SOLUTIONS TO THE EXERCISES OF LECTURE 9

ULM TEAM

Exercise 9.1. *Let (Ω, μ) be a measure space, $1 \le p \le \infty$, and let $A \in \mathcal{L}(L_p(\mu))$ be positive, i.e., $Au \ge 0$ for all $u \in L_p(\mu)$ with $u \ge 0$.*
(a) Show that $|Au| \le A\,|u|$ for all $u \in L_p(\mu)$.
(b) Show that

$$\|A\| = \sup\{\|Au\|_p\,; u \in L_p(\mu), u \ge 0, \|u\|_p \le 1\}.$$

Proof. (a) We first show the assertion for $\mathbb{K} = \mathbb{R}$: In this case we have

$$|Af| = |Af^+ - Af^-| \le |Af^+| + |Af^-| = Af^+ + Af^- = A|f|.$$

In the case $\mathbb{K} = \mathbb{C}$, the proof is a bit more involved: First observe that for each $z \in \mathbb{C}$ we have

$$|z| = \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} (\operatorname{Re} e^{-i\theta} z) = \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} (\cos\theta \operatorname{Re} z + \sin\theta \operatorname{Im} z).$$

This implies that we also have

$$|f| = \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} (\cos\theta \operatorname{Re} f + \sin\theta \operatorname{Im} f) \tag{1}$$

for each $f \in L_p(\mu; \mathbb{C})$; here the supremum is taken in the ordered space $L_p(\mu; \mathbb{R})$, where $\le$ is the canonical ordering. Using that A is positive and formula (1) we obtain

$$\begin{aligned}
|Af| &= \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} (\cos\theta \operatorname{Re} Af + \sin\theta \operatorname{Im} Af) = \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} (\cos\theta A \operatorname{Re} f + \sin\theta A \operatorname{Im} f)\\
&= \sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} A(\cos\theta \operatorname{Re} f + \sin\theta \operatorname{Im} f) \le A\Bigg(\sup_{\theta\in[0,2\pi]\cap\mathbb{Q}} \cos\theta \operatorname{Re} f + \sin\theta \operatorname{Im} f\Bigg)\\
&= A\,|f|\,.
\end{aligned}$$

(b) Let $\varepsilon > 0$ and choose $f \in L_p(\mu)$ such that $\|f\|_p = 1$ and $\|Af\|_p \ge \|A\| - \varepsilon$. Using (a) we obtain that $\|A\,|f|\|_p \ge \||Af|\|_p = \|Af\|_p \ge \|A\| - \varepsilon$. Since $|f|$ is positive and has norm 1, this shows that

$$\sup\{\|Au\|_p : u \in L_p(\mu),\ u \ge 0\ \ \|u\|_p \le 1\} \ge \|A\|\,.$$

The converse inequality is obvious. □

Exercise 9.2. *Let (Ω, μ) be a measure space, $\check{C} \subset \mathbb{K}$ convex and closed, $0 \in \check{C}$; let $\check{P} \colon \mathbb{K} \to \check{C}$ be the minimising projection. Then clearly*

$$C := \{u \in L_2(\mu); u(x) \in \check{C} \text{ for } \mu\text{-a.e. } x\} \ne \emptyset$$

is convex and closed. Show that the minimising projection $P \colon L_2(\mu) \to C$ is given by $(Pu)(x) = \check{P}(u(x))$ $(x \in \Omega)$.

Proof. Let $u \in L_2(\mu)$ and define $w(x) = \check{P}u(x)$ for almost all $x \in \Omega$. We now show that $\hat{P}$ is a (non-linear) contraction and therefore a fortiori continuous. This implies that w is measurable as the composition of a continuous and a measurable function and further that $w \in L_2(\mu)$. By the characterization of the minimising projection given in (9.1) in Lecture 9 we have for $y_1, y_2 \in \mathbb{K}$

$$\mathrm{Re}\,(\hat{P}y_2 - \hat{P}y_1) \cdot \overline{(y_1 - \hat{P}y_1)} \le 0$$

or equivalently

$$-\mathrm{Re}\,\hat{P}y_2\overline{\hat{P}y_1} \le -|\hat{P}y_1|^2 + (\hat{P}y_1 - \hat{P}y_2)\overline{y_1}.$$

By interchanging the two variables, we further obtain the analogous inequality

$$-\mathrm{Re}\,\hat{P}y_1\overline{\hat{P}y_2} \le -|\hat{P}y_2|^2 + (\hat{P}y_2 - \hat{P}y_1)\overline{y_2}.$$

Using these two inequalities, we obtain the estimate

$$\begin{aligned}|\hat{P}y_1 - \hat{P}y_2|^2 &= |\hat{P}y_1|^2 + |\hat{P}y_2|^2 - 2\,\mathrm{Re}\,\hat{P}y_1\hat{P}y_2 = (\hat{P}y_1 - \hat{P}y_2)(y_1 - y_2)\\ &\le |\hat{P}y_1 - \hat{P}y_2|\,|y_1 - y_2|\,,\end{aligned}$$

which immediately implies that the minimising projection is a (non-linear) contraction.

