Solutions to the Exercises of Lecture 6

Marrakesh team

Solution to Exercice 6.1.

Show that A is a compact operator if and only if A maps weakly convergent sequences in G to convergent sequences in H.

$\Rightarrow$) Let $(x_n) \subset G$ and $x_n \rightharpoonup x$, we show that

$$Ax_n \to Ax. \tag{1}$$

Now we suppose that the condition (1) is false, then

$$\exists \varepsilon > 0,\, \forall N \in \mathbb{N}, \exists n_N \geq N \;\; \| Ax_{n_N} - Ax \| > \varepsilon.$$

Since (x_n) is bounded and A is compact then $\exists (x_{n_k})_{k\in\mathbb{N}} \,/\, Ax_{n_k} \to y$.
We have $(Ax_{n_k}|z) \to (y|z)$, then $(x_{n_k}|A^*z) \to (y|z)$. Since
$(x_{n_k}|A^*z) \to (x|A^*z)$, then $(Ax|z) = (y|z)$.
We have $Ax_{n_k} \to Ax$, i.e.,

$$\exists N_1 \in \mathbb{N},\, \forall n \geq N_1,\;\; \| Ax_{n_k} - Ax \| < \varepsilon.$$

Which contradicts (1) and ends the proof.
$\Leftarrow$) We have $(x_n) \subset G$ and $x_n \rightharpoonup x$, such that

$$Ax_n \to Ax. \tag{2}$$

Let (x_n) be a bounded sequence of G. Since G is a Hilbert space, then $\exists x_{\varphi_n}, / \;\; x_{\varphi_n} \rightharpoonup z$, and by (2) we have $Ax_{\varphi_n} \to Az$.
Consequently, we have $A(\overline{B_G(0,1)})$ is compact which is equivalent to A is compact.

Solution to Exercice 6.2.

(a) Show that $dom(A_0)$ is dense in $L^2(\Omega,\mu)$ and $A_0^* = M_{\bar{m}}$.

• Let

$$\mathcal{E} = lin\{1_E : E \subseteq \Omega,\; \text{a mesurable set such that } \mu(E) < \infty\}.$$

It is well known that $\mathcal{E}$ is a dense subspace of $L^2(\Omega,\mu)$ (basic result from mesure theory and integration) and $dom(A_0) \subseteq \mathcal{E}$.
Let E be a mesurable subset of Ω such that $\mu(E) < \infty$ and for $n \in \mathbb{N}$, we set

$$E_n = \{x \in E : |m(x)| \leq n\}.$$

Then $1_E = \lim_{n\to\infty} 1_{E_n}$ and by the Lebesgue's Dominated Convergence Theorem this limit belongs to $L^2(\Omega,\mu)$.
Consequently, $dom(A_0)$ is dense in $L^2(\Omega,\mu)$, which achieves the proof.
• Let $g \in dom(M_{\bar{m}})$, for every $f \in dom(A_0)$ we have :

$$(A_0 f, g) = \int_\Omega m f \bar{g} d\mu = \int_\Omega f(m\bar{g}) d\mu = (f, M_{\bar{m}} g),$$

then $g \in dom(A_0^*)$ and $A_0^* g = \bar{m} g$. i.e., $dom(M_{\bar{m}}) \subseteq dom(A_0^*)$ and $A^*_{0|dom(M_{\bar{m}})} = M_{\bar{m}}$.
On the other hand, let $g \in dom(A_0^*)$, then for all $f \in dom(A_0)$,

$$\left| \int_\Omega f(m\bar{g}) d\mu \right| = |(A_0 f, g)| = |(f, A_0^* g)| \leq c\|f\|_{L^2(\Omega,\mu)}, \tag{3}$$

and by the density of $dom(A_0^*)$ in $L^2(\Omega,\mu)$, (3) holds for all $f \in L^2(\Omega,\mu)$ with $c = \|A_0^* g\|_{L^2(\Omega,\mu)}$.
Finaly we can see that $f \mapsto \int_\Omega f(m\bar{g}) d\mu$ is a linear form over $L^2(\Omega,\mu)$ and by Riesz representation theorem, there exists $h \in L^2(\Omega,\mu)$ such that

$$\int_\Omega f(m\bar{g}) d\mu = \int_\Omega f\bar{h} d\mu, \qquad \forall f \in L^2(\Omega,\mu),$$

then we obtain $\bar{m} g = \bar{h} \in L^2(\Omega,\mu)$, i.e., $g \in dom(M_{\bar{m}})$. □
(b) Show that $M^*_{\bar{m}} = M_m$.
• By a similar argument as above, we show that $dom(M_m) \subseteq dom(M^*_{\bar{m}})$ and $M^*_{\bar{m}|dom(M_m)} = M_m$.
Now let $g \in dom(M^*_{\bar{m}})$, for all $f \in dom(M_{\bar{m}})$, we have :

