EXERCISE LECTURE 8

TÜBINGEN TEAM

Exercises 8.1. Prove Proposition 8.7

Solution. We first show that A is a linear operator. In order to do this we split the proof into two parts.

A is a function: By definition, A is a relation and not empty because $(0,0) \in A$, thus we only have to prove that if $(x,y),(x,y') \in A$ for certain $x,y,y' \in H$, then $y = y'$. Observe that

$$\begin{aligned} (x,y) \in A &\quad\Rightarrow\quad \exists u_y \in V : j(u_y) = x,\ a(u_y,v) = \big(y|j(v)\big) \ \ \forall v \in V, \\ (x,y') \in A &\quad\Rightarrow\quad \exists u_{y'} \in V : j(u_{y'}) = x,\ a(u_{y'},v) = \big(y'|j(v)\big) \ \ \forall v \in V. \end{aligned}$$

Now, $j(u_y) = x = j(u_{y'}) \Rightarrow j(u_y - u_{y'}) = 0 \Rightarrow (u_y - u_{y'}) \in \ker(j)$. Observe that for $v \in V$ we have $a(u_y - u_{y'}, v) = \big(y - y'|j(v)\big)$. From this relation we deduce that for $v \in \ker(j)$ holds $a(u_y - u_{y'}, v) = 0$. Hence the condition (8.3) tells us that $u_y - u_{y'} = 0$ and so $u_y = u_{y'}$. From this we infer that for $v \in V$,

$$0 = a(u_y - u_{y'}, v) = a(u_y, v) - a(-u_{y'}, v) = \big(y|j(v)\big) - \big(y'|j(v)\big) = \big(y - y'|j(v)\big).$$

We know that a, j are continuous by hypothesis. Therefore the application $V \ni v \longmapsto \big(y - y'|j(v)\big)$ is continuous. It follows that $(y - y'|h) = 0 \ \forall h \in H$ since j has dense range. This means $y - y' = 0$ and $y = y'$, so A is effectively a function.

A is linear Let $(x_1,y_1),(x_2,y_2) \in A$. Then

$$\begin{aligned} (x_1,y_1) \in A &\quad\Rightarrow\quad \exists u_1 \in V : j(u_1) = x_1,\ a(u_1,v) = \big(y_1|j(v)\big) \ \ \forall v \in V \\ (x_2,y_2) \in A &\quad\Rightarrow\quad \exists u_2 \in V : j(u_2) = x_2,\ a(u_2,v) = \big(y_2|j(v)\big) \ \ \forall v \in V \end{aligned}$$

For $\lambda, \mu \in \mathbb{K}$, consider $u := \lambda u_1 + \mu u_2$. Then $j(u) = \lambda x_1 + \mu x_2$. Moreover $a(u,v) = \lambda a(u_1,v) + \mu a(u_2,v) = \lambda\big(y_1|j(v)\big) + \mu\big(y_2|j(v)\big) = \big(\lambda y_1 + \mu y_2|j(v)\big)$. This holds for all $v \in V$, hence $\lambda(x_1,y_1) + \mu(x_2,y_2) \in A$.

Let us turn to the second part of the proposition. Under the assumptions we know that both A and B are linear operators. By Remark 6.2(a), we can prove that $B \subset A^*$ and $A \subset B^*$ by showing that $\forall x \in \mathrm{Dom}(A)$ and $\forall y \in \mathrm{Dom}(B)$ the condition $(Ax|y) = (x|By)$ holds. For sake of clarity, we recall that $A \subset H \times H$ and $B \subset H \times H$ are given by:

$$\begin{aligned} A &:= \{(x,y) : \exists u \in V : j(u) = x,\ a(u,v) = \big(y|j(v)\big)\ \forall v \in V\}, \\ B &:= \{(x,y) : \exists u \in V : j(u) = x,\ a^*(u,v) = \big(y|j(v)\big)\ \forall v \in V\} \\ &= \{(x,y) : \exists u \in V : j(u) = x,\ a(v,u) = \big(j(v)|y\big)\ \forall v \in V\}. \end{aligned}$$

