

**Exercise 7.1.** Since  $\beta \geq 0$ , by Theorem 7.15, we know that  $\|T_\beta(t)\| \leq 1$  for all  $t \geq 0$ . We wish to show that there exists  $\omega > 0$  such that the semigroup  $T_\beta^\omega$  associated with the shifted operator  $\Delta_\beta + \omega$  is still contractive, i.e.  $\|T_\beta^\omega(t)\| \leq 1$  for all  $t \geq 0$ , since then  $T_\beta(t) = e^{-\omega t} T_\beta^\omega(t)$  and

$$\|T_\beta(t)\| \leq e^{-\omega t} \|T_\beta^\omega(t)\| \leq e^{-\omega t} \quad (1)$$

for all  $t \geq 0$ .

Now it is clear that the form associated with  $-\Delta_\beta - \omega$  is given by

$$a_\beta^\omega(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} \beta u \overline{v} \, d\sigma - \omega \int_\Omega u \overline{v}$$

for  $u, v \in H^1(\Omega)$  (cf. Remark 5.6). As in Theorem 7.15, to show that  $\|T_\beta^\omega(t)\| \leq 1$ , it suffices to show that

$$a_\beta^\omega(u) \geq 0 \quad (2)$$

for all  $u \in H^1(\Omega)$ . So our task reduces to showing that there exists  $\omega > 0$  with this property. To that end, we define the quantities

$$\mathcal{R}(u) := \frac{a_\beta(u)}{\|u\|_{L^2(\Omega)}^2} = \frac{\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta |u|^2 \, d\sigma}{\int_\Omega |u|^2} \quad (3)$$

for  $u \in H^1(\Omega) \setminus \{0\}$  and

$$\lambda_1 := \inf \{ \mathcal{R}(u) : u \in H^1(\Omega) \setminus \{0\} \}. \quad (4)$$

Then obviously  $\lambda_1 \geq 0$  since  $\beta \geq 0$ , and by construction, if  $\lambda_1 > 0$ , then (2) holds for all  $\omega \in [0, \lambda_1]$ . So in fact we wish to show that  $\lambda_1 > 0$ .

We claim that there exists  $u \in H^1(\Omega) \setminus \{0\}$  such that  $\mathcal{R}(u) = \lambda_1$ . Indeed, let  $0 \neq u_n \in H^1(\Omega)$  be such that  $\mathcal{R}(u_n) \rightarrow \lambda_1$ , without loss of generality assume  $\|u_n\|_{L^2(\Omega)} = 1$  for all  $n$ . Then the boundedness of  $\mathcal{R}(u_n)$  implies that  $(u_n)$  is bounded in  $H^1(\Omega)$  and hence, since the unit ball in  $H^1(\Omega)$  is weakly compact, there exists  $u_0 \in H^1(\Omega)$  such that  $u_n \rightharpoonup u_0$  weakly in  $H^1(\Omega)$ . As the embedding  $H^1(\Omega) \hookrightarrow L^2(\Omega)$  is compact by Theorem 7.11, we also have  $u_n \rightarrow u_0$  strongly in  $L^2(\Omega)$  strongly, and in particular  $\|u_0\|_{L^2(\Omega)} = 1$ , so  $u_0 \neq 0$ . Moreover, weak convergence in  $H^1(\Omega)$  implies that

$$\int_\Omega |\nabla u_0|^2 \leq \liminf_{n \rightarrow \infty} \int_\Omega |\nabla u_n|^2$$

and, since the trace operator is continuous (and  $\beta \geq 0$ ),

$$\int_{\partial\Omega} \beta |u_0|^2 \, d\sigma \leq \liminf_{n \rightarrow \infty} \int_{\partial\Omega} \beta |u_n|^2 \, d\sigma.$$