$$\left| \int_\Omega f\bar{m}\bar{g} d\mu \right| = |(\bar{m} f, g)| = |(f, M^*_{\bar{m}} g)| \leq c'\|f\|_{L^2(\Omega,\mu)}, \tag{4}$$

and by the density of $dom(M_{\bar{m}})$ in $L^2(\Omega,\mu)$, (4) holds for all $f \in L^2(\Omega,\mu)$ with $c' = \|M^*_{\bar{m}} g\|_{L^2(\Omega,\mu)}$.
This yields to $mg \in L^2(\Omega,\mu)$ and then $g \in dom(M_m)$. □
(c) Assume that m is real-valued. Show that A_0 is essentially self-adjoint, and $\bar{A}_0 = M_m$.
Following (b) and since m is real-valued we have M_m is self adjoint (note that M_m is closed), then A_0 is symmetric. Therefore A_0 is closable and $A_0^* = \bar{A}$ (see remarque (6.5)). However, from (a) we have $A_0^* = M_{\bar{m}} = M_m$. We conclude that $\bar{A} = M_m$ is self adjoint. □

Solution to Exercice 6.3.

(a) Let X be a vector space, Y an n-dimensional subspace, Z an $(n-1)$-codimensional subspace (i.e., there exists an $(n-1)$-dimensional subspace Z_0 of X such that $Z \cap Z_0 = \{0\}$, $Z + Z_0 = X$). Show that $Y \cap Z \neq \{0\}$.

• Assume that $Y \cap Z = \{0\}$ and let $\{e_1, \cdots, e_n\}$ be a basis of Y, then for every $i \in \{1, \cdots, n\}$ there exists $(f_i, g_i) \in Z \times Z_0$ such that $e_i = f_i + g_i$. Let's show that $\{g_1, \cdots, g_n\}$ is a linear

independent finite sequence on Z_0, which contradicts the fact that $\dim(Z_0) < n$.
Let $\{\alpha_1, \cdots, \alpha_n\} \subseteq \mathbb{K}$ such that

$$\sum_{i=1}^{n} \alpha_i g_i = 0,$$

then

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i e_i &= \sum_{i=1}^{n} \alpha_i f_i + \sum_{i=1}^{n} \alpha_i g_i \\ &= \sum_{i=1}^{n} \alpha_i f_i \in Y \cap Z = \{0\}. \end{aligned}$$

Therefore $\sum_{i=1}^{n} \alpha_i e_i = 0$ and so $\alpha_1 = \cdots = \alpha_n = 0$. $\Box$
(b) Show the min-max principle:

$$\lambda_n = max\{I(x_1, ..., x_{n-1}),\ x_1, ..., x_{n-1} \in H\}.$$

• First, we start by fixing some notations (see examples (6.6) and (6.16)),

$$I(x_1, \cdots, x_{n-1}) = \inf\{a(x) : x \in \{x_1, \cdots, x_{n-1}\}^{\perp} \cap V,\ \|x\| = 1\},$$

$$a(x) = (Ax, x), \qquad Ax = \sum_{n=0}^{\infty} \lambda_n (x|e_n) e_n,$$

$$V = \{u \in H : \sum_{n=0}^{\infty} \lambda_n |(u|e_n)|^2 < \infty\}$$

and

$$dom(A) = \{u \in H : \sum_{n=0}^{\infty} |\lambda_n|^2 |(u|e_n)|^2 < \infty\}.$$

Here A is a self-adjoint diagonal operator in H associated with the form a, $(\lambda_n)_{n\in\mathbb{N}}$ is the positive increasing sequence of eignvalues of A and $(e_n)_{n\in\mathbb{N}}$ the orthonormal basis of H, such that for every $n \in \mathbb{N}$, e_n is an eignfunction of A associated with λ_n.
Now let $n \in \mathbb{N}$, $\{x_1, \cdots, x_{n-1}\} \subset H$ and $E_n = lin\{e_1, \cdots, e_n\}$. According to (a) we have

$$E_n \cap \{x_1, \cdots, x_{n-1}\}^{\perp} \cap V = E_n \cap \{x_1, \cdots, x_{n-1}\}^{\perp} \neq \{0\},$$

then

$$x = \sum_{i=0}^{n} (x|e_i) e_i \in E_n \cap \{x_1, \cdots, x_{n-1}\}^{\perp}\ s.\ t.\ \|x\| = \sum_{i=0}^{n} |(x|e_i)|^2 = 1.$$

We have

$$\begin{aligned} a(x) = (Ax, x) &= \sum_{i=0}^{n} \lambda_i |(x|e_i)|^2 \\ &\le \lambda_n \sum_{i=0}^{n} |(x|e_i)|^2 \\ &\le \lambda_n. \end{aligned}$$

