# ISEM18: Solutions to Exercises Lecture 13

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**13.1** For the solution we use the following general fact. Let $H$ be a general Hilbert space and let $V$ be a dense subspace of $H$. For $u \in H$, we claim that

$$\|u\|_H = \sup_{\substack{\|v\|_H=1\\ v\in V}} |(u \mid v)_H|. \tag{1}$$

To prove this, note that by the Cauchy-Schwarz inequality $|(u \mid v)_H| \leq \|u\|_H\|v\|_H = \|u\|_H$, so

$$\sup_{\substack{\|v\|_H=1\\ v\in V}} |(u \mid v)_H| \leq \|u\|_H.$$

For the opposite inequality take a sequence $u_n \in V$ so that $u_n \to u$ in $H$. Such $u_n$ exists because $V$ is assumed to be dense in $H$. If we set $v_n := u_n/\|u_n\|_H$, then $\|v_n\|_H = 1$ and $v_n \to u/\|u\|$ in $H$. Hence, by the continuity of the inner product

$$|(u \mid v_n)| \to \big|(u \mid u/\|u\|)\big| = \frac{\|u\|_H^2}{\|u\|_H} = \|u\|_H.$$

and therefore

$$\sup_{\substack{\|v\|_H=1\\ v\in V}} |(u \mid v)_H| \geq \|u\|_H.$$

Now (1) follows.

(a) By the definition of operator norm and the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we obtain the isometry (and injectivity) of $(I-\Delta)$ as follows:

$$\begin{aligned}
\|(I-\Delta)u\|_{H^{-1}(\Omega)} &= \sup_{\|\phi\|_{H_0^1}=1} \big\{|\langle (I-\Delta)u,\phi\rangle_{H^{-1},H_0^1}|\colon \phi\in C_c^\infty(\Omega)\big\}\\
&= \sup_{\|\phi\|_{H_0^1}=1} \big\{|\langle Iu,\phi\rangle_{H^{-1},H_0^1} + \langle -\Delta u,\phi\rangle_{H^{-1},H_0^1}|\colon\ \phi\in C_c^\infty(\Omega)\big\}\\
&= \sup_{\|\phi\|_{H_0^1}=1} \big\{|(u\mid\phi)_{L_2(\Omega)} + (\nabla u\mid\nabla\phi)_{L_2(\Omega)}|\colon \phi\in C_c^\infty(\Omega)\big\}\\
&= \sup_{\|\phi\|_{H_0^1}=1} \big\{|(u\mid\phi)_{H_0^1(\Omega)}|\colon \phi\in C_c^\infty(\Omega)\big\}\\
&= \|u\|_{H_0^1(\Omega)}.
\end{aligned}$$

For each $f \in H^{-1}(\Omega)$, Riesz-Fréchet states that $\exists!\ u \in H_0^1(\Omega)$ such that

$$(f \mid v)_{L_2(\Omega)} =: \langle f, v\rangle_{H^{-1},H_0^1} = (u \mid v)_{H_0^1(\Omega)} \quad (v \in H_0^1). \tag{2}$$

Using the definitions of $H_0^1$-norm and distributional derivatives, we obtain the surjectivity of $(I-\Delta)$ as the right-hand side becomes

$$\begin{aligned}
(u \mid v)_{H_0^1(\Omega)} &= (u \mid v)_{L_2(\Omega)} + (\nabla u \mid \nabla v)_{L_2(\Omega)}\\
&= (u \mid v)_{L_2(\Omega)} + (-\Delta u \mid v)_{L_2(\Omega)}\\
&= (u - \Delta u \mid v)_{L_2(\Omega)}.
\end{aligned}$$

So for each $f \in H^{-1}(\Omega)$, there exists $u \in H_0^1(\Omega)$ such that $f = (I - \Delta)u$. Thus $(I - \Delta) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