It therefore remains to to show that $w = Pu$. For almost all $x \in \Omega$ and all $z \in \check{C}$ we have

$$\mathrm{Re}\,(z - \check{P}u(x)) \cdot \overline{(u(x) - \check{P}u(x))} \le 0$$

by the characterization of the minimising projection given in (9.1) in Lecture 9. Now, let $v \in C$. We have $v(x) \in \check{C}$ for almost all $x \in \Omega$, and hence

$$\mathrm{Re}(v - w|u - w) = \int_\Omega \mathrm{Re}\ (v(x) - \check{P}u(x)) \cdot \overline{(u(x) - \check{P}u(x))}\,d\mu \le \int_\Omega 0\,d\mu = 0.$$

Using again the characterization of Pu given by inequality (9.1), this implies that $w = Pu$. □

For the solution of the next two Exercises 9.3 and 9.4 we need the following result:

Proposition. *Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and has C^1-boundary. Then we have for all $u, v \in H^1(\Omega; \mathbb{R})$:*

(a) If $0 \ge u$, then $\mathrm{tr}(u) \ge 0$.
(b) $\mathrm{tr}(u^+) = (\mathrm{tr}(u))^+$, $\mathrm{tr}(u^-) = (\mathrm{tr}(u))^-$, $\mathrm{tr}(|u|) = |\,\mathrm{tr}(u)|$.
(c) $\mathrm{tr}(u \wedge v) = \mathrm{tr}(u) \wedge \mathrm{tr}(v)$.
(d) If $uv = 0$, then $\mathrm{tr}(u)\,\mathrm{tr}(v) = 0$.

Proof. (a)–(c) All three assertions are clear for $u, v \in C^1(\overline{\Omega}; \mathbb{R})$. For $u, v \in H^1(\Omega; \mathbb{R})$ the assertions then follow from the density assertion in Theorem 7.7.

(d) If $uv = 0$, then $|u| \wedge |v| = 0$. Hence, (b) and (c) imply that $|\,\mathrm{tr}(u)| \wedge |\,\mathrm{tr}(v)| = 0$ and thus $\mathrm{tr}(u)\,\mathrm{tr}(v) = 0$. □

Exercise 9.3. *Let Δ_β be the Robin Laplacian from Section 7.5.*

(a) Let β be real-valued. Show that the C_0-semigroup generated by Δ_β is positive.
(b) Let $\beta \ge 0$. Show that the C_0-semigroup generated by Δ_β is sub-Markovian.

Proof. Throughout the exercise we assume that $\mathbb{K} = \mathbb{R}$. By the proof of Theorem 7.15, $-\Delta_\beta$ is the operator associated with the form

$$a\colon H^1 \times H^1 \to \mathbb{R}, \quad a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \beta uv.$$

(a) Let C be the positive cone in $L^2(\Omega; \mathbb{R})$. We want to apply Theorem 9.20. The minimising projection P is given by $Pu = u^+$ for all $u \in L_2(\Omega; \mathbb{R})$. It follows from Theorem 9.14 that $P(H^1(\Omega; \mathbb{R})) \subset H^1(\Omega; \mathbb{R})$ and thus we only have to show that estimate (9.8) holds true. Using the notation $u^- := (-u)^+$, we obtain for all $u \in H^1(\Omega;\mathbb{R})$ that

$$a(Pu, u - Pu) = a(u^+, -u^-) = -\int_\Omega \nabla u^+ \cdot \nabla u^- - \int_{\partial\Omega} \beta u^+ u^- = 0.$$

For the last equality, we used that $\nabla u^+ \cdot \nabla u^- = \mathbb{1}_{[u>0]} \nabla u \cdot \mathbb{1}_{[u<0]} \nabla u = 0$ according to Theorem 9.14, as well as the fact that $\operatorname{tr}(u^+)\operatorname{tr}(u^-) = 0$ according to assertion (d) of the above proposition.

We showed that condition (ii) of Theorem 9.20 is fulfilled and thus conclude that the positive cone C is invariant with respect to the semigroup generated by Δ_β. Hence, the semigroup generated by the Robin Laplacian is positive.