Putting this all together, we conclude that

$$\lambda_1 \leq \mathcal{R}(u_0) \leq \liminf_{n \rightarrow \infty} \mathcal{R}(u_n) = \lambda_1,$$

proving our claim. Finally, we claim that  $\mathcal{R}(u_0) \neq 0$  since  $u_0 \neq 0$ . Indeed, suppose this is not true. Then by definition of  $\mathcal{R}$ ,

$$\int_{\Omega} |\nabla u_0|^2 = 0$$

and hence  $\nabla u_0 = 0$  almost everywhere. It follows from the hint that  $u_0$  is constant:  $u_0 = c \neq 0$ . Then

$$\int_{\partial\Omega} \beta |u_0|^2 d\sigma = |c|^2 \int_{\partial\Omega} \beta d\sigma > 0$$

by assumption on  $\beta$ , meaning  $\mathcal{R}(u_0) > 0$ , a contradiction. To summarise: we have shown that  $\lambda_1 > 0$ , and hence, by our earlier considerations, there exists  $\omega > 0$  satisfying (2) and hence also (1).

*Remark.* The notation  $\lambda_1$  is not coincidental: this is the smallest eigenvalue of the self-adjoint operator  $-\Delta_{\beta}$  (cf. Theorem 6.17 and Corollary 6.18). Moreover, the function  $u_0$  is a corresponding eigenfunction. In fact, in the setting of the corollary, one always has

$$\lambda_1 = \inf \left\{ \frac{a(u)}{\|j(u)\|_H^2} : u \in V \setminus \{0\} \right\} \quad (5)$$

with equality being attained by the preimage in  $V$  under the map  $j$  of the corresponding eigenfunctions in  $H$ . The quantity  $\mathcal{R}(u)$  is usually called the *Rayleigh quotient* at  $u$ . Moreover, one can show as above that the associated semigroup satisfies the bound

$$\|T(t)\| \leq e^{-\lambda_1 t} \quad \text{for all } t \geq 0;$$

in fact, there is equality, as  $T(t)\varphi_1 = e^{-\lambda_1 t}\varphi_1$  for any eigenfunction  $\varphi$  associated with  $\lambda_1$ .

We finally note that there are similar formulae to (5) for the higher eigenvalues: the so-called min-max (or max-min) formulae, usually associated with the names of Courant (as in Courant–Hilbert, authors of the weighty book *Methods of mathematical physics*) and Fischer. Actually, these formulae are always given in the literature only for the case when  $j$  is an embedding, and not just a bounded linear operator with dense range; we leave it as an exercise to check that (5) still holds when  $j$  is not injective.

**Exercise 7.2.** We shall interpret the exercise as referring to the operator  $\Delta_{D,\beta}$  defined by

$$\begin{aligned} \text{dom}(\Delta_{D,\beta}) &= \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), u|_{\Gamma_1} = 0, \partial_{\nu} u|_{\Gamma_2} + \beta u|_{\Gamma_2} = 0\}, \\ \Delta_{D,\beta} u &= \Delta u. \end{aligned}$$

Here the outer normal derivative  $\partial_\nu u|_{\Gamma_2}$  on  $\Gamma_2$  is defined in the natural way as being the function  $h \in L^2(\Gamma_2)$ , if one exists, satisfying

$$\int_{\Omega} (\Delta u)v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Gamma_2} hv \quad (6)$$

for all  $v \in C_c^\infty(\Omega \cup \Gamma_2)$ , i.e., for all smooth  $v$  vanishing in a neighbourhood of  $\Gamma_1$  (where we will impose the Dirichlet condition). We will introduce a Sobolev space related to these special test functions:

$$\tilde{H}_0^1(\Omega) := \overline{C_c^\infty(\Omega \cup \Gamma_2)}^{H^1(\Omega)},$$

the closure of  $C_c^\infty(\Omega \cup \Gamma_2)$  in the  $H^1$ -norm; then clearly,  $\tilde{H}_0^1(\Omega)$  is a Hilbert space and a subspace of  $H^1(\Omega)$  (and a superspace of  $H_0^1(\Omega)$ ). Note that we could equivalently demand in the definition of  $\partial_\nu u|_{\Gamma_2}$  that (6) hold for all  $v \in \tilde{H}_0^1(\Omega)$ .