(b) Providing $H_0^1(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}$, we proceed as before for the isometry of $(-\Delta)$:

$$\begin{aligned}
\|(-\Delta)u\|_{H^{-1}(\Omega)} &= \sup_{\|\phi\|_{H_0^1}=1} \{|\langle(-\Delta)u, \phi\rangle_{H^{-1},H_0^1}| : \phi \in C_c^\infty(\Omega)\} \\
&= \sup_{\|\phi\|_{H_0^1}=1} \{(\nabla u \mid \nabla\phi)_{L_2(\Omega)}| : \phi \in C_c^\infty(\Omega)\} \\
&= \sup_{\|\phi\|_{H_0^1}=1} \{(\nabla u \mid \nabla\phi)_{H_0^1(\Omega)}| : \phi \in C_c^\infty(\Omega)\} \\
&= \|u\|_{H_0^1(\Omega)}.
\end{aligned}$$

By Riesz-Fréchet, we again obtain (2) where, equipped with the new scalar product, the right-hand side becomes $(u \mid v)_{H_0^1(\Omega)} = (\nabla u \mid \nabla v)_{L_2(\Omega)} = (-\Delta u \mid v)_{L_2(\Omega)}$. This confirms that for each $f \in H^{-1}(\Omega)$, there exists $u \in H_0^1(\Omega)$ such that $f = (-\Delta)u$. Thus $(-\Delta) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism with the provided $H_0^1(\Omega)$-norm.

**13.2** (a) Since $\Omega$ is bounded, $1 \in L_2(\Omega)$ and for all $u \in L_2(\Omega)$, $u \perp 1$ if and only if $\int_\Omega u = 0$, that is, if and only if $u \in L_2^0(\Omega)$. As the orthogonal complement of any set in a Hilbert space is closed we conclude that $L_2^0(\Omega) = \{1\}^\perp$ is a closed subspace of $L_2(\Omega)$.

To show $L_{2,c}^0(\Omega)$ is dense in $L_2^0(\Omega)$ we use the fact that a subset of a Hilbert space is dense if and only if its orthogonal complement is zero. We show that $L_{2,c}^0(\Omega)^\perp = 0$ by contrapositive; given $u \in L_2^0(\Omega), u \neq 0$ we find $v \in L_{2,c}^0(\Omega)$ with $\mathrm{Re} \int_\Omega u\bar{v} > 0$. Assume first $u$ is real-valued. Then $[u > 0]$ and $[u < 0]$ are both of nonzero measure. Define

$$\Omega_k := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \tfrac{1}{k}\}.$$

Then $\overline{\Omega}_k$ is compact, $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_k \Omega_k$. Hence there exists $k \geq 1$ such that both $A := \Omega_k \cap [u > 0]$ and $B := \Omega_k \cap [u < 0]$ have nonzero measure. Define

$$v := \mathbf{1}_A - \alpha\mathbf{1}_B \quad \text{where} \quad \alpha = \frac{\lambda(A)}{\lambda(B)}.$$

Then clearly $v \in L_2(\Omega)$ and $\int_\Omega v = 0$. Moreover, $\mathrm{spt}\, v \subseteq \overline{\Omega}_k$, so $v \in L_{2,c}^0(\Omega)$, and

$$\int_\Omega u\bar{v} = \int_{\Omega_k \cap [u>0]} u + \alpha \int_{\Omega_k \cap [-u>0]} (-u) > 0.$$

If $u$ is complex-valued then write $u = u_1 + iu_2$ where $u_1, u_2$ are real-valued. By the above working there exist real-valued $v_1, v_2 \in L_{2,c}^0(\Omega)$ such that $\int u_j \bar{v}_j > 0$ for $j = 1, 2$. Defining $v = v_1 + iv_2$, we have

$$\mathrm{Re} \int_\Omega u\bar{v} = \int_\Omega u_1\bar{v}_1 + u_2\bar{v}_2 > 0,$$

completing the proof.

