ISEM 18: Solutions to the Exercises of Lecture 12

Delft Team

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Exercise 12.1 Let H be a complex Hilbert space, a an embedded sectorial form in H Show the following criteria for a being closed or closable:

(a) a is closed if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to u$ in H one has $\|u_n - u\| \to 0$.

(b) a is closable if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to 0$ in H one has $\|u_n\|_a \to 0$.

Proof of (a). Assume a closed. Let $(u_n)_n \in \mathbb{N}$ be a Cauchy sequence in $(dom(a), \|\cdot\|_a)$, with $u_n \to u$ in H. By the closedness of a, $\exists u_0 \in dom(a)$ s.t. $\|u_n - u_0\|_a \to 0$.
Observe that $j : dom(a) \hookrightarrow H$ is bounded. In fact, since a is sectorial, $\exists\, c \geq 0$ s.t. Re $a(u) \geq c^{-1}|\text{Im } a(u)|$ and hence $\|u\|_a = (\text{Re } a(u) + \|u\|_H^2)^{\frac{1}{2}} \geq \|u\|_H$.
This implies that $u_n \to u_0$ in H. By uniqueness of limits we can conclude that $u = u_0 \in H$ and hence $\|u_n - u\|_a \to 0$ as wanted.

Assume now that for every Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $(dom(a), \|\cdot\|_a)$, with $u_n \to u$ in H, we have $\|u_n - u\|_a \to 0$ as $n \to \infty$. This implies that $(dom(a), \|\cdot\|_a)$ is complete, and thus a is closed. □

Proof of (b). Let (V, q) be the completion of $(dom(a), \|\cdot\|_a)$, with $(\tilde{a}, \tilde{j})$ completion of (a, j), according to the definitions in Lecture 12.

Assume that a is closable. Then, by definition, $\tilde{j} : V \hookrightarrow H$ is injective. Recall also that $\|q(\cdot)\|_V = \|\cdot\|_a$.
Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(dom(a), \|\cdot\|_a)$ and assume that $u_n \to 0$ in H. By continuity of j, also $j(u_n) \to 0$ in H.
Observe that the equivalence of norms $\|u_n\|_a = \|q(u_n)\|_V$ yelds that also $(q(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in V. Thus, $\exists v \in V$ s.t. $q(u_n) \to v$ in V, and by continuity of $\tilde{j}$ we get $\tilde{j}(q(u_n)) \to \tilde{j}(v)$.
Therefore, $\tilde{j}(v) = \lim_{n\to\infty} \tilde{j}(q(u_n)) = \lim_{n\to\infty} j(u_n) = 0$ in H.
By injectivity of $\tilde{j}$, we can conclude that $v = 0$ and thus $\|u_n\|_a = \|q(u_n)\|_V \to 0$, as wanted.

For the converse, we need to show that $\tilde{j} : V \hookrightarrow H$ is injective.
Let $u \in V$ s.t. $\tilde{j}(u) = 0$ in H. We want to show that $u = 0$ in V. For this, observe that since $q : dom(a) \to V$ has dense range, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $dom(a)$ s.t. $q(u_n) \to u$ in V. By the continuity of $\tilde{j}$ we get that $j(u_n) = \tilde{j}(q(u_n)) \to \tilde{j}(u) = 0$ in H, from which holds by the continuity of j that $u_n \to 0$ in H.
By assumptions and norm equivalences, $\|q(u_n)\|_V = \|u_n\|_a \Longrightarrow 0$, which implies that $u = \lim_{n\to\infty} q(u_n) = q(0) = 0$. So it must be $u = 0$ and we can conclude that $\tilde{j}$ is injective and by definition a is closable. □

Exercise 12.2 Let $H := L_2(-1,1)$, a_1, a_2 in H defined by $\operatorname{dom}(a_j) = C_c^\infty(-1,1)$,

$$\begin{aligned} a_1(u,v) &:= u(0)\overline{v(0)}, \\ a_2(u,v) &:= \int_{-1}^1 u'(x)\overline{v'(x)}dx + u(0)\overline{v(0)} \end{aligned}$$

for all $u, v \in C_c^\infty(-1,1)$.

For $j = 1,2$ determine whether a_j is closable. Find the completion of $(\operatorname{dom}(a_j), \langle\cdot,\cdot\rangle_{a_j})$ and the operator associated with a_j.