We introduce the sesquilinear form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Gamma_2} \beta u \bar{v} \quad (7)$$

on  $\tilde{H}_0^1(\Omega)$ , and it is now hopefully clear how to proceed: the only ‘‘challenge’’ was to find the right Sobolev space. Mimicking the proof of Theorem 7.15, we have

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\beta\|_{L^\infty(\Gamma_2)} \|\operatorname{tr} u\|_{L^2(\Gamma_2)} \|\operatorname{tr} v\|_{L^2(\Gamma_2)}$$

for all  $u, v \in \tilde{H}_0^1(\Omega)$ , and since the trace is continuous (note that Theorem 7.9 also holds for  $\Gamma_2 \subseteq \partial\Omega$  instead of  $\partial\Omega$ ), the form  $a$  is bounded. We still have to show that  $a$  is  $L^2(\Omega)$ -elliptic, i.e., there exist  $\omega, \alpha \geq 0$  such that

$$\int_{\Omega} |\nabla u|^2 + \int_{\Gamma_2} \beta |u|^2 + \omega \int_{\Omega} |u|^2 \geq \alpha \|u\|_{H^1(\Omega)}^2 \quad (8)$$

holds for all  $u \in \tilde{H}_0^1(\Omega)$ . Note that  $\tilde{H}_0^1(\Omega)$  is a closed subspace of  $H^1(\Omega)$ . Therefore by Theorem 7.9 there exists  $c \geq 0$  such that

$$\|u|_{\Gamma_2}\|_{L^2(\Gamma_2)}^2 \leq \|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}^2 \leq c \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}$$

holds for all  $u \in \tilde{H}_0^1(\Omega)$ . We proceed as in the proof of Theorem 7.15 and obtain

$$\int_{\Omega} |\nabla u|^2 + \int_{\Gamma_2} \beta |u|^2 \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \left(\frac{1}{2} + c\right) \int_{\Omega} |u|^2$$

for  $c \geq 0$ . This proves (8). Hence our form is associated with an operator  $A$  on  $L^2(\Omega)$ , and it remains to establish the hopefully not surprising fact that

$A = -\Delta_{D,\beta}$ . This claim follows again from the same line of reasoning as in the proof of Theorem 7.15. Let  $(u, f) \in A$ , then

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Gamma_2} \beta u \bar{v} = \int_{\Omega} f \bar{v} \quad (9)$$

for all  $v \in \tilde{H}_0^1(\Omega)$ . Taking  $v \in C_c^\infty(\Omega \cup \Gamma_2)$  we see that  $-\Delta u = f$  and replacing  $f$  by  $-\Delta u$  we find

$$\int_{\Gamma_2} (\partial_\nu u) \bar{v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u) \bar{v} = - \int_{\Gamma_2} \beta u \bar{v}. \quad (10)$$

Thus  $u \in \text{dom } \Delta_{D,\beta}$ . Conversely, if  $u \in \text{dom } \Delta_{D,\beta}$  then (10) holds and if we choose  $f$  to be  $-\Delta u$  we obtain (9). Thus  $(u, f) \in A$ .

Now that we have associated  $\Delta_{D,\beta}$  with a form, we see the selfadjointness of  $\Delta_{D,\beta}$  follows (as in Theorem 7.15) from Theorem 6.10: since the shifted form  $a + \omega$  ( $\omega \geq 0$ ) is coercive for  $\omega$  large enough, the associated operator  $-\Delta_{D,\beta} + \omega I$  is self-adjoint, and this of course implies that the same is true of  $-\Delta_{D,\beta}$ .

*Remark.* It seems almost silly at this point *only* to show selfadjointness, since, having done all the hard work, we can establish many other nice properties of  $\Delta_{D,\beta}$  essentially for free. For example,  $\Delta_{D,\beta}$  has compact resolvent since  $\tilde{H}_0^1(\Omega)$  can be continuously embedded in  $H^1(\Omega)$  and hence compactly embedded in  $L^2(\Omega)$  (cf. Theorem 7.11). We (can) also obtain quasi-accretiveness similarly.