Moreover, $I(x_1, \cdots, x_{n-1}) \le a(x) \le \lambda_n$ for all $x_1, \cdots, x_{n-1} \in H$. Thus

$$\max\{I(x_1, \cdots, x_{n-1}) : x_1, \cdots, x_{n-1} \in H\} \le \lambda_n.$$

On the other hand, for each $n \in \mathbb{N}$ we have

$$I(e_1, \cdots, e_{n-1}) = \inf\{a(x) : x \in \{e_1, \cdots, e_{n-1}\}^{\perp} \cap V \ and \ \|x\| = 1\}$$

and for every $x \in \{e_1, \cdots, e_{n-1}\}^{\perp} \cap V$ with $\|x\| = 1$, we have

$$a(x) = \sum_{i=n}^{\infty} \lambda_i |(x|e_i)|^2 \ge \lambda_n \sum_{i=n}^{\infty} |(x|e_i)|^2 = \lambda_n \|x\|^2 = \lambda_n$$

then

$$I(e_1, \cdots, e_{n-1}) \ge \lambda_n.$$

Finally

$$\max\{I(x_1, \cdots, x_{n-1}) : x_1, \cdots, x_{n-1} \in H\} \ge \lambda_n. \qquad \square$$

(c) Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$ be bounded open sets, and let Δ_j be the Dirichlet-Laplacian on Ω_j and $(\lambda_k^j)_{k\in\mathbb{N}}$ the corresponding increasing sequence of eigenvalues, for $j \in \{1, 2\}$. Show the domain monotonicity of eigenvalues: $\lambda_k^1 \ge \lambda_k^2$ for all $k \in \mathbb{N}$.

• Let $j \in \{1, 2\}$, by Theorem 6.17 and Theorem 6.21, $-\Delta_j$ is a positive diagonal self-adjoint operator with increasing eigenvalues $(\lambda_n^j)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \lambda_n^j = +\infty$. From Example 5.13, $-\Delta_j$ is associated to the sesquilinear form a_j defined on $V_j = H_0^1(\Omega_j)$ by

$$a_j(u, v) = \int_{\Omega_j} \nabla u.\nabla \bar{v} dx, \quad j \in \{1, 2\}.$$

Then by the min-max principle (See question (b) above) :

$$\lambda_n^j = max\{I_j(u_1, ..., u_{n-1}) : u_1, ..., u_{n-1} \in L^2(\Omega_j)\}, \quad j \in \{1, 2\}, \quad n \in \mathbb{N}.$$

On the other hand, we have $L^2(\Omega_1) \subseteq L^2(\Omega_2)$ and $V_1 \subseteq V_2$.

Indeed, let $u \in L^2(\Omega_1)$ (resp. $u \in V_1$) and $\tilde{u}$ the extension of u to Ω_2 by zero, then $\tilde{u} \in L^2(\Omega_2)$ (resp. $\tilde{u} \in V_2$) (See exercise 4.2). Moreover, we have $a_1(u) = a_2(\tilde{u})$ for all $u \in V_1$ and for all $u \in L^2(\Omega_1)$. Hence, $\|u\|_1 = \|\tilde{u}\|_2$.

Now let $n \in \mathbb{N}$, $(u_1, ..., u_{n-1}) \subset L^2(\Omega_2)$ and for all $k \in \{1, ..., n-1\}$, we denote by $u_k^{(1)}$ the restriction of u_k on Ω_1, then $u_k^{(1)} \in L^2(\Omega_1)$ and since $V_1 \subset V_2$, one has

$$\begin{aligned} I_2(u_1, ..., u_{n-1}) &= \inf\{a_2(u) : u \in \{u_1, ..., u_{n-1}\}^\perp \cap V_2, \|u\|_2 = 1\} \\ &\leq \inf\{a_2(u) : u \in \{u_1, ..., u_{n-1}\}^\perp \cap V_1, \|u\|_2 = 1\} \\ &= \inf\{a_1(u) : u \in \{u_1, ..., u_{n-1}\}^\perp \cap V_1, \|u\|_1 = 1\} \\ &= \inf\{a_1(u) : u \in \{u_1^{(1)}, ..., u_{n-1}^{(1)}\}^\perp \cap V_1, \|u\|_1 = 1\} \\ &= I_1(u_1^{(1)}, ..., u_{n-1}^{(1)}). \end{aligned}$$

We obtain then the following conclusion : For arbitrary finite-sequence $\{v_1, ..., v_{n-1}\} \subset L^2(\Omega_1)$ and for arbitrary extension $\{\tilde{v}_1, ..., \tilde{v}_{n-1}\} \subset L^2(\Omega_2)$ (of $\{v_1, ..., v_{n-1}\}$), we have

$$I_1(v_1, ..., v_{n-1}) \geq I_2(\tilde{v}_1, ..., \tilde{v}_{n-1}).$$

Applying the maximum to the previous inequality, we get

$$\lambda_k^1 \geq \lambda_k^2. \qquad \square$$

Solution to Exercice 6.4.

