Solutions to the Exercises of Lecture 11

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Exercise 11.1 Let $(a_{jk}) \in \mathbb{R}^{n\times n}$, $\alpha > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}\xi_k\xi_j \geq \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. Show that

$$\operatorname{Re}\sum_{j,k=1}^{n} a_{jk}\xi_k\overline{\xi_j} \geq \alpha|\xi|^2$$

for all $\xi \in \mathbb{C}^n$.

Solution. We define $\eta = \operatorname{Re}\xi$, $\zeta = \operatorname{Im}\xi$, $A = (a_{ij})_{i,j=1}^n$. Let the symbol * denote the conjugate transpose of a vector. Then we have

$$\begin{aligned}\operatorname{Re}(\xi^* A\xi) &= \operatorname{Re}[(\eta - \mathrm{i}\zeta)^\top A(\eta + \mathrm{i}\zeta)] \\ &= \operatorname{Re}[\eta^\top A\eta + \zeta^\top A\zeta + \mathrm{i}(\eta^\top A\zeta - \zeta^\top A\eta).]\end{aligned}$$

Since the vectors η and ζ as well as the Matrix A are all real, the real part of the last two terms is zero and we can conclude that

$$\begin{aligned}\operatorname{Re}(\xi^* A\xi) &= \eta^\top A\eta + \zeta^\top A\zeta \\ &\geq \alpha|\eta|^2 + \alpha|\zeta|^2 \\ &= \alpha|\xi|^2\end{aligned}$$

which is what was to be shown.

Exercise 11.2 Let $\Omega \subseteq \mathbb{R}^2$ be open, $(a_{jk}) = \begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}$, $b = c = 0$, $d = 0$.

(a) Show that $A_D = -\Delta_D$.

(b) Assume that Ω is bounded with C^1-boundary. Find the conormal derivative corresponding to A_N; cf. Example 11.4(b). Find Ω with $A_N \neq -\Delta_N$. Can one see that for all of these $\Omega \neq \emptyset$ one has $A_N \neq -\Delta_N$?

Solution. For (a), we start by defining

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1\end{bmatrix}$$

and consider the form

$$a\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K},\quad a(u,v)=\int_\Omega \nabla u\cdot M^\top\nabla\bar v\,\mathrm{d}x.$$

This form can be written in the following way:

$$a(u,v)=\int_\Omega\big(\partial_1 u\partial_1\bar v-\partial_1 u\partial_2\bar v+\partial_2 u\partial_1\bar v+\partial_2 u\partial_2\bar v\big)\,\mathrm{d}x. \tag{1}$$

Now we consider the form a_D which we define as the restriction of a to the subspace $H_0^1(\Omega)$. We want to show that this is in fact the classical Dirichlet form. To do this, we choose $v\in C_c^\infty(\Omega)$. We can thus apply integration by parts and exchange the order of differentiation to see that

$$\begin{aligned}\int_\Omega \partial_1 u\partial_2\bar v\mathrm{d}x&=-\int_\Omega u\partial_1\partial_2\bar v\,\mathrm{d}x\\&=-\int_\Omega u\partial_2\partial_1\bar v\,\mathrm{d}x\\&=\int_\Omega \partial_2 u\partial_1\bar v\,\mathrm{d}x.\end{aligned}$$

The bilinear form

$$H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{K},\quad (u,v)\mapsto\int_\Omega\partial_1 u\partial_2\bar v\,\mathrm{d}x$$

is continuous with respect to the $H^1(\Omega)$-Norm and by combining this with the fact that $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we conclude from equation (1) that

$$a_D(u,v)=\int_\Omega\big(\partial_1 u\partial_1\bar v+\partial_2 u\partial_2\bar v\big)\,\mathrm{d}x$$

which is the classical Dirichlet form. Thus we have $A_D=-\Delta_D$.