**Exercise 7.3.** (a) We know that  $a$  is continuous, therefore it is sufficient to prove that  $b$  is continuous. Using

$$|b(u, v)| \leq c \|u\|_V \|j(v)\|_H \quad (u, v \in V). \quad (11)$$

we obtain

$$|b(u, u)| \leq c \|u\|_V \|j(u)\|_H \leq c \|j\|_{\mathcal{L}(V, \mathcal{H})} \|u\|_V^2.$$

This implies the continuity of  $(a + b)$ .

To establish the  $j$ -ellipticity we will use *Euclid's inequality*, also, and possibly better, known under the name *Young's inequality*,

$$|ab| \leq \frac{1}{2} \left( \varepsilon^2 a^2 + \frac{1}{\varepsilon^2} b^2 \right) \quad (12)$$

for  $a, b \in \mathbb{R}$  and  $\varepsilon > 0$ . We start out by using the  $j$ -ellipticity of  $a$  to obtain

$$\begin{aligned} \text{Re} [(a + b)(u)] &= \text{Re} [(a)(u)] + \text{Re} [(b)(u)] \\ &\geq \alpha \|u\|_V^2 - \omega \|j(u)\|_H^2 - C \|u\|_V \|j(u)\|_H, \end{aligned}$$

where  $\alpha > 0$  and  $\omega \in \mathbb{R}$ . To control the term  $-C\|u\|_V\|j(u)\|_H$  we use (12) to get

$$\begin{aligned} \operatorname{Re}[(a+b)(u)] &\geq \alpha\|u\|_V^2 - \omega\|j(u)\|_H^2 - \frac{1}{2}\left(\varepsilon^2\|u\|_V^2 + \frac{1}{\varepsilon^2}\|j(u)\|_H^2\right) \\ &\geq \left(\alpha - \frac{\varepsilon^2 C}{2}\right)\|u\|_V^2 - \left(\omega + \frac{C}{2\varepsilon^2}\right)\|j(u)\|_H^2. \end{aligned}$$

This is exactly the  $j$ -ellipticity for  $a+b$  if  $\varepsilon$  is chosen small enough that the coefficient in front of  $\|u\|_V^2$  is positive.

(b) We will first decompose  $\langle Ax, x \rangle$  into real and imaginary parts. Let  $u \in V$  such that  $j(u) = x$ . By definition we obtain

$$\langle Ax, x \rangle = (a+b)(u) = a(u) + b(u).$$

Using that  $a$  is symmetric we get what we expect:

$$\begin{aligned} \operatorname{Re}\langle Ax, x \rangle &= \operatorname{Re}(a+b)(u) = a(u) + \operatorname{Re}b(u), \\ \operatorname{Im}\langle Ax, x \rangle &= \operatorname{Im}(a+b)(u) = 0 + \operatorname{Im}b(u). \end{aligned}$$

We have to show that there exist constants  $c_1, c_2 \in \mathbb{R}$  such that for all  $x \in \operatorname{dom}(A)$  with  $\|x\| = 1$  the following inequality holds:

$$(\operatorname{Im}\langle Ax, x \rangle)^2 \leq c_1 \operatorname{Re}\langle Ax, x \rangle + c_2.$$

Since  $j(u) = x$  and  $\|x\|_H = 1$  it follows that  $\|j(u)\|_H^2 = 1$ . Using (11) we get

$$(\operatorname{Im}\langle Ax, x \rangle)^2 \leq |b(u)|^2 \leq c^2\|u\|_V^2 \cdot \|j(u)\|_H^2 = c^2\|u\|_V^2 \cdot 1 \quad (13)$$

We know that  $a+b$  is  $j$ -elliptic. Therefore we obtain

$$\operatorname{Re}\langle Ax, x \rangle \geq \alpha\|u\|_V^2 - \omega\|j(u)\|_H^2 = \alpha\|u\|_V^2 - \omega,$$

where  $\alpha > 0$  and  $\omega \in \mathbb{R}$ . Now we combine this inequality and (13) to get the final estimate

$$(\operatorname{Im}\langle Ax, x \rangle)^2 \leq c^2\|u\|_V^2 \leq \frac{c^2}{\alpha}(\operatorname{Re}\langle Ax, x \rangle + \omega)$$