We note that if $(\rho_k)$ is a $\delta$-sequence then

$$\int \rho_k * f(x)\, \mathrm{d}x = \iint \rho_k(x - y)f(y)\, \mathrm{d}y\, \mathrm{d}x = \int f(y) \int \rho_k(x - y)\, \mathrm{d}x\, \mathrm{d}y = \int f(y)\, \mathrm{d}y = 0$$

for all $f \in L_2^0(\Omega)$. In particular, taking $f \in L_{2,c}^0(\Omega)$ and $\rho_k \in C_c^\infty(\mathbb{R}^n)$ for sufficiently large $k$ we can see that $L_2^0(\Omega) \cap C_c^\infty(\Omega)$ is also dense in $L_2^0(\Omega)$.

(b) **First Solution** We remove the condition that $\Omega$ is connected, and instead assume that for every choice of non-empty subsets $A, B \subseteq \{1, \ldots, m\}$ with $A \cup B = \{1, \ldots, m\}$ we have

$$\Big(\bigcup_{j\in A} \Omega_j\Big) \cap \Big(\bigcup_{k\in B} \Omega_k\Big) \neq \emptyset \tag{3}$$

Clearly (3) holds if $\Omega$ is connected by definition of a connected set.

We proceed by induction. The proof of the $m = 2$ case is identical to the proof given in Theorem 13.9, since in this case condition (3) is equivalent to $\Omega_1 \cap \Omega_2 \neq \emptyset$. Assume $m > 2$ and that the result holds for smaller $m$. Clearly condition (3) implies there exists $j < m$ such that $\Omega_j \cap \Omega_m \neq \emptyset$. Put $\Omega'_j = \Omega_j \cup \Omega_m$ and $\Omega'_k = \Omega_k$ for $k \leq m-1, k \neq j$. We now have $\Omega = \bigcup_{k=1}^{m-1} \Omega'_k$ and certainly condition (3) holds for this collection. Hence by induction hypothesis for every $f \in L_2^0(\Omega)$ there exists $f'_1, \ldots, f'_{m-1} \in L_2^0(\Omega)$ such that $[f'_k \neq 0] \in \Omega'_k$ and $f = \sum_{k=1}^{m-1} f'_k$. Moreover, condition (3) holds for $\Omega'_j = \Omega_j \cup \Omega_m$ since $\Omega_j \cap \Omega_m \neq \emptyset$, so there exists $f_j, f_m \in L_2^0(\Omega)$ such that $[f_k \neq 0] \in \Omega_k$ and $f'_j = f_j + f_m$. Putting $f_k = f'_k$ for $k \neq j, m$ we have $f = \sum_{k=1}^{m} f_k$ and $[f_k \neq 0] \in \Omega_k$, so by induction the proof is complete.

**Second Solution** Define $L := \sum_{j=1}^{m} L_2^0(\Omega_j)$. To show that $L = L_2^0(\Omega)$, it suffices to show that $L$ is dense in $L_2^0(\Omega)$ because $L$ is a closed subspace (being a finite sum of closed subspaces). Thus, we let $u \in L_2^0(\Omega), u \neq 0$ and show that

$$\text{There exists } v \in L \text{ such that } \int_\Omega u\bar{v}\, dx \neq 0. \tag{4}$$

As in Question 13.2(a), we may assume without loss of generality that $u$ is real valued.

To show (4), we note the following fact: there exists a $k \in (1, 2, \ldots, m)$ such that $u$ is not constant on $\Omega_k$. This follows by contradiction. Indeed, suppose that $u$ is constant almost everywhere on each $\Omega_k$. Then in fact, $u$ is globally constant almost everywhere as a result of the chain characterisation of connected sets: that is, fix an arbitrary open cover $\mathcal{V}$ of a connected set $\Omega$. Then for any two open sets $X$ and $Y$ being elements of $\mathcal{V}$, there exists a finite sequence $\{V_1, V_2, \ldots, V_n\}$ of $\mathcal{V}$ such that $V_1 = X$ and $V_n = Y$ with $V_i \cap V_{i+1} \neq \phi$ for all $1 \leq i \leq n-1$. However, this implies that $u = 0$ almost everywhere, which is a contradiction.