Take $x \in \mathrm{Dom}\,A$ and $z \in \mathrm{Dom}\,B$. Then

- $\exists u_x \in V :\ j(u_x) = x,\ a(u_x,v) = \big(Ax|j(v)\big) = \big(Aj(u_x)|j(v)\big)\ \forall v \in V$;
- $\exists u_z \in V :\ j(u_z) = z,\ a(v,u_z) = \big(j(v)|Bz\big) = \big(j(v)|Bj(u_z)\big)\ \forall v \in V$;

In our case, we can compute

$$\begin{aligned}(Ax|z) &= \big(Aj(u_x)|j(u_z)\big) \stackrel{\dagger}{=} a(u_x,u_z)\\ (x|Bz) &= \big(j(u_x)|Bj(u_z)\big) \stackrel{\dagger}{=} a(u_x,u_z)\end{aligned}$$

where the two equalities hold because they hold for all $v \in V$ and in particular for u_x and u_z. Thus, $(Ax|z) = a(u_x,u_z) = (x|Bz)$ for all $x \in \operatorname{Dom} A$ and $z \in \operatorname{Dom} B$.

If a is symmetric, i.e., $a = a^*$, this means that $A = B$. From the previous point we get $A \subset A^*$. If we also assume that $\operatorname{Dom} A$ is dense, we immediately obtain that A is symmetric. □

Exercises 8.2.

Solution. (a) Let $A : D(A) \subseteq G \to H$ be densely defined and $B \in \mathcal{L}(G,H)$.

We note that $A+B : D(A) \subseteq G \to H$ and $(A+B)^* : D\big((A+B)^*\big) \subseteq H \to G$ is the adjoint operator of $A+B$.
Let $x \in D((A+B)^*)$, i.e., there exists $g \in G$ such that

$$(x,(A+B)y) = (g,y)$$

for all $y \in D(A)$. We set $(A+B)^*x = g$. Further, $D((A+B)^*) \subseteq D(A^*)$ holds. We obtain

$$\begin{aligned}(g,y) &= (x,(A+B)y) = (x,Ay+By) = (x,Ay)+(x,By)\\ &= (A^*x,y)+(B^*x,y) = ((A^*+B^*)x,y).\end{aligned}$$

Therefore we conclude that $(A+B)^* = A^*+B^*$.

(b) We have the operator $BA \in \mathcal{L}(F,H)$ and its adjoint $(BA)^* \in \mathcal{L}(H,F)$.
For each $x \in H$ and $y \in F$ we have

$$(A^*B^*x,y) = (B^*x,Ay) = (x,BAy) = ((BA)^*x,y).$$

Thus, we have $(BA)^* = A^*B^*$.

(c) We show that there exists an operator $B \in \mathcal{L}(H)$ such that $A^* \cdot B = I_H$ and $B \cdot A^* = I_H$. Let $x,y \in H$. We obtain

$$(x,y) = (AA^{-1}\cdot x,y) = (A^{-1}x,A^*y) = (x,(A^{-1})^*A^*y)$$

since $A \in \mathcal{L}(H)$ invertible and

$$(x,y) = (A^{-1}\cdot A\cdot x,y) = (A\cdot x,(A^{-1})^*\cdot y) = (x,A^*\cdot(A^{-1})^*\cdot y)$$

Hence,

$$(A^{-1})^*\cdot A^* = I_H,$$

and

$$A^*\cdot(A^{-1})^* = I_H.$$

So, $(A^{-1})^* = (A^*)^{-1}$ is the inverse of the operator A^*.

(d) We can decompose the Hilbert space H uniquely via $H = H_0 \oplus H_0^\perp$ since $H_0 \subseteq H$ is a closed subspace. Let $J : H_0 \to H$ be the embedding. Show that $J^* : H \to H_0$ is the orthogonal projection. We show that

(i) $(J^*)^2 = J^*$; since

$$((J^*)^2x,y) = (J^*x,Jy) = (J^*x,y)$$

for all $x \in H$ and $y \in H_0$.