As for part (b), we can in fact show that for all bounded Ω with C^2-boundary there exists a function u such that u is in $\operatorname{dom}(-\Delta_N)$ but not in $\operatorname{dom}(A_N)$. To do this, we fix a point $p_0\in\partial\Omega$. We suppose that $\partial\Omega$ is a normal C^2-graph around p_0, therefore we have an open set $W\subseteq\partial\Omega$ such that $W=\{(x,g(x));x\in W'\}$ for some open set $W'\subseteq\mathbb{R}$ and some C^2-Function $g\colon W'\to\mathbb{R}$. Let $\nu(x,g(x))$ denote the outer normal at the point $(x,g(x))\in W$. We note that from Remark 7.3, we have the formula

$$\nu(x,g(x))=\frac{1}{\sqrt{g(x)^2+1}}\begin{bmatrix}-g'(x)\\1\end{bmatrix}. \tag{2}$$

Before constructing the desired function u, we consider the mapping

$$\varphi\colon W'\times\mathbb{R}\to\mathbb{R}^2,\quad (x,t)\mapsto(x,g(x))+t\nu(x,g(x)).$$

Since g is supposed to be C^2 it follows from (2) that φ is C^1. Because $p_0\in W$ there is $x_0\in W'$ such that $p_0=(x_0,g(x_0))=\varphi(x_0,0)$. A straightforward calculation yields

$$\det(\mathrm{d}\varphi(x_0,0))=\sqrt{g'(x_0)^2+1}\neq 0 \tag{3}$$

so that, by the inverse function theorem, there are open neighbourhoods U and V in $\mathbb{R}^2$ of $(x_0, 0)$ and p_0, respectively, such that $\varphi\colon U \to V$ is a C^1-diffeomorphism. Note that $\varphi(U \cap (\mathbb{R} \times \{0\})) = V \cap \partial\Omega$.

In order to construct the function u let $\rho, \lambda \in C_c^\infty(\mathbb{R})$ be non-trivial such that $\rho'(x_0) \neq 0, \lambda(0) = 1, \lambda'(0) = 0$, and

$$\rho \otimes \lambda\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \rho(x)\lambda(y)$$

satisfies $\operatorname{spt}(\rho \otimes \lambda) \subseteq U$. Now we set $u := (\rho \otimes \lambda) \circ \varphi^{-1} \in C_c^1(V)$ and extend this function by zero to $\mathbb{R}^2$. Clearly, $u \in C_c^1(\mathbb{R}^2)$ with $\operatorname{spt} u \subseteq V, u(p_0) = \rho(x_0)\lambda(0) = \rho(x_0)$.

For $q \in V \cap \partial\Omega$ we have $q = \varphi(x, 0) = (x, g(x))$ for suitable $x \in \mathbb{R}$ such that by the chain rule and $\lambda(0) = 1, \lambda'(0) = 0$

$$\begin{aligned}
\mathrm{d}u(q) &= \mathrm{d}(\rho \otimes \lambda)(\varphi^{-1}(q)) \circ \mathrm{d}(\varphi^{-1})(q) \\
&= \begin{bmatrix} \rho'(x)\lambda(0) & \rho(x)\lambda'(0) \end{bmatrix} \begin{bmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 \end{bmatrix}^{-1} (\varphi^{-1}(q)) \\
&= \frac{1}{\det(\mathrm{d}\varphi)(x, 0)} \begin{bmatrix} \rho'(x) & 0 \end{bmatrix} \begin{bmatrix} \partial_2\varphi_2(x, 0) & -\partial_2\varphi_1(x, 0) \\ -\partial_1\varphi_2(x, 0) & \partial_1\varphi_1(x, 0) \end{bmatrix} \\
&= \frac{\rho'(x)}{\det(\mathrm{d}\varphi)(x, 0)} \begin{bmatrix} \partial_2\varphi_2(x, 0) & -\partial_2\varphi_1(x, 0) \end{bmatrix} \\
&= \frac{\rho'(x)}{\det(\mathrm{d}\varphi)(x, 0)} \begin{bmatrix} \nu_2(x, g(x)) & -\nu_1(x, g(x)) \end{bmatrix}
\end{aligned} \tag{4}$$

where we used the inversion formula for 2×2-matrices in the third equality and the definition of φ in the last equality. For the normal derivative $\partial_\nu u$ we thus obtain

$$\partial_\nu u(q) = \frac{\rho'(x)}{\det(\mathrm{d}\varphi)(x, 0)} (\nu_2(x, g(x))\nu_1(x, g(x)) - \nu_1(x, g(x))\nu_2(x, g(x)) = 0.$$

Since $q \in V \cap \partial\Omega$ was chosen arbitrarily and because $\operatorname{spt} u \subseteq V$ it follows that $u \in \operatorname{dom}(-\Delta_N) = \{v \in H^1(\Omega); \partial_\nu v = 0\}$