Solution for $j = 1$: **Claim:** a_1 is not closable. The completion of $(\operatorname{dom}(a_1), \langle\cdot,\cdot\rangle_{a_1})$ is $L_2(-1,1)\oplus\mathbb{C}$ (direct sum of Hilbert spaces) with $q : C_c^\infty(-1,1) \longrightarrow L_2(-1,1)\oplus\mathbb{C}, u \mapsto (u, u(0))$.

Proof. First note that

$$\langle u, v\rangle_{a_1} = (u|v)_{L_2} + u(0)\overline{v(0)}, \qquad u, v \in C_c^\infty(-1,1).$$

So we may view $(\operatorname{dom}(a_1), \langle\cdot,\cdot\rangle_{a_1})$ as subspace of the Hilbert space $H \oplus \mathbb{C}$ via the mapping $\operatorname{dom}(a_1) \longrightarrow H \oplus \mathbb{C}, u \mapsto (u, u(0))$. Since $\operatorname{dom}(a_1)$ is a dense in $H \oplus \mathbb{C}$, the second part of the claim follows. For the part that a_1 is not closable we just have the note that, in the notation of Remark 12.3(d), $\tilde{j} : H \oplus \mathbb{C} \longrightarrow H$ is the projection onto the first coordinate and in particular not injective. □

Solution for $j = 2$: **Claim:** a_2 is closable. The completion of $(\operatorname{dom}(a_2), \langle\cdot,\cdot\rangle_{a_2})$ is $H_0^1(-1,1)$, when equipped with the inner product

$$\langle u, v\rangle = (u|v)_{L_2} + (u'|v')_{L_2} + u(0)\overline{v(0)}, \qquad u, v \in H_0^1(-1,1), \tag{0.1}$$

with q and $\tilde{j}$ both the natural inclusion.

Proof. We only need to show the second part; the natural inclusion $\tilde{j}$ is injective so that a_2 will be closable. For $u \in C_c^\infty(-1,1)$ we have

$$\begin{aligned} \|u\|_{a_2} &= \|u\|_{L_2(-1,1)} + \sqrt{\Re a_2(u)} = \|u\|_{L_2(-1,1)} + \sqrt{\|u'\|^2_{L_2(-1,1)} + |u(0)|^2} \\ &\approx \|u\|_{L_2(-1,1)} + \|u'\|_{L_2(-1,1)} + |u(0)| \\ &\approx \|u\|_{H_0^1(-1,1)} + |u(0)|. \end{aligned}$$

Since $H_0^1(-1,1) \longrightarrow \mathbb{C}, u \mapsto u(0)$ is bounded, it follows that $\|u\|_{a_2} \approx \|u\|_{H_0^1(-1,1)}$ for all $u \in C_c^\infty(-1,1)$. Since $C_c^\infty(-1,1)$ is dense in the complete space $H_0^1(-1,1)$, it follows that the completion of $(\operatorname{dom}(a_j), \langle\cdot,\cdot\rangle_{a_j})$ is $H_0^1(-1,1)$ with $\tilde{j}$ the inclusion mapping. Furthermore, note that (0.1) is just an extension of the inner product $\langle\cdot,\cdot\rangle_{a_2}$ to $H_0^1(-1,1)$. □

Exercise 12.3 Let $\Omega := (-1,0) \cup (0,1)$.

(a) Determine the relation tr of Section 12.5, and show that dom(tr) is not dense.

(b) Find $\partial_\nu u$ for those $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ for which $\partial_\nu u \in L_2(\partial\Omega)$.

(c) Determine the Robin-Laplacian for $\beta = 1$.

Solution of 12.3(a). **Claim:** $\operatorname{dom}(\mathrm{tr}) = C[-1,1] \cap H^1(\Omega) = H^1(-1,1)$ and $\mathrm{tr}\, u = \{u|_{\partial\Omega}\} = \{u|_{\{-1,0,1\}}\}$ for every $u \in C[-1,1] \cap H^1(\Omega)$.

As a consequence of this claim, dom(tr) is a closed proper subspace of $H^1(\Omega)$, and thus certainly not dense.