(ii) $\ker J^* = H_0^\perp$ and range$J^* = H_0$; for $x \in H$ we have

$$0 = (J^*x, y) = (x, Jy) = (x, y) \quad \forall y \in H_0$$

iff $x \in H_0^\perp$. We obtain range$J^* = H_0$ since $H = \ker J^* \oplus \text{range}J^*$ and the above decomposition is unique. Hence, the operator J^* is the orthogonal projection onto H_0.

□

Excercise 8.3. Let $\Omega \subset \mathbb{R}^n$ be open, $m \in L_\infty(\Omega)$ real-valued. Show that $\Delta_D + m$ is self-adjoint, bounded above and has compact resolvent.

Solution. We first recall that the Dirichlet Laplacian, as in Subsection 4.2.1., is defined on $\text{Dom}(\Delta_D) := \{u \in H_0^1(\Omega) : \Delta u \in L_2(\Omega)\}$ as

$$\begin{aligned} \Delta_D : L_2(\Omega) \supset \text{Dom}(\Delta_D) &\longrightarrow L_2(\Omega), \\ \Delta_D u &:= \Delta u, \end{aligned}$$

or equivalently, $\Delta_D := \{(u, f) \in L_2(\Omega) \times L_2(\Omega) \ : \ u \in H_0^1(\Omega),\ \Delta u = f\}$. The operator $M : L_2(\Omega) \longrightarrow L_2(\Omega)$ defined by $M(f) := mf$ is a bounded linear operator with

$$\|mf\|_2 := \left(\int_\Omega |mf|^2\right)^{\frac{1}{2}} = \left(\int_\Omega |m|^2|f|^2\right)^{\frac{1}{2}} \le \|m\|_\infty \|f\|_2 < +\infty.$$

Thus $\Delta_D + M$ is defined on $\text{Dom}(\Delta_D + M) = \text{Dom}(M) \cap \text{Dom}(\Delta_D) = L_2(\Omega) \cap \text{Dom}(\Delta_D) = \text{Dom}(\Delta_D)$:

$$\Delta_D + M = \{(u, f) \in L_2(\Omega) \times L_2(\Omega) \ : \ u \in H_0^1(\Omega),\ f = \Delta u + mu\}.$$

We start our proof with some very simple remarks. Let G, H be Hilbert spaces over $\mathbb{C}$ and $A : G \longrightarrow H$ a linear operator. Then:

- A is self-adjoint $\Leftrightarrow -A$ is self-adjoint:
  Indeed $A = A^* \Leftrightarrow -A = -A^* = (-A)^*$ and the last equality holds since

$$\begin{aligned} (-A)^* &:= \{(y, x) \in H \times G : \ \forall x_1 \in \text{Dom}(-A),\ (-Ax_1|y)_H = (x_1|x)_G\}; \\ (A)^* &:= \{(y, x) \in H \times G : \ \forall x_1 \in \text{Dom}(A),\ (Ax_1|y)_H = (x_1|x)_G\}. \end{aligned}$$

  Observing that $\text{Dom}(-A) = \text{Dom}(A)$, one has for every $x_2 \in Dom(A)$

$$\begin{aligned} (y, x) \in (-A)^* &\Leftrightarrow (-Ax_1|y)_H = (x_1|x)_G \Leftrightarrow -(Ax_1|y)_H = (x_1|x)_G \\ &\Leftrightarrow (Ax_1|y)_H = -(x_1|x)_G \Leftrightarrow (Ax_1|y)_H = (x_1| - x)_G \\ &\Leftrightarrow (y, -x) \in A^* \Leftrightarrow (x, y) \in -A^*. \end{aligned}$$

- A is bounded above $\Leftrightarrow -A$ is bounded below: directly from the definition.

We shall now show the following:

$\Delta_D + M$ is self-adjoint: We can immediately see that $M^* = M$, because m is real-valued. From Theorem 4.18 we know that $-\Delta_D$ is sectorial of angle 0 and by Theorem 6.1 we have that $-\Delta_D$ is a positive and self-adjoint operator. Furthermore Δ_D is self-adjoint. Then

$$\Delta_D + M = \Delta_D^* + M^* = (\Delta_D + M)^*$$

where the last equality holds by Exercise 2(a).