On the other hand, denoting the conormal derivative by δ_ν, we obtain by (4) and (3)

$$\begin{aligned}
\delta_\nu u(p_0) &= \delta_\nu u(p_0) - \partial_\nu u(p_0) = \mathrm{d}u(p_0)\big((M^\top - I)\nu(p_0)\big) \\
&= -\frac{\rho'(x_0)}{\det(\mathrm{d}\varphi)(x_0, 0)} |\nu(x_0, g(x_0))|^2 = -\frac{\rho'(x_0)}{\det(\mathrm{d}\varphi)(x_0, 0)} \neq 0
\end{aligned}$$

so that $u \notin \operatorname{dom}(A_N)$.

Remark. In order to emphasize that the property of Ω being a C^2-domain is essential for our proof we briefly sketch how to construct a C^1-domain where the above construction fails. Our construction of the function u relies on the fact that the mapping

$$\psi\colon W \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2, \quad (p, t) \mapsto p + t\nu(p) \tag{5}$$

is injective for some open subset $W \subseteq \partial\Omega$ and an $\varepsilon > 0$. We construct a C^1-domain where this is not the case.

Let $r\colon [0, 2\pi] \to \mathbb{R}$ be a continuously differentiable function with equal values and derivatives at 0 and 2π such that r' is nowhere $\frac{1}{2}$-Hölder. A way to construct such a function is to integrate a typical path of the Brownian motion between 0 and 2π and then subtract a quadratic correction term to obtain proper boundary conditions. We can assume that r has values in $[1/2, 3/2]$ because scaling and translating by non-zero constants does not change smoothness properties.

We let

$$\Omega := \{(x,y) \in \mathbb{R}^2; x^2 + y^2 < \big(r(\arg(x,y))\big)^2\},$$

where $\arg(x,y)$ is the angle between the positive x-axis and the vector (x,y) (and we set $\arg(0,0) = 0$ for the sake of well-definedness). Ω could be described as a "disc with a fuzzy boundary".

We claim:

i) Ω is a C^1-domain.

ii) For all open $W \subseteq \partial\Omega$ and all $\varepsilon > 0$, the map ψ given in (5) is not injective.

Ad i): This is a consequence of the implicit function theorem: The map

$$F \colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto x^2 + y^2 - \big(r(\arg(x,y))\big)^2$$

is C^1 off the origin, $\{(x,y) \in \mathbb{R}^2; F(x,y) = 0\} = \partial\Omega$, and $\frac{\partial}{\partial y} F(0, r(\pi/2)) \neq 0$. Thus $(0, r(\pi/2))$ has a neighbourhood $W \subseteq \partial\Omega$ such that W is the graph of a C^1 function $g \colon W' \to \mathbb{R}$ on an open interval W'. For all other boundary points, the same argument applies after a suitable rotation.

Ad ii): Upfront, we fix an open $W \subseteq \partial\Omega$ and $\varepsilon > 0$. We need to find the outer normal vector. Since there is a bijection $\gamma \colon [0, 2\pi) \to \partial\Omega$, $\gamma(\phi) := (r(\phi)\cos(\phi), r(\phi)\sin(\phi))$, we can parametrize the boundary by $\phi \in [0, 2\pi)$ and find the outer normal at the point corresponding to the angle ϕ as

$$\tilde{\nu}(\phi) := \nu(\gamma(\phi)) = \frac{1}{\sqrt{(r(\phi))^2 + (r'(\phi))^2}} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} r(\phi) \\ -r'(\phi) \end{bmatrix},$$

as one checks by computing $\tilde{\nu}(\phi) \cdot \gamma'(\phi) = 0$.

We note that $\tilde{\nu}$ is nowhere differentiable, because if $\tilde{\nu}$ was differentiable in ϕ_0, then

$$(r'(\phi))^2 = (r(\phi))^2 \Big(\Big(\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \cdot \tilde{\nu}(\phi) \Big)^{-2} - 1 \Big) \qquad (\phi \in [0, 2\pi))$$

would be differentiable in ϕ_0, too. But that contradicts the fact that r is not $\frac{1}{2}$-Hölder:

$$\limsup_{\phi \to \phi_0} \Big| \frac{(r'(\phi))^2 - (r'(\phi_0))^2}{\phi - \phi_0} \Big| = \limsup_{\phi \to \phi_0} \frac{|r'(\phi) - r'(\phi_0)|}{|\phi - \phi_0|^{1/2}} \frac{|r'(\phi) + r'(\phi_0)|}{|\phi - \phi_0|^{1/2}} = \infty.$$