Proof of claim. We first show that $\operatorname{dom}(\operatorname{tr}) \subset C[-1,1] \cap H^1(\Omega)$ and that $\operatorname{tr} u \subset \{u|_{\{-1,0,1\}}\}$ for all $u \in \operatorname{dom}(\operatorname{tr})$. Let $u \in \operatorname{dom}(\operatorname{tr})$, that is, $u \in H^1(\Omega)$ with $\operatorname{tr} u \neq \emptyset$; say $g \in \operatorname{tr} u$. Then there exists a sequence $(u_k)_k \subset C([-1,1]) \cap H^1(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$ and $u_k|_{\partial\Omega} \to g$ in $L_2(\partial\Omega)$. Since

$$H^1(\Omega) = H^1(-1,0) \oplus H^1(0,1) \hookrightarrow C[-1,0] \oplus C[0,1]$$

while $C[-1,1]$ is a closed subspace of $C[-1,0] \oplus C[0,1]$, it follows that $u = \lim_{k\to\infty} u_k$ in $C[-1,1] \cap H^1(\Omega)$. So $u \in C[-1,1] \cap H^1(\Omega)$ with $u|_{\{-1,0,1\}} = \lim_{k\to\infty} u_k|_{\{-1,0,1\}} = g$ in $C[\{-1,0,1\}] \cong L^2(\partial\Omega)$. As this holds for arbitrary $g \in \operatorname{tr} u$, this shows $\operatorname{tr} u \subset \{u|_{\{-1,0,1\}}\}$.

Next we show that $\{u|_{\partial\Omega}\} \subset \operatorname{tr} u$ for every $u \in C[-1,1] \cap H^1(\Omega)$; this in particularly shows that $C[-1,1] \cap H^1(\Omega) \subset \operatorname{dom}(\operatorname{tr})$. Let $u \in C[-1,1] \cap H^1(\Omega)$. Then the inclusion $\{u|_{\partial\Omega}\} \subset \operatorname{tr} u$ follows by simply taking $u_k = u$ in the definition of $\operatorname{tr} u$.

Finally we show that $C[-1,1] \cap H^1(\Omega) = H^1(-1,1)$. The inclusion '$\supset$' follows directly from the fact that $H^1(-1,1) \subset C[-1,1]$. For the reverse inclusion, recall that

$$H^1(a,b) = \left\{ u \in C[a,b]; \exists g \in L_2(a,b) : u(x) - u(y) = \int_y^x g(t)dt \right\}, \quad -\infty < a < b < \infty; \quad (0.2)$$

see Proposition 4.6 and Theorem 4.9. Let $u \in H^1(\Omega) = H^1(-1,1) \oplus H^1(0,1)$. Then there exist $g_l \in L_2(-1,0)$ and $g_r \in L_2(0,1)$ representing $u_l := u|_{(-1,0)} \in H^1(-1,0)$ and $u_r := u|_{(0,1)} \in H^1(0,1)$, respectively, as in (0.2). Now it is not difficult to see that $g := g_l + g_r \in L_2(-1,0) \oplus L_2(0,1) = L_2(-1,1)$ represents $u \in C[-1,1]$ as in (0.2). Hence, $u \in H^1(-1,1)$. □

Solution of 12.3(b). First observe that $u \in H^2(\Omega)$ if and only if $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$. So we need to find $\partial_\nu u$ for those $u \in H^2(\Omega)$ for which $\partial_\nu u \in L_2(\partial\Omega)$.

Claim: Let $u \in H^2(\Omega) = H^2(-1,0) \oplus H^2(0,1) \hookrightarrow C^1[-1,0] \oplus C^1[0,1]$. Then for the function $h \in L_2(\partial\Omega)$ given by

$$h(x) = \begin{cases} -u'(-1), & x = -1; \\ u'(0+) - u'(0-), & x = 0; \\ u'(1), & x = 1, \end{cases}$$

where $u'(0+) = u'|_{]0,1[}(0)$ and $u'(0-) = u'|_{]-1,0[}(0)$, we have $\partial_\nu u = h$.