$\Delta_D + M$ is bounded above: As written before, we know that $-\Delta_D$ is a positive operator and then $(-\Delta_D u|u) \geq 0$ $\forall x \in \mathrm{Dom}(\Delta_D)$. We deduce that $(\Delta_D u|u) \leq 0$ $\forall u \in \mathrm{Dom}(\Delta_D)$. Take now $u \in \mathrm{Dom}(\Delta_D)$ with $\|u\|_2 = 1$. Then

$$\big((\Delta_D + M)u|u\big) = (\Delta_D u|u) + (Mu|u) \leq (Mu|u) \leq \|M\|_\infty \|u\|_2^2 = \|M\|_\infty,$$

and so $\Delta_D + M$ is bounded above.

<u>$\Delta_D + M$ has compact resolvent:</u> We know from Example 6.19 that Δ_D has compact resolvent. Moreover we know by Theorem 4.18 that $\pm\Delta_D$ is the generator of a contractive C_0-semigroups and from Theorem 2.7 that $\forall \lambda \in \mathbb{R}_{>0}$ holds

$$\|R(\lambda, \Delta_D)^n\| \leq \frac{1}{\lambda^n}.$$

In particular, for all $\lambda > \|M\|_\infty$, the Neumann series

$$\sum_{n=0}^{\infty} \big(R(\lambda, \Delta_D) \circ M\big)^n$$

converges to an operator in $\mathcal{L}\big(L_2(\Omega)\big)$. Namely

$$\begin{aligned}\Big\| \sum_{n=0}^{\infty} \big(R(\lambda, \Delta_D) \circ M\big)^n \Big\| &\leq \sum_{n=0}^{\infty} \big\| \big(R(\lambda, \Delta_D) \circ M\big)^n \big\| \leq \sum_{n=0}^{\infty} \big(\|R(\lambda, \Delta_D)\| \|M\|\big)^n \\ &\leq \sum_{n=0}^{\infty} \Big(\frac{\|M\|}{\lambda}\Big)^n < +\infty.\end{aligned}$$

Observe now that $\sum_{n=0}^{\infty} \big(R(\lambda, \Delta_D) \circ M\big)^n = \big(I - R(\lambda, \Delta_D) \circ M\big)^{-1}$ for every $\lambda > \|M\|_\infty$. Moreover

$$\begin{aligned}\big(I - R(\lambda, \Delta_D) \circ M\big)^{-1} R(\lambda, \Delta_D) &= \Big((\lambda - \Delta_D)\big(I - R(\lambda, \Delta_D) \circ M\big)\Big)^{-1} \\ &= (\lambda - \Delta_D - M)^{-1} \\ &= R(\lambda, \Delta_D + M).\end{aligned}$$

Thus, $(\|M\|_\infty, \infty) \subset \rho(\Delta_D + M)$ and $R(\lambda, \Delta_D + M)$ is compact since the operator $\big(I - R(\lambda, \Delta_D) \circ M\big)^{-1}$ is bounded on $L_2(\Omega)$ and the resolvent $R(\lambda, \Delta_D)$ is compact. Applying the resolvent equation, we can conclude that $R(\lambda, \Delta_D + M)$ is compact for every $\lambda \in \rho(\Delta_D + M)$. □

Exercise 8.4. Let $-\infty < a < b < \infty$.

(a) Compute the Dirichlet-to-Neumann operator D_0 for $\Omega = (a, b)$, and compute the C_0-semigroup generated by $-D_0$.

We first notice that $\partial\Omega = \{a, b\}$, thus $L^2(\partial\Omega) = \mathbb{C}^2$.