Define $A := (\int_{\Omega_k} u\, dx)/\lambda(\Omega_k)$ and observe that $u - A \in L_2^0(\Omega_k)$ and $u - A$ is not identically zero on $\Omega_k$. We therefore use the same construction as in Question 13.2(a) to obtain a function $v_k \in L_2^0(\Omega_k)$ such that

$$\int_{\Omega_k} u\bar{v}_k \,\mathrm{d}x > A \int_{\Omega_k} \bar{v}_k \,\mathrm{d}x = 0.$$

By setting $v_j = 0$ for all $j \neq k$, we observe that $v = \sum_{j=1}^{m} v_j$ satisfies (4) as required.

**13.3 Claim 1: The** spt $\Phi$ **is contained in the convex hull of** $(\text{spt}\phi + \text{y}) \cup (\text{spt}\phi + \text{z})$**.** First, we use the substitution $t \mapsto t-1$ to obtain that

$$\begin{aligned} \Phi(x) &= \left[\int_{-\infty}^{0} \phi(x + (t-1)z - ty)\, dt - \int_{-\infty}^{1} \phi(x + (t-1)z - ty)\, dt\right](z - y) \\ &= \int_0^1 \phi(x + (t-1)z - ty)\, dt(y - z). \end{aligned} \tag{5}$$

Denote the convex hull of $(\text{spt}\phi + \text{y}) \cup (\text{spt}\phi + \text{z})$ by $E$.

We prove the contrapositive claim. We let $x \notin E$ and show that $\Phi(x) = 0$. Since $t \in (0, 1)$ in equation (5), this follows immediately from the fact that $h := x + (t-1)z - ty \notin \operatorname{spt}(\phi)$. Indeed, if $h$ were in the $\operatorname{spt}(\phi)$, then we obtain by simple rearrangement that $x = t(h+y) + (1-t)(h+z)$ contradicting our assumption on $x$. This concludes the proof of the claim.

**Claim 2: We have that** $\operatorname{div}\Phi(x) = \phi(x-z) - \phi(x-y)$**.** The result follows from the definition of the total derivative and the fundamental theorem of calculus. Observe that we may pass the derivative through the integral sign because $\phi$ is smooth.

$$\begin{aligned}
\operatorname{div}\Phi(x) &= \int_{\infty}^{0} \sum_{k=1}^{N} \partial_k \left(\varphi(x + t(z-y) - z) - \varphi(x + t(z-y)) - y\right)(z_k - y_k)\, dt \\
&= \int_{-\infty}^{0} \nabla\left(\varphi(x + t(z-y) - z) - \varphi(x + t(z-y) - y)\right) \cdot (z-y)\, dt \\
&= \int_{-\infty}^{0} \frac{d}{dt}\left[\varphi(x + t(z-y) - z) - \varphi(x + t(z-y) - y)\right] dt \\
&= \varphi(x-z) - \varphi(x-y).
\end{aligned} \tag{6}$$

In the last equality, we used the fact that since $\varphi$ has compact support in $\mathbb{R}^N$, then for sufficiently large negative $t$, we have $x + t(z-y) - z$ and $x + t(z-y) - y \notin \operatorname{spt}(\varphi)$.

**Claim 3: We have** $\Phi \in C_c^\infty(\mathbb{R}^N)$**.** Since $\varphi$ is smooth with compact support, the dominated convergence theorem allows us to differentiate under the integral sign. Clearly this also holds for any mixed partial derivatives of $\varphi$ as well, so $\Phi$ is infinitely differentiable. Compact support follows immediately from Claim 1.