Proof. Fix $u \in H^2(\Omega) = H^2(-1,0) \oplus H^2(0,1) \hookrightarrow C^1[-1,0] \oplus C^1[0,1]$. For $v \in C(\overline{\Omega}) \cap H^1(\Omega)$ we compute, using the fact that the product of two $H^1(\Omega)$-functions is an $H^1(\Omega)$-function again with derivative given via the Leibniz rule (referentie geven?),

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega (\Delta u) v \, dx &= \int_\Omega u'v' \, dx + \int_\Omega u'' v \, dx \\ &= \int_\Omega [u'v' + u''v] \, dx \\ &= \int_\Omega (u'v)' \, dx \\ &= \int_{]-1,0[} (u'v)' \, dx + \int_{]0,1[} (u'v)' \, dx \\ &= u'(0-)v(0) - u'(-1)v(-1) + u'(1)v(1) - u'(0+)v(0) \\ &= \int_{\partial\Omega} hv \, d\sigma. \end{aligned}$$

This proofs the claim. □

Solution of 12.3(c). **Claim:** The Robin-Laplacian Δ_1 is the linear operator A given by

$$\begin{aligned} A &= \{(u,f) \in H^1(-1,1) \times L_2(\Omega); -\Delta u = f \text{ on } \Omega, u|_{\partial\Omega} = -\partial_\nu u\} \\ &= \{(u,f) \in [H^1(-1,1) \cap H^2(\Omega)] \times L_2(\Omega); -\Delta u = f \text{ on } \Omega, u|_{\partial\Omega} = -\partial_\nu u\}, \end{aligned}$$

which is well-defined thanks to (b).

Proof. Recall that the Robin-Laplacian Δ_1 is defined as the relation

$$\Delta_1 = \{(u,f) \in H^1(\Omega) \times L_2(\Omega); -\Delta u = f, \exists g \in \operatorname{tr} u : \partial_\nu u = -g\}.$$

The inclusion $A \subset \Delta_1$ follows directly from (a). For the reverse inclusion, let $(u,f) \in \Delta_1$. Then there exists $g \in \operatorname{tr} u$ such that $\partial_\nu u = -g$. By (a) we must thus have $u \in \operatorname{dom}(\operatorname{tr}) = C[-1,1] \cap H^1(\Omega) = H^1(-1,1)$ with $u|_{\partial\Omega} = g = -\partial_\nu u$. This shows $\Delta_1 \subset A$, as desired. □

Exercise 12.4 Let $S := [0,1] \times \{0\} \subset \mathbb{R}^2$, and let (x_n) be a bounded sequence in $\mathbb{R}^2 \setminus S$ having the set S as its cluster values. Let (r_n) be a sequence in $(0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$, $B[x_n, r_n] \cap S = \varnothing$ $(n \in \mathbb{N})$ and $B[x_n, r_n] \cap B[x_m, r_m] = \varnothing$ $(m, n \in \mathbb{N}, m \neq n)$. Let $\Omega := \bigcup_{n\in\mathbb{N}} B(x_n, r_n)$.

(a) Determine $\partial\Omega$ and $\sigma_1(\partial\Omega)$ (1-dimensional Hausdorff measure).

(b) Show that dom(tr) is dense in $H^1(\Omega)$ and that $\operatorname{tr} 0 = L_2(S)$ (where $L_2(S) \subset L_2(\partial\Omega)$ is the natural embedding).

(c) Let D_0 be the Dirichlet-to-Neumann operator for Ω. Show that $L_2(S) \subset \operatorname{dom}(D_0)$ and that $D_0|_{L_2(S)} = 0$.

Solution of 12.4(a). **Claim:** $\partial\Omega = \bigcup_{n\in\mathbb{N}} \partial B[x_n, r_n] \cup S$ and $\sigma_1(\partial\Omega) = 1 + 2\pi \sum_{n\in\mathbb{N}} r_n$..

Proof. We first determine $\overline{\Omega}$. Define $A := \bigcup_{n\in\mathbb{N}} B[x_n, r_n] \cup S$. We claim that $A = \overline{\Omega}$. The assumption that S consists of cluster values of $\{x_n\}_{n\in\mathbb{N}}$ yields $A \subseteq \overline{\Omega}$.