Then by definition

$$D_0 := \Bigg\{ (g, h) \in \mathbb{C}^2 \times \mathbb{C}^2; \exists u \in H^1((a, b)) : \Delta u = 0, \\ \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} -u'(a) \\ u'(b) \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \Bigg\}.$$

It is well-known that every harmonic function on (a, b) has the form

$$u(x) = c_1 x + c_2,$$

for $c_1, c_2 \in \mathbb{C}$. Thus

$$D_0 = \left\{ (g,h) \in \mathbb{C}^2 \times \mathbb{C}^2 ; \exists c_1, c_2 \in \mathbb{C} : \begin{pmatrix} c_1 a + c_2 \\ c_1 b + c_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} -c_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\}$$
$$= \left\{ (g,h) \in \mathbb{C}^2 \times \mathbb{C}^2 : \begin{pmatrix} \frac{g_1 - g_2}{b-a} \\ \frac{g_2 - g_1}{b-a} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\}.$$

This means that D_0 is a 2×2 matrix given by

$$D_0 = \frac{1}{b-a} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The C_0-semigroup generated by $-D_0$ is

$$e^{-tD_0} = \sum_{n=0}^{\infty} \frac{(-tD_0)^n}{n!}.$$

In order to compute this, one first proves by induction that

$$D_0^n = \frac{1}{(b-a)^n} \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix} \quad \forall n \geq 1.$$

It thus follows

$$e^{-tD_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \sum_{n=1}^{\infty} \frac{(-t)^n 2^{n-1}}{(b-a)^n n!} & -\sum_{n=1}^{\infty} \frac{(-t)^n 2^{n-1}}{(b-a)^n n!} \\ -\sum_{n=1}^{\infty} \frac{(-t)^n 2^{n-1}}{(b-a)^n n!} & \sum_{n=1}^{\infty} \frac{(-t)^n 2^{n-1}}{(b-a)^n n!} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\left(e^{-\frac{2t}{b-a}} - 1\right) & -\frac{1}{2}\left(e^{-\frac{2t}{b-a}} - 1\right) \\ -\frac{1}{2}\left(e^{-\frac{2t}{b-a}} - 1\right) & \frac{1}{2}\left(e^{-\frac{2t}{b-a}} - 1\right) \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} e^{-(\frac{2}{b-a})t} + 1 & 1 - e^{-(\frac{2}{b-a})t} \\ 1 - e^{-(\frac{2}{b-a})t} & e^{-(\frac{2}{b-a})t} + 1 \end{pmatrix}.$$

Remark: The matrix D_0 is symmetric, thus diagonalizable and its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -\frac{2}{b-a}$. Thus the generated semigroup is not stable.

(b) For $a = -1$, $b = 1$, interpret the result in the light of Exercise 8.5.

The intervall $U_1 := (-1, 1)$ is the open unit ball in $\mathbb{R}$ and $\partial U_1 = \{-1, 1\}$ the unit sphere.

In this special case $C(\partial U_1) = \mathbb{C}^2 = L^2(\partial U_1)$ and the semigroup $(T(t))_{t \geq 0}$ defined in Exercise 8.5 is the semigroup generated by $-D_0$.

Exercise 8.5.

a) Let $\phi \in C(S_{n-1})$ and $z \in S_{n-1}$. Then $T(0)\phi(z) = u(z) = \phi(z)$.

For t, $s \geq 0$ we obtain

$$T(t+s)\phi(z) = u(e^{-(t+s)}z) = u(e^{-t}e^{-s}z) = [T(t)T(s)\phi](z).$$

Hence, $(T(t))_{t \geq 0}$ satisfies the functional equation. The family $(T(t))_{t \geq 0}$ is strongly continuous since ϕ is uniformly continuous on the compact set S_{n-1}, i.e.,

$$\|T(t)\phi - \phi\|_\infty = \sup_{z \in S_{n-1}} \left| u(e^{-t}z) - \phi(z) \right| \overset{t \searrow 0}{\longrightarrow} 0.$$

Further, we obtain

$$\|T(t)\phi\|_\infty = \sup_{z\in S_{n-1}} \left|u(e^{-t}z)\right| \leq \sup_{z\in B_n} |u(z)| = \|u\|_\infty \leq \|\phi\|_\infty .$$

b) For D to be a core for A, it suffices to show that $D \subset C(S_{n-1})$ is dense and $(T(t))$-invariant. The invariance is obvious by the definition of D. To see density, take some arbitrary $\epsilon > 0$ and $\phi \in C(S_{n-1})$. Then there exists $t_0 > 0$ such that

$$\|T(t_0)\phi - \phi\|_\infty = \sup_{z\in S_{n-1}} \left|[G\phi](e^{-t_0}z) - \phi(z)\right| < \epsilon$$

and the assertion follows. In particular, we obtain $\overline{A_{min}}^{\|\cdot\|_\infty} = A$, where the closure is taken with respect to the graph norm on $C(S_{n-1})$.