(b) Now let $C := \{u \in L_2(\Omega;\mathbb{R}) : u \le \mathbb{1}\}$. We use Theorem 9.20 to show that C is invariant under the semigroup generated by Δ_β. The minimising projection P for the set C is given by $Pu = u \wedge \mathbb{1}$ for all $u \in L_2(\Omega;\mathbb{R})$. It thus follows from Theorem 9.14 that $PH^1(\Omega;\mathbb{R}) \subset H^1(\Omega;\mathbb{R})$ and it remains to show that the estimate (9.8) holds true. For all $u \in H^1(\Omega;\mathbb{R})$, we have $u - Pu = u - u \wedge \mathbb{1} = (u - \mathbb{1})^+$, and hence

$$\begin{aligned} a(Pu, u - Pu) &= \int_\Omega \nabla(u \wedge \mathbb{1}) \cdot \nabla(u - \mathbb{1})^+ + \int_{\partial\Omega} \beta(u \wedge \mathbb{1})(u - \mathbb{1})^+ = \\ &= \int_\Omega \mathbb{1}_{[u<1]} \nabla u \cdot \mathbb{1}_{[u>\mathbb{1}]} \nabla u + \int_{\partial\Omega} \beta(u \wedge \mathbb{1})(u - \mathbb{1})^+ = \int_{\partial\Omega} \beta(u \wedge \mathbb{1})(u - \mathbb{1})^+. \end{aligned}$$

The last integrand is, more precisely, given by $\beta(\operatorname{tr}(u) \wedge \mathbb{1})(\operatorname{tr}(u) - \mathbb{1})^+$; this follows from the above Proposition. Since $\beta \ge 0$, the integrand is clearly positive, so we conclude that $a(Pu, u - Pu) \ge 0$ for each $u \in H^1(\Omega;\mathbb{R})$. □

Exercise 9.4. *(a) Assume that H, V, a, j are as in Proposition 5.5, and such that minus the operator associated with (a, j) is a generator. Let $\emptyset \ne C \subseteq H$ be convex and closed, P the minimising projection onto C. Let $\hat{P}\colon V \to V$ be a mapping satisfying $Pj = j\hat{P}$. Further assume*

$$a(u, u - \hat{P}u) \ge 0 \qquad (u \in V).$$

Show that C is invariant under the C_0-semigroup associated with (a, j).

(b) Show that the C_0-semigroup generated by the Dirichlet-to-Neumann operator of Section 8.1 is sub-Markovian.

Proof. (a) Let A be the operator associated to the form a.

(i) Let us assume that $a(u, u - \hat{P}u) \ge 0$ for all $u \in V$. We show that the estimate (9.2) is fulfilled for the operator $-A$, so that we can apply Proposition 9.4. To this end, let $x \in D(A)$. Then there is $u \in V$ such that $j(u) = x$ and such that $a(u, v) = (Ax, j(v))$ for all $v \in V$. Hence, we have

$$\begin{aligned} \operatorname{Re}(-Ax|x - Px) &= -\operatorname{Re}(Ax|j(u)) + \operatorname{Re}(Ax, j\hat{P}u) = \\ &= -\operatorname{Re} a(u, u) + \operatorname{Re} a(u, \hat{P}u) = -\operatorname{Re} a(u, u - \hat{P}u) \le 0. \end{aligned}$$

Thus the estimate (9.2) is fulfilled for $\omega = 0$ and we conclude from Proposition 9.4 that the semigroup generated by $-A$ leaves C invariant.

(ii) Now, for the more ambitious part not stated above, now assume that $a(\hat{P}u, u - \hat{P}u) \ge 0$ for all $u \in V$. Again, we want to employ Proposition 9.4, so let $x \in D(A)$. As above, there is $u \in V$ such that $j(u) = x$ and such that $a(u, v) = (Ax, j(v))$ for all $v \in V$. Using the same computation as above, we obtain

$$\begin{aligned} &\operatorname{Re}(-Ax, x - Px) = -\operatorname{Re} a(u, u - \hat{P}u) = \\ &= -\operatorname{Re} a(\hat{P}u, u - \hat{P}u) - \operatorname{Re} a(u - \hat{P}u, u - \hat{P}u) \le -\operatorname{Re} a(u - \hat{P}u) \end{aligned}$$

Since a is j-elliptic, there are constants $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(v) + \omega ||j(v)||_H^2 \geq \alpha ||v||_V^2 \geq 0$$

for all $v \in V$. Hence, we conclude that

$$\operatorname{Re}(-Ax, x - Px) \leq \omega ||j(u - \hat{P}u)||_H^2 = \omega ||x - Px||_H^2.$$

Hence, estimate (9.2) is fulfilled and Proposition 9.4 yields that the semigroup generated by $-A$ leaves C invariant.