Now let $y \in \overline{\Omega}$. Then there exists a sequence $\{y_k\} \in \Omega$ that converges to y. If there is some $n \in \mathbb{N}$ such that $B(x_n, r_n)$ contains infinitely many terms of the sequence, then the limit $y \in B[x_n, r_n] \subseteq A$. In the other case, we have for every n only finitely many terms of $\{y_k\}_{k\geq 1}$ in $B(x_n, r_n)$. Therefore, we can find for any $m \in \mathbb{N}$ numbers $n(m), k(m) \in \mathbb{N}$ such that $r_{n(m)} \leq m^{-1}$ and $y_{k(m)} \in B(x_{n(m)}, r_{n(m)})$. By assumption $y_{k(m)}$ converges to y, which implies that $x_{n(m)}$ converges to y. But S is the set of cluster values of $\{x_n\}_{n\geq 0}$, which yields $y \in S$.

This proves $\overline{\Omega} = \bigcup_{n\in\mathbb{N}} B[x_n, r_n] \cup S$. All the sets in the union are disjoint by assumption. This together with the fact that Ω is open gives $\partial\Omega = \overline{\Omega} \setminus \Omega = \bigcup_{n\in\mathbb{N}} \partial B(x_n, r_n) \cup S$.

We now determine $\sigma_1(\partial\Omega)$. As $\partial\Omega = \bigcup_{n\in\mathbb{N}} \partial B(x_n, r_n) \cup S$ as a disjoint union, we obtain

$$\sigma_1(\partial\Omega) = \sigma_1(S) + \sum_{n\in\mathbb{N}} \sigma_1(\partial B(x_n, r_n)).$$

Since σ_1 coincides with the Lebesgue measure on $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$, we find $\sigma_1(S) = 1$. All balls $B(x_n, r_n)$ have C^1 boundary, so $\sigma_1(\partial B(x_n, r_n)) = 2\pi r_n$. We conclude

$$\sigma_1(\partial\Omega) = 1 + 2\pi \sum_{n\in\mathbb{N}} r_n.$$

□

Notations and preparations for 12.4(b). Writing $B_n := B(x_n, r_n)$ for each n, we have the canonical isometric isomorphisms

$$H^1(\Omega) = \bigoplus_{n=1}^{\infty} H^1(B_n) \quad \left(= \left(\bigoplus_{n=1}^{\infty} H^1(B_n) \right)_{\ell^2} \right) \tag{0.3}$$

and

$$L_2(\partial\Omega \setminus S) = \bigoplus_{n=1}^{\infty} L_2(\partial B_n) \quad \left(= \left(\bigoplus_{n=1}^{\infty} L_2(\partial B_n) \right)_{\ell^2} \right).$$

Denote by $(\pi_n)_n$ and (P_n) the sequence of coordinate projections and its sequence of partial sum projections, respectively, associated with $H^1(\Omega)$; here $P_n = \pi_1 + \ldots + \pi_n$.

By Theorem 7.9, for each n we have a trace operator $\mathrm{tr}_n \in \mathcal{B}(H^1(B_n), L_2(\partial B_n))$. So we can define the closed linear operator

$$\mathrm{tr}_{\partial\Omega\setminus S} := \oplus_{n=1}^{\infty} \mathrm{tr}_n : \mathrm{dom}(\mathrm{tr}_{\partial\Omega\setminus S}) \subset H^1(\Omega) \longrightarrow L_2(\partial\Omega \setminus S), u \mapsto (\mathrm{tr}_n \pi_n u)_n$$

with its natural domain

$$\begin{aligned} \mathrm{dom}(\mathrm{tr}_{\partial\Omega\setminus S}) &:= \{u \in H^1(\Omega) : (\mathrm{tr}_n \pi_n u)_n \in L_2(\partial\Omega \setminus S)\} \\ &= \left[\bigoplus_{n=1}^{\infty} \left(H^1(B_n), \|\cdot\|_{\mathrm{dom}(\mathrm{tr}_n)} \right) \right]_{\ell^2} \end{aligned} \tag{0.4}$$

Since each trace operator tr_n is the unique bounded linear extension of the classical restriction operator $u \mapsto u|_{\partial B_n}, C(\overline{B}_n) \cap H^1(B_n) \longrightarrow C(\partial B_n)$, we have

$$C(\overline{\Omega}) \cap H^1(\Omega) \subset \mathrm{dom}(\mathrm{tr}_{\partial\Omega\setminus S}) \quad \text{with} \quad \mathrm{tr}_{\partial\Omega\setminus S} u = u|_{\partial\Omega\setminus S} \quad \text{for } u \in C(\overline{\Omega}) \cap H^1(\Omega). \tag{0.5}$$