(c) We first show that  $j$  is an isomorphism (bijective, continuous linear map with continuous inverse). We know that  $a$  is  $j$ -elliptic and  $a = 0$ . This means there exist  $\alpha > 0$  and  $\omega \in \mathbb{R}$  such that

$$\omega\|j(u)\|_H^2 \geq \alpha\|u\|_V^2.$$

In addition we know that  $j \in \mathcal{L}(V, H)$ . Therefore exists a constant  $C > 0$  such that

$$\frac{1}{C}\|u\|_V^2 \leq \|j(u)\|_H^2 \leq C\|u\|_V^2. \quad (14)$$

Obviously  $j$  is injective, because of this inequality. Now we show that  $j$  is surjective. Take  $f \in H$ . Since  $j$  has dense range, there exists a sequence  $u_n \in V$  with  $j(u_n) \xrightarrow{H} f$ . Obviously  $j(u_n)$  is a Cauchy sequence. Using (14) we see that  $u_n$  is a Cauchy sequence, too. We know that  $V$  is a Banach space, and so  $u_n \xrightarrow{V} u \in V$  for some  $u \in V$ . Now we use that  $j$  is continuous and we obtain  $j(u) = f$ . It follows now directly from (14) that  $j^{-1}$  is continuous. Hence  $j$  is an isomorphism.

The last statement to be proved is that  $B$  is bounded. To do so it clearly suffices to show that  $\text{dom}(B) = H$  and there exists a constant  $C > 0$  such that

$$\langle Bx, y \rangle \leq C\|x\|_H\|y\|_H$$

for all  $x, y \in H$ . We know by definition of the operator  $B$  that for a given  $x \in \text{dom}(B)$  there exists  $u \in V$  such that  $j(u) = x$  and

$$b(u, v) = \langle Bx, j(v) \rangle \quad \text{for all } v \in V.$$

Since  $j$  is bijective, for any fixed  $v \in V$  there exists  $y \in H$  with  $j(v) = y$ . Using (11) we obtain

$$|\langle Bx, y \rangle| = |\langle Bx, j(v) \rangle| = |b(u, v)| \leq c\|u\|_V\|j(v)\|_H = c\|j^{-1}(x)\|_V\|y\|_H. \quad (15)$$

This shows that  $B$  is bounded on  $\text{dom}(B)$ . However, since  $\text{dom}(B)$  is necessarily dense in  $H$  as well as being a Banach space when equipped with the graph norm (which is seen by (15) to be equivalent to the norm on  $H$ ), it follows that  $\text{dom}(B) = H$  and  $B$  is bounded on  $H$ .

**Exercise 7.4.** We remark that in the proof of Theorem 7.15 the assumption that  $\beta$  is real valued was *not* used to show that  $A \sim (a, j)$ . So instead of giving another proof which is a trivial modification of Theorem 7.15, we shall simply assume that we have proved this and write  $\langle Au, v \rangle = a(u, v)$  for all  $u \in \text{dom}(A) \subset V = H^1(\Omega) \ni v$  (where  $a$  is the usual form, as in the proof of Theorem 7.15, and we identify  $v \in H^1(\Omega)$  with its image in  $H = L^2(\Omega)$  under the embedding  $j : V \hookrightarrow H$ ).

