Solution to the Exercises of Lecture 10

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Exercise 10.1: For this exercise let S be the strip

$$S := \{z \in \mathbb{C}; -1/2 < Re\, z < 1/2\}.$$

Let $h : \overline{S} \mapsto \mathbb{C}$ be continuous and holomorphic on S. Assume that h is bounded on ∂S, and that there exists $\alpha < \pi$ such that

$$|h(z)| \leq c e^{e^{\alpha |Im\, z|}} \qquad (z \in S).$$

Show that h is bounded by $\|h\|_{\partial S}$.

Solution: For $n \in \mathbb{N}$, the function $\psi_n(z) := e^{-\frac{1}{n}(e^{i\beta z}+e^{-i\beta z})}$, with $\alpha < \beta < \pi$, is continuous on $\overline{S}$ and holomorphic on S. By the maximum principle one obtains

$$\|\psi_n h\|_{\overline{S}_k} \leq \|\psi_n h\|_{\partial S_k},$$

where $S_k := \{z \in S : |Im\, z| < k\}$, $k \in \mathbb{N}$. Observe that $\|\psi_n h\|_{\{z \in \partial S_k : |Im\, z| = k\}} \to 0\,(k \to \infty)$. In fact,

$$\begin{aligned} |\psi_n h(z)| \leq & c \left| e^{-\frac{1}{n}(e^{i\beta z}+e^{-i\beta z})} \right| e^{e^{\alpha |Im\, z|}} \\ = & c\, e^{-\frac{1}{n}(e^{-\beta Im\, z}+e^{\beta Im\, z})\cos(\beta |Re\, z|)} e^{e^{\alpha |Im\, z|}} \\ \leq & c\, e^{-\frac{\cos(\beta/2)}{n}(e^{-\beta Im\, z}+e^{\beta Im\, z})} e^{e^{\alpha |Im\, z|}} \end{aligned}$$

Hence,

$$\begin{aligned} \|\psi_n h\|_{\{z \in \partial S_k : |Im\, z| = k\}} & \leq c\, e^{-\frac{\cos(\beta/2)}{n}(e^{-\beta k}+e^{\beta k})} e^{e^{\alpha k}} \\ & = c\, e^{-\frac{\cos(\beta/2)}{n} e^{-\beta k}} e^{e^{\beta k}\left(-\frac{\cos(\beta/2)}{n}+e^{(\alpha-\beta)k}\right)} \end{aligned}$$

For big k, we have that $e^{(\alpha-\beta)k} < \varepsilon$. Thus, choosing $\varepsilon < \frac{\cos(\beta/2)}{2n}$, so that $e^{(\alpha-\beta)k} - \frac{\cos(\beta/2)}{n} < -\varepsilon$, we have that, in the limit as $k \to \infty$, the right-hand side of the inequality tends to 0.
Therefore,

$$\|\psi_n h\|_{\overline{S}} \leq \|\psi_n h\|_{\partial S}.$$

Letting $n \to \infty$ we obtain the assertion.

Exercise 10.2: Let (Ω, μ) be a measure space. Show that

$$\left(\|\frac{1}{2}(f+g)\|_p^p + \|\frac{1}{2}(f-g)\|_p^p \right)^{\frac{1}{p}} \leq 2^{-\frac{1}{p}} \left(\|f\|_p^p + \|g\|_p^p \right)^{\frac{1}{p}}$$

for all $f, g \in L_p(\mu)$, $2 \leq p \leq \infty$.