(b) We assume that $\mathbb{K} = \mathbb{R}$. Let $a\colon H^1(\Omega;\mathbb{R}) \times H^1(\Omega;\mathbb{R}) \to \mathbb{R}$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$. Moreover, let $j = \operatorname{tr}\colon H^1(\Omega;\mathbb{R}) \to L_2(\partial\Omega;\mathbb{R})$ be the trace operator. By definition, the Dirichlet-to-Neumann operator D_0 is associated with (a, j). Let $C = \{v \in L_2(\partial\Omega;\mathbb{R}) : v \leq \mathbb{1}\}$. Then the minimising projection P for C is given by $Pv = v \wedge \mathbb{1}$ for all $v \in L_2(\partial\Omega;\mathbb{R})$. Now, define $\hat{P}\colon H^1(\Omega;\mathbb{R}) \to H^1(\Omega;\mathbb{R})$ similarly by $\hat{P}u = u \wedge \mathbb{1}$. It follows from the Proposition before Exercise 9.3 that $Pj = j\hat{P}$, so according to (a), we only have to check that $a(u, u - \hat{P}u) \geq 0$ for all $u \in H^1(\Omega;\mathbb{R})$. So let $u \in H^1(\Omega;\mathbb{R})$ and observe that $u - \hat{P}u = u - u \wedge \mathbb{1} = (u - \mathbb{1})^+$. Thus,

$$a(u, u - \hat{P}u) = \int_\Omega \nabla u \cdot \nabla(u - \mathbb{1})^+ = \int_\Omega \nabla u \cdot \mathbb{1}_{[u>\mathbb{1}]} \nabla u = \int_{[u>\mathbb{1}]} |\nabla u|^2 \geq 0.$$

Hence, (a) implies that C is invariant under the semigroup generated by $-D_0$. □

Exercise 9.5. *Let $\Omega \subseteq \mathbb{R}^n$ be open, let $b \in L_\infty(\Omega;\mathbb{R}^n)$, and define the operator A in $L_2(\Omega)$ by*

$$\begin{aligned} \operatorname{dom}(A) &\coloneqq \{u \in H_0^1(\Omega); -\Delta u + b \cdot \nabla u \in L_2(\Omega)\}, \\ Au &\coloneqq -\Delta u + b \cdot \nabla u \qquad (u \in \operatorname{dom}(A)). \end{aligned}$$

(a) Show that A is associated with an H-elliptic form on $V \times V$, with $V \coloneqq H_0^1(\Omega) \subseteq H \coloneqq L_2(\Omega)$ (and therefore $-A$ generates a quasi-contractive C_0-semigroup). Show that the semigroup generated by $-A$ is holomorphic of angle $\pi/2$, if $\mathbb{K} = \mathbb{C}$ (cf. Exercises 7.3 and 7.4).

(b) Show that the C_0-semigroup generated by $-A$ is sub-Markovian.

Proof. (a) Define the form $a\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{K}$ by

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (b \cdot \nabla u)\overline{v}.$$

With Young's inequality we obtain for each $u \in L_2(\Omega)$ that

$$\begin{aligned} \operatorname{Re} a(u) &\geq \|\nabla u\|_{L_2(\Omega)}^2 - \|b\|_\infty \|u\|_{L_2(\Omega)} \|\nabla u\|_{L_2(\Omega)} \\ &\geq \|\nabla u\|_{L_2(\Omega)}^2 - \frac{1}{2}\|\nabla u\|_{L_2(\Omega)}^2 - \frac{1}{2}\|b\|_\infty^2 \|u\|_{L_2(\Omega)}^2 \\ &= \frac{1}{2}\|\nabla u\|_{L_2(\Omega)}^2 - \frac{1}{2}\|b\|_\infty^2 \|u\|_{L_2(\Omega)}^2 . \end{aligned}$$

From the above estimate it follows directly that a is $L_2(\Omega)$-elliptic. Now, let B be the operator associated with a. We have to show that $B = A$. First, let $u \in D(B)$. Then there is $y = Bu \in L_2(\Omega)$ such that $(y|v)_{L_2(\Omega)} = a(u,v)$ for all $v \in H_0^1(\Omega)$. In particular this is true for each $v \in \mathcal{D}(\Omega) \coloneqq C_c^\infty(\Omega)$ (the space of test functions topologized in the usual way) and for each such v we obtain

$$\begin{aligned} \int_\Omega y\overline{v} = (y|v)_{L_2(\Omega)} = a(u,v) &= \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (b \cdot \nabla u)\overline{v} \\ &= \langle -\Delta u + b \cdot \nabla u, v\rangle_{\mathcal{D}(\Omega)', \mathcal{D}(\Omega)}. \end{aligned}$$

Hence, the distribution $-\Delta u + b \cdot \nabla u$ is contained in $L_2(\Omega)$ and coincides with $y = Bu$. By definition of A, this implies $B \subset A$.