(a) We first show that  $-\Delta_\beta$  is  $L^2(\Omega)$ -elliptic; we proceed (for a change) as in the proof of Theorem 7.15. Using that  $\beta \in L^\infty(\partial\Omega)$  together with the

Trace Theorem 7.9, if  $u \in \text{dom}(A)$  and  $\omega \geq 0$ , then

$$\begin{aligned}
\text{Re} \langle Au, u \rangle + \omega \|u\|_{L^2(\Omega)} &= \text{Re} a(u) + \omega \|u\|_{L^2(\Omega)} \\
&= \int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} \text{Re} \beta |u|^2 + \omega \int_{\Omega} |u|^2 \\
&\geq \int_{\Omega} |\nabla u|^2 - \|\text{Re} \beta\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u|^2 + \omega \int_{\Omega} |u|^2 \\
&\geq \int_{\Omega} |\nabla u|^2 - \|\beta\|_{L^\infty(\partial\Omega)} \|u\|_{L^2(\partial\Omega)}^2 + \omega \int_{\Omega} |u|^2 \\
&\geq \int_{\Omega} |\nabla u|^2 - c\gamma \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} + \omega \int_{\Omega} |u|^2
\end{aligned}$$

with  $\gamma = \|\beta\|_{L^\infty(\partial\Omega)}$ . If  $c = 0$  or  $\gamma = 0$ , then we choose  $\omega = 1$  and we are done. Otherwise, we continue by using Euclid's (or Young's) inequality (and introduce the notation  $\int_{\Omega} |\nabla v|^2 = |v|_{H^1(\Omega)}^2$ )

$$\begin{aligned}
\text{Re} \langle Au, u \rangle + \omega \|u\|_H &\geq |u|_{H^1(\Omega)}^2 - c\gamma \left( \frac{c\gamma}{2} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2c\gamma} \|u\|_{H^1(\Omega)}^2 \right) + \omega \|u\|_{L^2(\Omega)}^2 \\
&= \frac{1}{2} |v|_{H^1(\Omega)}^2 + \left( \omega - \frac{(c\gamma)^2 + 1}{2} \right) \|u\|_{L^2(\Omega)}^2 \geq \frac{1}{2} \|u\|_{H^1(\Omega)}^2,
\end{aligned}$$

where the last inequality follows by choosing  $\omega = \omega_0 > \frac{(c\gamma)^2 + 1}{2}$ . This is the  $j$ -ellipticity. Moreover, if  $\text{Re} \beta \geq 0$ , we can estimate  $\gamma$  from below by 0 to see that the operator is accretive.

So we still need to show that  $\text{ran}(I - \Delta_\beta) = H$  if  $\text{Re} \beta \geq 0$ . We proceed as in proof of Theorem 5.7 and introduce the form  $b : V \times V \rightarrow \mathbb{C}$ ,  $b(u, v) := a(u, v) + (j(u), j(v))$  for all  $u, v \in V = H^1(\Omega)$ , where  $j$  is the embedding of  $H^1(\Omega)$  into  $L^2(\Omega)$ . Using again Theorem 7.9 together with Cauchy–Schwarz, we have

$$\begin{aligned}
|b(u, v)| &\leq \int_{\Omega} |\nabla u \overline{\nabla v}| + \left| \int_{\partial\Omega} \beta u \overline{v} \right| + \int_{\Omega} |u \overline{v}| \\
&\leq |u|_{H^1(\Omega)} |v|_{H^1(\Omega)} + \gamma \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq 2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + c\gamma \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2} \\
&\leq (2 + c\gamma) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

with the constants as in part (a) and

$$\text{Re} b(u, v) = \int_{\Omega} |\nabla v|^2 + \int_{\partial\Omega} \text{Re} \beta |u|^2 + \int_{\Omega} |u|^2 \geq \|v\|_{H^1(\Omega)}^2.$$

That is,  $b$  is coercive and bounded. Now one may proceed exactly as in the proof of Theorem 5.7.