For $\phi \in D$ we can write $\phi = T(t_0)\psi$ for some $\psi \in C(S_{n-1})$ and $t_0 > 0$. Take $z \in S_{n-1}$. Then

$$\begin{aligned} A\phi(z) &= \lim_{h\searrow 0} h^{-1}(T(h)\phi(z) - \phi(z)) \\ &= \lim_{h\searrow 0} h^{-1}(T(t_0+h)\psi(z) - T(t_0)\psi(z)) \\ &= -\lim_{h\searrow 0} h^{-1}(G\psi(e^{-t_0}z) - G\psi(e^{-(t_0+h)}z)) = -[\partial_\nu G\psi](e^{-t_0}z), \end{aligned}$$

the derivative of $G\psi$ at the point $e^{-t_0}z$ in the direction of the (radially) outer normal. Since $(T(t))_{t\geq 0}$ shrinks the argument by the factor e^{-t}, we conclude that $[\partial_\nu G\psi](e^{-t_0}z) = [\partial_\nu G\phi](z)$ and the assertion follows.

c) Using an maximum principle analogue for L^2-functions (see [1, Thm. 1', p.463] for the two-dimensional case) we can extend the semigroup $(T(t))_{t\geq 0}$ to a C_0-contraction semigroup $(T_2(t))_{t\geq 0} \subset \mathcal{L}(L^2(S_{n-1}))$. We denote its generator by $(A_2, D(A_2))$ and $A_2|_D = A_{min}$ holds since A_2 is an extension of A. The subset D is a core for A_2 since $D \subset L^2(S_{n-1})$ is dense and $(T_2(t))_{t\geq 0}$-invariant. Hence, $A_2 = \overline{A_{min}}^{\|\cdot\|_2}$.

Define $A_{min} := A|_D : D \subset L^2(S_{n-1}) \to L^2(S_{n-1})$. We can extend every continuous function $\phi \in C(S_{n-1})$ to an harmonic function in $C(\overline{B_n})$ such that ϕ is the restriction to the boundary. Hence, the inclusion $D \subset D(D_0)$ holds. Let $\phi \in D$. Then

$$A_{min}\phi(z) = [\partial_\nu G\phi](z) = -D_0\phi(z)$$

by definition. Therefore, $-A_{min}$ is a restriction of the Dirichlet-to-Neumann operator D_0. Moreover,

$$A_2 = \overline{A_{min}}^{\|\cdot\|_2} \subset -D_0$$

since D_0 is closed (having non-empty resolvent set). In fact, for $x \in D(A_2)$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that

$$x_n \to x, \quad -D_0x_n = A_{min}x_n \to A_2x.$$

Hence $x \in D(D_0)$ and $A_2x = -D_0x$.

By Theorem 8.4 in Lecture 8, the Dirichlet-to-Neumann operator D_0 is positive, i.e., $\langle D_0x, x\rangle \geq 0$ for all $x \in D(D_0)$. Thus, $-D_0$ is dissipative and $\lambda + D_0$ is injective for each $\lambda > 0$. We obtain $A_2 = -D_0$ since $\lambda - A_2$ is surjective for all $\lambda > 0$ and $\lambda - A_2 \subset \lambda + D_0$.

We conclude that $-D_0 = A_2 = \overline{A_{min}}^{\|\cdot\|_2}$ is the generator of $(T_2(t))_{t\geq 0}$.

References

[1] P.D. Lax, *Functional analysis*. Wiley-Interscience, New York, 2002