In the light of Rademacher's theorem, which states that Lipschitz functions are differentiable almost everywhere, we see that the nowhere differentiable $\tilde{\nu}$ cannot be Lipschitz, not even locally. Thus, there is a dense set of points $\phi_0 \in [0, 2\pi]$ such that

$$\limsup_{\phi \to \phi_0} \Big| \frac{\tilde{\nu}(\phi) - \tilde{\nu}(\phi_0)}{\phi - \phi_0} \Big| = \infty.$$

Pick one of these ϕ_0 such that $p_0 := \gamma(\phi_0) \in W \subseteq \partial\Omega$. We immediately get a sequence $\phi_n \in [0, 2\pi] \setminus \{\phi_0\}$ with $\phi_n \to \phi_0$ as $n \to \infty$ and

$$\Big| \frac{\tilde{\nu}(\phi_n) - \tilde{\nu}(\phi_0)}{\phi_n - \phi_0} \Big| \geq n$$

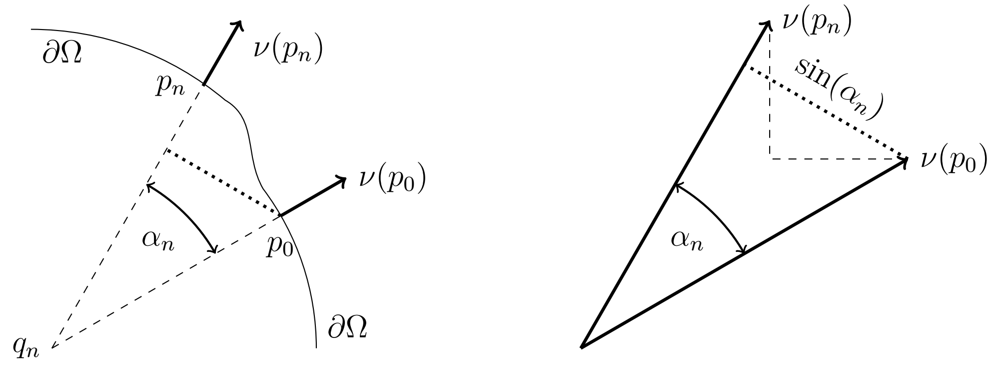

Figure 1: Left: The dotted line has length $|p_n - q_n| \sin(\alpha_n) \leq |p_n - p_0|$. Right: By Pythagoras, each of the dashed lines has length $|\nu(p_n) - \nu(p_0)|/\sqrt{2} \leq \sin(\alpha_n)$.

for $n \in \mathbb{N}$. This implies in particular $\nu(p_n) \neq \nu(p_0)$, for $p_n := \gamma(\phi_n)$, so that the intersection of the lines

$$\{q_n\} := \big(p_0 + \mathbb{R}\nu(p_0)\big) \cap \big(p_n + \mathbb{R}\nu(p_n)\big)$$

is well-defined. We show that the infimum $\tau := \inf\{|q_n - p_0|; n \in \mathbb{N}\}$ of the distance of the intersection of lines to the boundary point p_0 vanishes. Denote by α_n the angle between $\nu(p_0)$ and $\nu(p_n)$. From figure 1, we learn that

$$|p_n - p_0| \geq |q_n - p_0| \sin(\alpha_n) \geq \tau |\nu(p_n) - \nu(p_0)|/\sqrt{2}.$$

This rewrites to

$$\tau \leq \sqrt{2} \frac{|p_n - p_0|}{|\nu(p_n) - \nu(p_0)|} = \sqrt{2} \frac{|\gamma(\phi_n) - \gamma(\phi_0)|}{|\phi_n - \phi_0|} \frac{|\phi_n - \phi_0|}{|\tilde{\nu}(\phi_n) - \tilde{\nu}(\phi_0)|} \leq \frac{2|\gamma'(\phi_0)|}{n}$$

for all n sufficiently large. Therefore, $\tau = 0$. Because $p_n \to p_0$, this implies in particular that there exists an $n \in \mathbb{N}$ such that $p_n \in W$ and $t, s \in (-\varepsilon, \varepsilon)$ with $\psi(p_0, t) = q_n = \psi(p_n, s)$. Thus, ψ is not injective.