**13.4** Let $(u_n)$ be a sequence in $\operatorname{ran}(\nabla)$ such that $u_n \to u \in H^{-1}(\Omega)^n$. We show $u \in \operatorname{ran}(\nabla)$. There exists $f_n \in L_2(\Omega)$ such that $\nabla f_n = u_n$, and moreover, we may assume $f_n \in L_2^0(\Omega)$. Indeed, if $f \notin L_2^0(\Omega)$ then set

$$g_n := f_n - \frac{1}{\lambda(\Omega)} \int_\Omega f_n$$

and note that $g_n \in L_2^0(\Omega)$ and $\nabla g_n = \nabla f_n = u_n$. Now using the inequality (13.9) we have

$$\|f_n\|_2 \le c \|\nabla f_n\|_{H^{-1}(\Omega)^n} = c\|u_n\|_{H^{-1}(\Omega)^n}$$

which is convergent, and thus bounded. Hence $(f_n)$ is bounded in $L_2$, so $f_n \rightharpoonup f$ weakly along a subsequence. Thus we find $\nabla f_n \rightharpoonup \nabla f$ weakly, and hence $u = \nabla f \in \operatorname{ran}(\nabla)$ as claimed.

**13.5** Let $R > 0$ be large enough that both $\Omega$ and the support of $\rho$ are contained in $B(0,R)$. For any $x \in \Omega$, if $s \ge 2R$ then $\rho(x + s\frac{z}{|z|}) = 0$ and if $|z| \ge 2R$ then $f(x-z) = 0$. For $s \in (0, 2R)$ and $z \in B(0,2R)$ we have $(s + |z|)^{n-1} \le 2^{n-2}(s^{n-1} + |z|^{n-1}) \le (2R)^{n-1}$, so we find

$$\begin{aligned}
|Bf(x)| &\le \int \int_0^\infty |f(x-z)| \rho\left(x + s\frac{z}{|z|}\right) \mathbf{1}_{(0,2R)}(s) \mathbf{1}_{B(0,2R)}(z)(s + |z|)^{1-n}\, \mathrm{d}s\, \mathrm{d}z \\
&\le (2R)^n \|\rho\|_\infty \int_{B(0,2R)} |f(x-z)||z|^{1-n}\, \mathrm{d}z.
\end{aligned}$$

Setting $M := (2R)^n \|\rho\|_\infty$, this gives us an $L^\infty$-$L^\infty$ bound:

$$\|Bf\|_\infty \le \left(M \int_{B(0,2R)} |z|^{1-n}\, \mathrm{d}z\right) \|f\|_\infty$$

and integrating over $x \in \Omega$ gives us an $L^1$-$L^1$ bound:

$$\begin{aligned}
\|Bf\|_1 &\leq M \int_\Omega \int_{B(0,2R)} |f(x-z)||z|^{1-n} \, \mathrm{d}z \, \mathrm{d}x \\
&= M \int_{B(0,2R)} \left( \int_\Omega |f(x-z)| \, \mathrm{d}x \right) |z|^{1-n} \, \mathrm{d}z \\
&\leq \left( M \int_{B(0,2R)} |z|^{1-n} \, \mathrm{d}z \right) \|f\|_1.
\end{aligned}$$

To verify that $c := M \int_{B(0,2R)} |z|^{1-n} \, \mathrm{d}z$ is finite, convert to polar coordinates:

$$\begin{aligned}
z_1 &= r \cos \phi_1, \\
z_2 &= r \sin \phi_1 \cos \phi_2, \\
&\vdots \\
z_{n-1} &= r \sin \phi_1 \dots \sin \phi_{n-2} \cos \phi_{n-1}, \\
z_n &= r \sin \phi_1 \dots \sin \phi_{n-2} \sin \phi_{n-1}
\end{aligned}$$

and note that the Jacobian of the transformation is $r^{n-1} F(\phi_1, \dots, \phi_{n-2})$ for some continuous (and bounded) function $F$. It hence follows by the Riesz-Thorin Theorem that $\|Bf\|_2 \leq c\|f\|_2$ for all $f \in C_c^\infty(\Omega) \cap L_2(\Omega)$.