbf{1}_C(s) = \begin{cases} 1, & \text{if } s \in C, \\ 0, & \text{else,} \end{cases} \quad (s \in (0,1)).$$

Then, $u \in L_1(0,1;\ell_2(0,1))$ with $\|u(t)\|_{\ell_2(0,1)} = 1$ for every $t \in (0,1)$, while for any $g \in \ell_2(0,1)$

$$(u(\cdot) \,|\, g)|_{(0,1)\setminus S_g} = 0.$$

In other words: we have that $(u(\cdot) \,|\, g) = 0$ a.e. for all $g \in \ell_2(0,1)$, which shows that separability was a necessary condition in (a).

Exercise 14.4

Let V and H Hilbert spaces over the field $\mathbb{K}$ with $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let $\tau > 0$ and $a : [0,\tau] \times V \times V \to \mathbb{K}$ such that

(a) $\forall t \in [0,\tau] : a(t,\cdot,\cdot) : V \times V \to \mathbb{K}$ is sesquilinear,

(b) $\exists M \geq 0 \,\forall t \in [0,\tau], u,v \in V : |a(t,u,v)| \leq M\|u\|_V\|v\|_V$,

(c) $\exists \omega \geq 0, \alpha > 0 \,\forall t \in [0,\tau], u \in V : \operatorname{Re} a(t,u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2$,

(d) $\forall u,v \in V : a(\cdot,u,v) : [0,\tau] \to \mathbb{K}$ is measurable.

For $t \in [0,\tau]$ define the operator $\mathcal{A}(t) \in \mathcal{L}(V,V^*)$ by

$$\forall u,v \in V : \langle \mathcal{A}(t)u, v\rangle = a(t,u,v)$$

and define $\mathcal{A} : L_2(0,\tau;V) \to L_2(0,\tau;V^*)$ by

$$(\mathcal{A}u)(t) \coloneqq \mathcal{A}(t)u(t) \quad (t \in (0,\tau))$$

for $u \in L_2(0,\tau;V)$ (cp. Proposition 14.14).

Theorem. *Let $u_0 \in H$ and $f \in L_2(0,\tau;V^*)$. Then there exists a unique $u \in \mathrm{MR}(0,\tau) = H^1(0,\tau;V^*) \cap L_2(0,\tau;V)$ such that*

$$\begin{aligned} u' + \mathcal{A}u &= f, \\ u(0) &= u_0. \end{aligned} \tag{2}$$

Proof. We define the mapping $\widetilde{a} : [0,\tau] \times V \times V \to \mathbb{K}$ by

$$\widetilde{a}(t,u,v) \coloneqq a(t,u,v) + \omega(u \,|\, v)_H$$

for $t \in [0,\tau], u,v \in V$. Then, $\widetilde{a}$ satisfies the assumptions (i)-(iv) of Section 14.5. Indeed, $\widetilde{a}(t,\cdot,\cdot)$ is sesquilinear for each $t \in [0,\tau]$ and $\widetilde{a}(\cdot,u,v)$ is measurable for each $u,v \in V$, which shows (i) and (iv). Moreover,

$$\begin{aligned} |\widetilde{a}(t,u,v)| &\leq |a(t,u,v)| + \omega|(u \,|\, v)_H| \\ &\leq M\|u\|_V\|v\|_V + \omega\|u\|_H\|v\|_H \\ &\leq (M + \omega C)\|u\|_V\|v\|_V \end{aligned}$$

for each $t \in [0,\tau], u, v \in V$, where $C > 0$ is such that $\|x\|_H \leq C\|x\|_V$ for each $x \in V$ (recall $V \hookrightarrow H$), which is (ii). Finally, we have for each $t \in [0,\tau], u \in V$

$$\operatorname{Re}\widetilde{a}(t,u,u) = \operatorname{Re} a(t,u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2,$$

which gives (iii). For $t \in [0,\tau]$ we denote by $\widetilde{A}(t) \in \mathcal{L}(V, V^*)$ the operator associated with $\widetilde{a}(t,\cdot,\cdot)$ and set $\widetilde{\mathcal{A}} : L_2(0,\tau;V) \to L_2(0,\tau;V^*)$ with

$$(\widetilde{\mathcal{A}}u)(t) = \widetilde{\mathcal{A}}(t)u(t) \quad (t \in (0,\tau), u \in L_2(0,\tau;V)).$$

For $t \in [0,\tau], u, v \in V$ we have that

$$\langle \widetilde{\mathcal{A}}(t)u, v\rangle = \widetilde{a}(t,u,v) = a(t,u,v) + \omega(u \,|\, v)_H = \langle (\mathcal{A}(t) + \omega)u | v\rangle,$$

which gives $\widetilde{\mathcal{A}}(t) = \mathcal{A}(t) + \omega$ and consequently, $\widetilde{\mathcal{A}} = \mathcal{A} + \omega$. By Theorem 14.16 we find a unique $v \in \mathrm{MR}(0,\tau)$ such that

$$\begin{aligned} v' + \widetilde{\mathcal{A}}v &= \mathrm{e}^{-\omega\cdot} f \\ v(0) &= u_0, \end{aligned} \tag{3}$$

where we have used that $\mathrm{e}^{-\omega\cdot} f \in L_2(0,\tau;V^*)$, since $\int_0^\tau \mathrm{e}^{-2\omega t}\|f(t)\|_{V^*}^2 \,\mathrm{d}t \leq \|f\|_{L_2(0,\tau;V^*)}^2$. Let $u := \mathrm{e}^{\omega\cdot} v$. Then, $u \in \mathrm{MR}(0,\tau)$. Indeed, we have that $u \in L_2(0,\tau;H)$ since $\int_0^\tau \mathrm{e}^{2\omega t}\|v(t)\|_H^2 \,\mathrm{d}t \leq \mathrm{e}^{2\omega\tau}\|v\|_{L_2(0,\tau;H)}^2$ and thus, $u' = \omega u + \mathrm{e}^{\omega\cdot} v' \in L_2(0,\tau;V^*)$. Moreover, we have that $u(0) = v(0) = u_0$ and

$$\begin{aligned} u' &= \omega u + \mathrm{e}^{\omega\cdot} v' \\ &= \omega u + \mathrm{e}^{\omega\cdot}(\mathrm{e}^{-\omega\cdot} f - \widetilde{\mathcal{A}}v) \\ &= \omega u + f - \mathrm{e}^{\omega\cdot}\widetilde{\mathcal{A}}(\cdot)v(\cdot) \\ &= \omega u + f - \widetilde{\mathcal{A}}u \\ &= f - \mathcal{A}u, \end{aligned}$$

which shows that u is a solution of (2). Moreover, if $u \in \mathrm{MR}(0,\tau)$ is a solution of (2), then $v := \mathrm{e}^{-\omega\cdot} u$ is a solution of (3), which follows by the same argumentation as above. Hence, the uniqueness follows by the uniqueness statement in Theorem 14.16. □