Solution: For $2 \leq p \leq \infty$ consider

$$\begin{array}{rcl} T_p : L_p(\mu) \times L_p(\mu) & \longrightarrow & L_p(\mu) \times L_p(\mu) \\ (f,g) & \mapsto & \left(\frac{1}{2}(f+g), \frac{1}{2}(f-g)\right). \end{array}$$

We endow $L_p(\mu) \times L_p(\mu)$, $2 \leq p \leq \infty$, with the norms

$$\begin{array}{rcll} \|(f,g)\|_{p\times p} & = & (\|f\|_p^p + \|g\|_p^p)^{\frac{1}{p}}, & f, g \in L_p(\mu),\ 2 \leq p < \infty, \\ \|(f,g)\|_\infty & = & \max(\|f\|_\infty, \|g\|_\infty), & f, g \in L_\infty(\mu). \end{array}$$

We first estimate the norm of the operator T on $L_2(\mu) \times L_2(\mu) = L_2(\mu \otimes \mu)$ and $L_\infty(\mu) \times L_\infty(\mu) = L_\infty(\mu \otimes \mu)$. For $p = 2$ one has

$$\begin{array}{rcl} \|T(f,g)\|_{2\times 2} & = & \frac{1}{2}\|f+g, f-g\|_{2\times 2} \\ & = & \frac{1}{2}\left(\|f+g\|_2^2 + \|f-g\|_2^2\right)^{\frac{1}{2}} \\ & = & \frac{1}{2}\left(\|f\|^2 + 2(f|g) + \|g\|^2 + \|f\|^2 - 2(f|g) + \|g\|^2\right)^{\frac{1}{2}} \\ & = & \frac{1}{\sqrt{2}}(\|f\|_2^2 + \|g\|_2^2)^{\frac{1}{2}} \\ & = & 2^{-\frac{1}{2}}\|(f,g)\|_{2\times 2}. \end{array}$$

And for $p = \infty$, one obtains

$$\begin{array}{rcl} \|T(f,g)\|_\infty & = & \frac{1}{2}\|f+g, f-g\|_\infty \\ & = & \frac{1}{2}\max\left(\|f+g\|_\infty, \|f-g\|_\infty\right) \\ & \leq & \frac{1}{2}\max\left(\|f\|_\infty + \|g\|_\infty, \|f\|_\infty + \|g\|_\infty\right) \\ & = & \frac{1}{2}(\|f\|_\infty + \|g\|_\infty) \\ & \leq & \max(\|f\|_\infty, \|g\|_\infty) \\ & = & \|(f,g)\|_\infty. \end{array}$$

Let $p \in (2, +\infty)$, $p_0 = \infty$, $p_1 = 2$, $M_0 = 1$, $M_1 = 2^{-\frac{1}{2}}$ and $\tau = \frac{2}{p} \in (0,1)$, then by Riesz-Thorin Theorem, see Corollary 10.6, one has

$$\|T(f,g)\|_p \leq M_\tau \|(f,g)\|_p, \qquad f, g \in S(\mathcal{A}_c) \tag{1}$$

with $M_\tau = M_0^{1-\tau} M_1^\tau = 2^{-\frac{1}{p}}$ and

$$\mathcal{A}_c = \{A \subset \Omega, \mathit{measurable} : \mu(A) < \infty\}.$$

By density of $S(\mathcal{A}_c)$ on $L_p(\mu)$, (1) holds for all $f, g \in L_p(\mu)$. Therefore,

$$\left(\|\frac{1}{2}(f+g)\|_p^p + \|\frac{1}{2}(f-g)\|_p^p\right)^{\frac{1}{p}} \leq 2^{-\frac{1}{p}}\left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}}, \quad f, g \in L_p(\mu).$$

Exercise 10.3: (a) Let $p \in (1, \infty)$, $r \in [0, \infty)$. Show that

$$r = \inf_{\alpha \in \mathbb{Q} \cap (0,\infty)} \left(\frac{1}{p} \alpha^{1-p} r^p + (1 - \frac{1}{p})\alpha \right). \tag{2}$$

(b) Let (Ω, μ) be a measure space, and let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian and substochastic. Show that S is L_p-contractive for all $p \in (1, \infty)$.
(c) Let (Ω, μ) be a measure space, and let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian, and assume that there exists $c > 0$ such that $\frac{1}{c}S$ is substochastic. Show that S interpolates to an operator $S_p \in \mathcal{L}(L_p(\mu))$ with $\|S_p\| \leq c^{\frac{1}{p}}$, for $1 < p < \infty$.