Exercise 11.3 Let a_{jk}, b_j, c_j, d be as in Section 11.2.

(a) Assume additionally that $b \in C_b^1(\Omega; \mathbb{K}^n)$. Let the formal elliptic operators $\mathcal{A}_1, \mathcal{A}_2$, in the sense of (11.2), be defined by

$$\mathcal{A}_1 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b \cdot (\nabla u), \ \mathcal{A}_2 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \operatorname{div}(bu) - (\operatorname{div} b)u.$$

Show that $A_{1,D} = A_{2,D}$.

(b) Assume additionally that $b, c \in C_b^1(\Omega; \mathbb{K}^n), c = b$, and let $\mathcal{A}$ be defined by

$$\mathcal{A}u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b \cdot (\nabla u) - \operatorname{div}(cu).$$

Show that A_D is associated with a formal elliptic operator without drift terms.

Solution. We first prove two auxiliary results:

i) For $u, g \in L_2(\Omega)$ and $1 \le j \le n$ we have $\partial_j u = g$ if and only if

$$\forall v \in H_0^1(\Omega)\colon\ -\int_\Omega gv\,\mathrm{d}x = \int_\Omega u\,\partial_j v\,\mathrm{d}x.$$

ii) Let $f \in C_b^1(\Omega), u \in H^1(\Omega)$. Then $fu \in H^1(\Omega)$ and $\partial_j(fu) = (\partial_j f)u + f\partial_j u\,(1 \le j \le n)$. Moreover, if $u \in H_0^1(\Omega)$ the same holds for fu.

Proof. i) By definition of derivatives in the sense of distributions the condition in i) is clearly sufficient for $g = \partial_j u$. Now, if $g = \partial_j u$ then the two linear mappings

$$\Phi_1\colon H_0^1(\Omega) \to \mathbb{K}, \quad v \mapsto -\int_\Omega gv\,\mathrm{d}x$$

and

$$\Phi_2\colon H_0^1(\Omega) \to \mathbb{K}, \quad v \mapsto \int_\Omega u\partial_j v\,\mathrm{d}x$$

are $\|\cdot\|_{H^1(\Omega)}$-continuous and coincide on $C_c^\infty(\Omega)$. Thus, they also coincide on the $\|\cdot\|_{H^1(\Omega)}$-closure of $C_c^\infty(\Omega)$, i. e.

$$\forall\, v \in H_0^1(\Omega)\colon\ -\int_\Omega gv\,\mathrm{d}x = \int_\Omega u\,\partial_j v\,\mathrm{d}x.$$

ii) Clearly, $f\varphi \in H_0^1(\Omega)$ for every $\varphi \in C_c^\infty(\Omega)$. Thus, applying i) we obtain

$$\begin{aligned}\forall\, \varphi \in C_c^\infty(\Omega)\colon\ -\int_\Omega u\,f\,\partial_j\varphi\,\mathrm{d}x &= -\int_\Omega u\big(\partial_j(f\varphi) - \varphi\,\partial_j f\big)\,\mathrm{d}x \\ &= \int_\Omega (\partial_j u\ f + u\,\partial_j f)\varphi\,\mathrm{d}x.\end{aligned}$$

Since $f \in C_b^1(\Omega)$ we have $\partial_j u\ f + u\,\partial_j f \in L_2(\Omega)$ showing $fu \in H^1(\Omega)$ and $\partial_j(fu) = (\partial_j f)u + f\partial_j u$. In fact, it follows that

$$M_f\colon H^1(\Omega) \to H^1(\Omega), \quad u \mapsto fu$$

is a continuous linear mapping. Obviously, $fu \in H_c^1(\Omega)$ whenever $u \in H_c^1(\Omega)$ so that the continuity of M_f implies $M_f(H_0^1(\Omega)) \subseteq H_0^1(\Omega)$. □

a) For every $u \in H_0^1(\Omega)$ we have by part ii) of the auxiliary results

$$\mathrm{div}(bu) - (\mathrm{div}\ b)u = \sum_{j=1}^n \partial_j(b_j u) - (\mathrm{div}\ b)u = (\mathrm{div}\ b)u + b\cdot\nabla u - (\mathrm{div}\ b)u = b\cdot\nabla u.$$