(b) We have to show that  $\text{num}(A) = \{(Av, v) \in \mathbb{C} : v \in \text{dom}(A), \|v\|_H = 1\}$  is contained in a subset of the complex plane which is bordered by a “parabola with vertex on the real axis and opened in the direction of the positive real axis.” Therefore, as in Ex. 7.3, we consider for  $u \in \text{dom}(A)$  (again using Theorem 7.9)

$$\begin{aligned} (\text{Im} \langle Au, u \rangle)^2 &= (\text{Im} a(u))^2 = \left( \int_{\partial\Omega} \text{Im}(\beta) |u|^2 \right)^2 \leq \left( \int_{\partial\Omega} \|\beta\|_{L^\infty(\partial\Omega)} |u|^2 \right)^2 \\ &\leq c^2 \gamma^2 \|u\|_{H^1(\Omega)}^2 \|u\|_{L^2(\Omega)}^2 \leq 2c^2 \gamma^2 (\text{Re} \langle Au, u \rangle + \omega_0), \end{aligned}$$

where the last inequality follows from part (a) (in particular  $\omega_0$  is the constant chosen in part (a), which depends on the domain  $\Omega$ ), and  $\|u\|_{L^2(\Omega)}^2 = 1$ .

(c) We will start by showing that  $-\Delta_\beta$  is quasi- $m$ -sectorial of any angle  $\varphi \in (0, \frac{\pi}{2})$ . (Note that we have to exclude the case  $\varphi = 0$ , since that would require that  $\text{num}(-\Delta_\beta)$  be real, and of course this is impossible if  $\text{Im} \beta \neq 0$ .)

It is already clear that the shifted form  $a + \omega$ , defined by

$$(a + \omega)(u, v) = a(u, v) + \omega(j(u), j(v))_H \text{ for all } u, v \in V,$$

is associated with  $-\Delta_\beta + \omega I$  and enjoys all the properties of  $a$ , being in particular coercive for  $\omega \geq 0$  large enough (this is the  $j$ -ellipticity of  $a$  from (a)) and obviously bounded. Moreover, we still have  $\text{ran}(I - \Delta_\beta + \omega I) = H$ .

We merely have to show that for fixed  $0 < \varphi < \frac{\pi}{2}$ , we can find  $\omega \geq 0$  for which  $\text{num}(-\Delta_\beta + \omega I)$  is contained in the sector  $\Sigma_{\varphi, 0}$ . Obviously,  $\text{num}(-\Delta_\beta + \omega I) = \text{num}(-\Delta_\beta) + \omega$ , so  $\text{num}(-\Delta_\beta + \omega I)$  is contained in a parabola formed by shifting the parabola in part (b) to the right (i.e. in the direction of the positive real axis) by a factor of  $\omega$ . Clearly, by making  $\omega$  large enough, we can ensure that this parabola is contained in the sector  $\Sigma_{\varphi, 0}$  (provided  $\varphi \neq 0!$ ). This is exactly what we wanted to show.

It remains to prove that  $\Delta_\beta = -A$  is the generator of a holomorphic semigroup of angle  $\pi/2$ . Fix  $0 < \phi < \pi/2$ . We have just proved that there exists  $\omega > 0$  such that  $-\Delta_\beta + \omega I$  (the operator associated with  $(a + \omega, j)$ ) is  $m$ -sectorial of angle  $\frac{\pi}{2} - \phi$ . Hence, Theorem 3.22 says that  $-(A + \omega I)$  is the generator of a holomorphic contractive semigroup  $T_{-\omega}$  of angle  $\phi$ . As in Ex.7.1, one easily concludes that the scaled holomorphic semigroup  $T(t) := e^{\omega t} T_{-\omega}(t)$  is generated by  $-A = \Delta_\beta$  and of angle  $\phi$  (use Lemma 3.10 with Ex. 2.2 and use that  $e^{\omega t} T_{-\omega}(t)$  is holomorphic). So, for any  $0 < \phi < \frac{\pi}{2}$ , we conclude that  $\Delta_\beta$  generates a holomorphic semigroup of angle  $\phi$ , i.e.  $T$  is holomorphic on  $\Sigma_\phi$ . But since

$$\Sigma_{\frac{\pi}{2}} = \bigcup_{0 < \phi < \frac{\pi}{2}} \Sigma_\phi,$$

it follows that  $T$  is holomorphic on the whole (open) sector  $\Sigma_{\frac{\pi}{2}}$ , that is, the operator  $-A = \Delta_\beta$  is the generator of a holomorphic semigroup of angle  $\pi/2$ , and we are done.