Solution of 12.4(b). We first show that dom(tr) is dense in $H^1(\Omega)$. In light of the direct sum decomposition (0.3),

$$D := \{u \in H^1(\Omega) : \pi_n u = 0 \text{ for only finitely many } n\} = \bigcup_{n\in\mathbb{N}} \mathrm{ran}(P_n)$$

defines a dense subspace of $H^1(\Omega)$. So it is enough to show that $D \subset \mathrm{dom}(\mathrm{tr})$. But for each $u \in D \subset \mathrm{dom}(\mathrm{tr}_{\partial\Omega\setminus S})$ it holds that $\mathrm{tr}_{\partial\Omega\setminus S} u \in \mathrm{tr}\, u$. Indeed, if $u \in \mathrm{ran}(P_N)$, then there exists a sequence $(u_k)_k \subset \bigoplus_{n=1}^{N} C^1(\overline{B}_n) = C^1(\bigcup_{n=1}^{N} \overline{B}_n) \subset C^1(\overline{\Omega})$ such that $u = \lim_{k\to\infty} u_k$ in $\bigoplus_{n=1}^{N} H^1(B_n) \hookrightarrow \mathrm{dom}(\mathrm{tr}_{\partial\Omega\setminus S})$. As $u_k|_S = 0$ for every k, we obtain $u = \lim_{k\to\infty} u_k$ in $H^1(\Omega)$ and $\mathrm{tr}_{\partial\Omega\setminus S}\, u = \lim_{k\to\infty} \mathrm{tr}_{\partial\Omega\setminus S}\, u_k \overset{(0.5)}{=} \lim_{k\to\infty} u_k|_{\partial\Omega}$ in $L_2(\partial\Omega)$.

Next we show that $\mathrm{tr}\, 0 = L_2(S)$. Being the pre-image of the closed set

$$\overline{\{(v, v_{|\partial\Omega}) : v \in C(\overline{\Omega}) \cap H^1(\Omega)\}}^{H^1(\Omega)\times L_2(\partial\Omega)}$$

under the the continuous mapping

$$L_2(\partial\Omega) \longrightarrow H^1(\Omega) \times L_2(\partial\Omega), g \mapsto (0, g),$$

$\operatorname{tr} 0$ is a closed subset of $L_2(\partial\Omega)$. By denseness of $C^1(S)$ in $L_2(S)$, it thus suffices to show that

$$C^1(S) \subset \operatorname{tr} 0 \subset L_2(S).$$

The second inclusion follows from the definition of $\operatorname{tr} 0$, (0.5) and the fact that $\operatorname{tr}_{\partial\Omega\setminus S}$ is a closed operator. For the first inclusion, fix $g \in C^1(S)$. Pick an extension $\tilde{g} \in C^1(\overline{\Omega})$ of g and define the sequence $(u_k)_k \subset C^1(\overline{\Omega})$ by $u_k := (1 - P_k)\tilde{g}$. By a combination of (0.4) and (0.5), we have $u_k \to 0$ in $\operatorname{dom}(\operatorname{tr}_{\partial\Omega\setminus S})$, or equivalently, $u_k \to 0$ in $H^1(\Omega)$ and $u_k|_{\partial\Omega\setminus S} \to 0 = g|_{\partial\Omega\setminus S}$ in $L_2(\partial\Omega\setminus S)$. Since $u_k|_S = g$ for every k, it follows that $g \in \operatorname{tr} 0$.

Solution of 12.4(c).

Proof. Recall that the Dirichlet-to-Neumann operator is defined by

$$D_0 = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega) : \Delta u = 0, g \in \operatorname{tr} u, h = \partial_\nu u\};$$

see Theorem 12.12 (which in particular says that it is an operator). Let $g \in L_2(S) \overset{(b)}{=} \operatorname{tr} 0$. Then $u = 0 \in H^1(\Omega)$ satisfies $\Delta u = 0$, $g \in \operatorname{tr} u$ and $u = \partial_\nu u$, whence $(g, 0) \in D_0$. □