For $u \in \mathrm{dom}(A_{1,D})$ and $v \in H_0^1(\Omega)$ we therefore have

$$\begin{aligned}\int_\Omega A_{1,D}u\,\overline{v}\,\mathrm{d}x &= \int_\Omega\Big(\sum_{j,k=1}^n a_{jk}\partial_k u\overline{\partial_j v} + (b\cdot\nabla u)\overline{v}\Big)\,\mathrm{d}x \\ &= \int_\Omega\Big(\sum_{j,k=1}^n a_{jk}\partial_k u\overline{\partial_j v} + \mathrm{div}(bu)\overline{v} - (\mathrm{div}\,b)u\overline{v}\Big)\,\mathrm{d}x,\end{aligned}$$

so that $u \in \operatorname{dom}(A_{2,D})$ and $A_{2,D}u = A_{1,D}u$. Analogously, one shows $\operatorname{dom}(A_{2,D}) \subseteq \operatorname{dom}(A_{1,D})$ which proves a).

b) Because $b = c$, by part ii) of the auxiliary results

$$b \cdot (\nabla u) - \operatorname{div}(cu) = b \cdot (\nabla u) - (\operatorname{div} b)u - b \cdot \nabla u = -(\operatorname{div} b)u$$

for every $u \in H_0^1(\Omega)$. Therefore, A_D does not contain drift terms.

Exercise 11.4 Prove Remark 11.17(b): Let Ω be bounded with C^1-boundary and let V be a closed vector sublattice of $H^1(\Omega)$ containing $H_0^1(\Omega)$ such that $u \wedge 1 \in V$ whenever $u \in V$. Moreover, let $\mathbf{1}_\Omega \in V$ and assume that $\operatorname{div} b = d$ on Ω, $b \cdot \nu = 0$ on $\partial\Omega$. Then T_V is stochastic, i. e. $\|T_V(t)u\|_1 = \|u\|_1$ for all $0 \leq u \in L_2 \cap L_1(\Omega), t \geq 0$.

Solution. For $u \in V$ we have

$$\begin{aligned} a(u, \mathbf{1}_\Omega) &= \int_\Omega \Big(\sum_{j,k=1}^n a_{jk} \partial_k u \overline{\partial_j \mathbf{1}_\Omega} + \sum_{j=1}^n (b_j \partial_j u \overline{\mathbf{1}_\Omega} + c_j u \overline{\partial_j \mathbf{1}_\Omega}) + du \overline{\mathbf{1}_\Omega} \Big) \, \mathrm{d}x \\ &= \int_\Omega \Big(\sum_{j=1}^n b_j \partial_j u + du \Big) \, \mathrm{d}x \\ &= \int_\Omega \Big(- \sum_{j=1}^n \partial_j b_j + d \Big) u \, \mathrm{d}x + \int_{\partial\Omega} (b \cdot \nu) u \, \mathrm{d}\sigma = 0, \end{aligned}$$

where in the third equality we used Gauss' Theorem as in the proof of Proposition 11.16. Thus, $a^*(\mathbf{1}_\Omega, u) = 0$ for every $u \in V$. From Theorem 6.10 it follows that $\mathbf{1}_\Omega \in \operatorname{dom}(A_V^*)$ and $A_V^* \mathbf{1}_\Omega = 0$. Since by Theorem 10.9 A_V^* is the generator of the adjoint C_0-semigroup $(T_V^*(t))_{t \geq 0}$ it follows from Theorem 1.10 a) together with Theorem 1.9 b) that $T_V^*(t)\mathbf{1}_\Omega = \mathbf{1}_\Omega$ for every $t \geq 0$, i. e. $(T_V^*(t))_{t \geq 0}$ is Markovian.

By Proposition 11.16 b) T_V is substochastic. A standard application of the Hahn-Banach Theorem together with the fact that $(L_\infty(\Omega), \|\cdot\|_\infty)$ is isometrically isomorphic to the dual of $(L_1(\Omega), \|\cdot\|_1)$ in the canonical way now gives for $0 \leq u \in L_2 \cap L_1(\Omega), t \geq 0$

$$\begin{aligned} \|u\|_1 \geq \|T_V(t)u\|_1 &= \sup_{\|v\|_\infty \leq 1} \Big| \int_\Omega T_V(t)u \, v \, \mathrm{d}x \Big| \geq \Big| \int_\Omega T_V(t)u \, \mathbf{1}_\Omega \, \mathrm{d}x \Big| = \Big| \int_\Omega u \, T_V^*(t)\mathbf{1}_\Omega \, \mathrm{d}x \Big| \\ &= \Big| \int_\Omega u \, \mathrm{d}x \Big| = \int_\Omega |u| \, \mathrm{d}x = \|u\|_1. \end{aligned}$$

Hence, T_V is indeed stochastic.