Solution: (a) Let p' be the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and let $\alpha \in \mathbb{Q} \cap (0, \infty)$. Applying Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

with $a = \alpha^{\frac{1-p}{p}} r$ and $b = \alpha^{\frac{p-1}{p}}$, one obtains

$$\begin{aligned} r = ab \quad &\leq \quad \frac{1}{p} (\alpha^{\frac{1-p}{p}} r)^p + \left(1 - \frac{1}{p}\right) \left(\alpha^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} \\ &\leq \quad \frac{1}{p} \alpha^{1-p} r^p + \left(1 - \frac{1}{p}\right) \alpha. \end{aligned}$$

Passing to the infinimum, one gets

$$r \leq \inf_{\alpha \in \mathbb{Q} \cap (0,\infty)} \left(\frac{1}{p} \alpha^{1-p} r^p + (1 - \frac{1}{p})\alpha \right).$$

On the other hand, consider $\alpha_n = \dfrac{E(nr)}{n} \in \mathbb{Q}$ for $n \geq 1$, where $E(nr)$ is the integer part of nr. Since $\lim_{n \longrightarrow \infty} \alpha_n = r$, it follows that

$$\lim_{n \longrightarrow \infty} \left(\frac{1}{p} \alpha_n^{1-p} r^p + \left(1 - \frac{1}{p}\right) \alpha_n \right) = r.$$

Finally,

$$r = \inf_{\alpha \in \mathbb{Q} \cap (0,\infty)} \left(\frac{1}{p} \alpha^{1-p} r^p + (1 - \frac{1}{p})\alpha \right).$$

(b) Let $u \in L_2(\mu) \cap L_p(\mu)$ and A be a measurable subset of Ω with finite measure. Then, by (2) (with $r = |u(x)|\chi_A(x)$) , it follows that

$$|u|\chi_A \leq \frac{1}{p} \alpha^{1-p} |u|^p \chi_A + (1 - \frac{1}{p}) \alpha \chi_A$$

for $\alpha \in \mathbb{Q} \cap (0, \infty)$. So, by the positivity and the L^∞-contractivity of S, one has

$$\begin{aligned} S(|u|\chi_A) \quad &\leq \quad \frac{1}{p} \alpha^{1-p} S(|u|^p \chi_A) + (1 - \frac{1}{p}) \alpha S \chi_A \\ &\leq \quad \frac{1}{p} \alpha^{1-p} S(|u|^p \chi_A) + (1 - \frac{1}{p}) \alpha. \end{aligned}$$

So, passing to the infinimum over $\alpha \in \mathbb{Q} \cap (0, \infty)$, one obtains

$$S(|u|\chi_A) \leq (S(|u|^p\chi_A))^{\frac{1}{p}}.$$

Let now $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable sets with finite measure such that $\cup_{n\in\mathbb{N}} A_n = \Omega$. Then, $|u|\chi_{A_n} \longrightarrow |u|$ and $|u|^p\chi_{A_n} \longrightarrow |u|^p$ as $n \longrightarrow \infty$. Since $S \in \mathcal{L}(L_2(\mu))$ and S can be extended to a bounded linear operator on $L_1(\mu)$ (using the fact that S is substochastic), it follows that

$$\lim_{n\longrightarrow\infty} S(|u|\chi_{A_n}) = S|u| \; and \; \lim_{n\longrightarrow\infty} S(|u|^p\chi_{A_n}) = S|u|^p.$$

Finally one has

$$S(|u|) \leq (S(|u|^p))^{\frac{1}{p}}, \qquad \forall u \in L_2(\mu) \cap L_p(\mu). \tag{3}$$

Therefore, by using the L_1-contractivity of S and the fact that $|Su| \leq S|u|$, one has

$$\int_\Omega |Su|^p d\mu \leq \int_\Omega (S|u|)^p d\mu \leq \int_\Omega S|u|^p d\mu \leq \int_\Omega |u|^p d\mu.$$

Thus,

$$\|Su\|_p^p \leq \|u\|_p^p, \quad \forall u \in L_2(\mu) \cap L_p(\mu).$$

By the density of $L_2(\mu) \cap L_p(\mu)$ in $L_p(\mu)$, the above estimate shows that S can be extended to a contractive operator over $L_p(\mu)$.