On the other hand, if $u \in D(A)$, then the same computation as above shows that $(Au|v)_{L_2(\Omega)} = a(u,v)$ for each $v \in C_c^\infty(\Omega)$. Using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, this implies that even $(Au|v)_{L_2(\Omega)} = a(u,v)$ for all $v \in H_0^1(\Omega)$. Hence, $u \in D(B)$, and we conclude that $A = B$.

Now, let $K = \mathbb{C}$ and consider the forms $\tilde{a}, \tilde{b}\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ given by

$$\tilde{a}(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad \tilde{b}(u,v) = \int_\Omega (b \cdot \nabla u)\overline{v}.$$

Then $\tilde{a}$ is symmetric, continuous and H-elliptic, and $\tilde{b}$ fulfills the estimate

$$|\tilde{b}(u,v)| \le \|b\|_\infty \|\nabla u\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \le \|b\|_\infty \|u\|_{H_0^1(\Omega)} \|u\|_{L_2(\Omega)}.$$

Moreover we have $a = \tilde{a} + \tilde{b}$, so Exercise 7.3(b) implies that the numerical range $\operatorname{num}(A)$ lies in a parabola with vertex on the real axis and opened in the direction of the positive real axis. Hence, if $\theta \in (0, \frac{\pi}{2})$, then we can find a number $\omega \in \mathbb{R}$ such that the numerical range of $A + \omega$ is contained in the sector with vertex 0 and angle $\frac{\pi}{2} - \theta$. Hence, $A+\omega$ is sectorial of angle $\frac{\pi}{2} - \theta$. Since $-A$ generates a C_0-semigroup we know that $A+\lambda$ is bijective if λ is sufficiently large. Hence, $A+\omega+\lambda$ is also surjective for large λ. Since $A+\omega$ is sectorial, it is in particular accretive, and so Lemma 3.19 implies that $A + \omega + 1$ is surjective, too. Hence, $A + \omega$ is m-sectorial of angle $\frac{\pi}{2} - \theta$, which in turn implies that $-A$ generates a quasi-contractive holomorphic semigroup of angle θ; this follows by a rescaling argument from Theorem 3.22. Since θ was an arbitrary angle in $(0, \frac{\pi}{2})$, the semigroup generated by $-A$ is in fact holomorphic of angle $\frac{\pi}{2}$.

(b) Let $C = \{u \in L_2(\Omega;\mathbb{R}) : u \le \mathbb{1}\}$. This is a closed convex set in $L_2(\Omega;\mathbb{K})$, and the corresponding minimising projection P is given by $Pu = (\operatorname{Re} u) \wedge \mathbb{1}$. We intend to use Theorem 9.20 to prove the assertion. By Theorem 9.14 we have $P(H_0^1(\Omega)) \subset H_0^1(\Omega)$. Now, let $u \in H_0^1(\Omega)$; denote $\tilde{u} := \operatorname{Re} u$ and observe that $\operatorname{Re} Pu = \tilde{u} \wedge \mathbb{1}$ and $\operatorname{Re}(u - Pu) = (\tilde{u} - \mathbb{1})^+$. We thus have

$$\operatorname{Re} a(Pu, u - Pu) = \int_\Omega \nabla(\tilde{u} \wedge \mathbb{1}) \cdot \nabla(\tilde{u} - \mathbb{1})^+ + \int_\Omega (b \cdot \nabla(\tilde{u} \wedge \mathbb{1}))(\tilde{u} - \mathbb{1})^+.$$

Similarly as in the previous exercises we can see that the first integral equals 0. Thus, we have

$$\operatorname{Re} a(Pu, u - Pu) = \int_\Omega (b \cdot \mathbb{1}_{[\tilde{u} < \mathbb{1}]} \nabla \tilde{u})(\tilde{u} - \mathbb{1})^+ = 0.$$

Hence, assertion (ii) of Theorem 9.20 is fulfilled, and we conclude that C is invariant under the semigroup generated by $-A$. ☐