Exercise 11.5

(a) Let $\mathbb{K} = \mathbb{R}$, (Ω, μ) a measure space, $H := L_2(\mu)$, V, W Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, $W \overset{\mathrm{d}}{\hookrightarrow} H$ and let $a \colon V \times V \to \mathbb{R}$, $b \colon W \times W \to \mathbb{R}$ be continuous bilinear forms, both H-elliptic. Denote by A the operator associated with a and by B the operator associated with b. Assume that the semigroups T generated by $-A$ and S generated by $-B$ are both positive. Assume

(i) $V \subseteq W$, and if $v \in V$, $w \in W$, $0 \leq w \leq v$, then $w \in V$;

(ii) $a(u, v) \geq b(u, v)$ for all $0 \leq u, v \in V$.

Show that $T(t) \leq S(t)$ $(t \geq 0)$.

(b) Prove Theorem 11.20.

Solution. The general idea for our proof of part (a) is to adapt the proof of Theorem 11.19. Thus we start by remarking that, due to the exponential formula of Theorem 2.12, it suffices to prove that

$$(\lambda + A)^{-1} f \leq (\lambda + B)^{-1} f$$

for sufficiently large λ and non-negative $f \in L^2(\mu)$. The forms $a + \lambda(\cdot\,,\,\cdot)$ and $b + \lambda(\cdot\,,\,\cdot)$ fullfill all the assumptions made in the exercise, therefore we can suppose without loss of generality that $\lambda = 0$. In particular, if the semigroup $T(t)$ associated with a is positive, so is the semigroup $e^{-\lambda t}T(t)$ associated with $a + \lambda(\cdot\,,\,\cdot)$. By the same argument, we can assume that b is coercive – if b is just H-elliptic but not coercive, we simply add a suitable λ to a and b.

Therefore, we now commence with the main part of our proof by taking an arbitrary positive $f \in L^2(\mu)$ and defining $u = A^{-1}f$, $v = B^{-1}f$. We have to show that $u \leq v$. We rewrite u as follows:

$$u = R(0, -A)f = \int_0^\infty T(t)f\mathrm{d}t = \lim_{r\to\infty} r \cdot \frac{1}{r}\int_0^r T(t)f\mathrm{d}t.$$

By Lemma 9.2, $\frac{1}{r}\int_0^r T(t)f\mathrm{d}t \geq 0$, and thus $r \cdot \frac{1}{r}\int_0^r T(t)f\mathrm{d}t = \int_0^r T(t)f\mathrm{d}t \geq 0$. Since the positive cone is closed, it follows that $u \geq 0$. By the same argument we conclude that $v \geq 0$ as well. Thus we have that $0 \leq (u-v)^+ \leq u$ and therefore $(u-v)^+ \in V$ by the ideal property of V. For any positive $g \in V$ we have

$$b(u, g) \leq a(u, g) = (f, g) = b(v, g)$$

and therefore

$$b(u - v, g) \leq 0, \text{ for all positive } g \in V$$

Specializing to $g = (u-v)^+$, we obtain

$$\begin{aligned}
0 &\geq b(u - v, (u - v)^+) \\
&= b((u-v)^+, (u-v)^+) - b((u-v)^-, (u-v)^+) \\
&= b((u-v)^+, (u-v)^+) - b((v-u)^+, (v-u)^-) \\
&\geq b((u-v)^+)
\end{aligned}$$

where the last inequality is obtained by using the fact that from the positivity of S, it follows that $b((v-u)^+, (v-u)^-) \leq 0$, see Theorem 10.12(b). Using the coercivity of b, we have that $b((u-v)^+) = 0$, again by coercivity we obtain that $(u-v)^+ = 0$ and thus we can conclude that $u \leq v$.

Part (b) is now a simple application of part (a). It was already shown in the lecture in Proposition 11.13 that the semigroups in question are positive and the properties (i) and (ii) are a direct consequence of the definition of A_V and A_D.