(b) Let $u \in L_2(\mu) \cap L_p(\mu)$, then from (3) one has $(S(|u|))^p \leq S(|u|^p)$. Thus,

$$\int_\Omega |Su|^p d\mu \leq \int_\Omega (S|u|)^p d\mu \leq \int_\Omega S|u|^p d\mu \leq c\int_\Omega |u|^p d\mu.$$

Therefore,

$$\|Su\|_p^p \leq c\|u\|_p^p, \quad \forall u \in L_2(\mu) \cap L_p(\mu).$$

By the density of $L_2(\mu) \cap L_p(\mu)$ in $L_p(\mu)$, it follows that S interpolates to an operator $S_p \in \mathcal{L}(L_p(\mu))$ and $\|S_p\| \leq c^{1/p}$ for all $1 < p < \infty$.

Exercise 10.4: Let the hypotheses be as in Exercise 9.5, and additionally $b \in C^1(\Omega)$. Assume that $\omega \in \mathbb{R}$ is such that div $b(x) \leq \omega$ for all $x \in \Omega$.

(a) Show that $\|T(t)u\|_1 \leq e^{\omega t}\|u\|_1$ for $u \in L_2(\Omega) \cap L_1(\Omega)$ and $t \geq 0$, where $T(\cdot)$ is the C_0-semigroup generated by the operator $-A$.

(b) Compute estimates for $\|T_p(t)\|$ in terms of $\omega := \sup \operatorname{div} b$ for $t \geq 0$, $1 \leq p < \infty$, where $T_p(\cdot)$ is the interpolated semigroup on $L_p(\Omega)$, analogous to Theorem 10.15(b).

Solution:

(a) In order to show that $\|T(t)u\|_1 \leq e^{\omega t}\|u\|_1$, we show that $e^{-\omega t}T(t)$ is substochastic. To do so we use the characterization in Theorem 10.12.(d). Hence we have to show that $a_\omega((u-1)^+, u \wedge 1) \geq 0$, where a_ω is the form associated with $A + \omega I$.

Let us consider $V = H_0^1(\Omega)$ and $u \in C_c^1(\Omega)$. Observe that if $u \in C_c^1(\Omega)$ then $u \wedge 1 \in V$, cf. [1, Proposition 4.11, p.113], and also $(u-1)^+ \in V$, since $u - u \wedge 1 = (u-1)^+$.

Integrating by parts and using the fact that div $b(x) \leq \omega$, we have

$$\begin{aligned}
a_\omega((u-1)^+, u \wedge 1) &= \int_\Omega \nabla(u-1)^+ \cdot \nabla(u \wedge 1)\, dx + \int_\Omega b \cdot \nabla(u-1)^+(u \wedge 1)\, dx \\
&\quad + \int_\Omega \omega(u-1)^+(u \wedge 1)\, dx \\
&\geq \int_\Omega (u-1)^+(\omega - \text{div } b)(u \wedge 1)\, dx - \int_\Omega (u-1)^+ b \cdot \nabla(u \wedge 1)\, dx \\
&= \int_\Omega (u-1)^+(\omega - \text{div } b)(u \wedge 1)\, dx \geq 0.
\end{aligned}$$

(b) We know that $\|T(t)\|_\infty \leq 1$ (Exercise 9.5(b)) and $\|T(t)\|_1 \leq e^{\omega t}$ with $\omega = \sup \text{div}(b)$. Applying Exercise 10.3.(c) we have

$$\|T(t)\|_p \leq e^{\frac{\omega t}{p}}, \qquad t \geq 0.$$

References

[1] E.M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Math. Soc. Monographs, Vol. 31. Princeton Univ. Press 2004.