(a) Let $f \in C(-1,1)$, and assume that $g_1 := \left(f_{|(-1,0)}\right)' \in L^1(-1,0)$, $g_2 := \left(f_{|(0,1)}\right)' \in L^1(0,1)$. Define $g \in L^1(-1,1)$ by $g_{|(-1,0)} := g_1$, $g_{|(0,1)} := g_2$. Show that $f' = g$ (all derivatives in the distributional sense).

• Let $\phi \in C_c^\infty(-1,1)$, $-1 < a < 0 < b < 1$ such that $supp(\phi) \subseteq [a,b]$ and let $n \geq 1$, then

$$\begin{aligned} \int_{-1}^{1} f(x)\phi'(x)dx &= \int_a^b f(x)\phi'(x)dx \\ &= \int_a^{-\frac{1}{n}} f(x)\phi'(x)dx + \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x)\phi'(x)dx + \int_{\frac{1}{n}}^{b} f(x)\phi'(x)dx. \end{aligned}$$

Applying the Green-Riemann formula on $[a, -\frac{1}{n}]$ and $[\frac{1}{n}, b]$ respectively, we get

$$\begin{aligned} \int_a^{-\frac{1}{n}} f(x)\phi'(x)dx &= [f(x)\phi(x)]_a^{-\frac{1}{n}} - \int_a^{-\frac{1}{n}} g_1(x)\phi(x)dx \\ &= f(-\frac{1}{n})\phi(-\frac{1}{n}) - \int_{-1}^{-\frac{1}{n}} g(x)\phi(x)dx, \end{aligned}$$

and

$$\begin{aligned} \int_{\frac{1}{n}}^{b} f(x)\phi'(x)dx &= [f(x)\phi(x)]_{\frac{1}{n}}^{b} - \int_{\frac{1}{n}}^{b} g_2(x)\phi(x)dx \\ &= -f(\frac{1}{n})\phi(\frac{1}{n}) - \int_{\frac{1}{n}}^{1} g(x)\phi(x)dx. \end{aligned}$$

Since f and ϕ are continuous, we get

$$\lim_{n\to\infty}\int_a^{-\frac{1}{n}} f(x)\phi'(x)dx = f(0)\phi(0) - \int_{-1}^{0} g(x)\phi(x)dx$$

and

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{b} f(x)\phi'(x)dx = -f(0)\phi(0) - \int_{0}^{1} g(x)\phi(x)dx.$$

Moreover

$$\lim_{n\to\infty}\int_{\frac{-1}{n}}^{\frac{1}{n}} f(x)\phi'(x)dx = 0.$$

Therefore

$$\int_{-1}^{1} f(x)\phi'(x)dx = -\int_{-1}^{0} g(x)\phi(x)dx - \int_{0}^{1} g(x)\phi(x)dx = -\int_{-1}^{1} g(x)\phi(x)dx$$

and then $f \in H^1(a,b)$ and $f' = g$ in the distributional sense. $\square$

(b) Let $a, b \in \mathbb{R}$, $a < b$. Show that

$$H_0^1(a,b) = \{f \in H^1(a,b) : f(a) = f(b) = 0\}.$$

• Let $f \in H_0^1(a,b)$, then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(a,b)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{H^1(a,b)} = 0.$$

By Theorem 4.9, $H^1(a,b) \subset C[a,b]$ with continuous embedding, then

$$\lim_{n\to\infty} \|f_n - f\|_{\infty,[a,b]} = 0,$$

and finally

$$f(a) = \lim_{n\to\infty} f_n(a) = 0, \quad f(b) = \lim_{n\to\infty} f_n(b) = 0.$$

On the other hand, let $f \in H^1(a,b)$ such that $f(a) = f(b) = 0$. In order to show that $f \in H_0^1(a,b)$, we show hint 2: For some $c \in (a,b)$,

$$\big(g \in H^1(a,b), g(a) = 0 \text{ and } g_{|(c,b)} = 0\big) \Longrightarrow g \in H_0^1(a,b). \tag{5}$$

Let $g \in H^1(a,b)$ satisfying $g(a) = 0$ and $g_{|(c,b)} = 0$ for some $c \in (a,b)$. For all $n \in \mathbb{N}$ satisfying $\frac{1}{n} < c - a$, we consider

$$g_n(x) = \begin{cases} 0, & x \in (a, a+\frac{1}{n}), \\ ng(a+\frac{2}{n})(x - a - \frac{1}{n}), & x \in (a+\frac{1}{n}, a+\frac{2}{n}), \\ g(x), & x \in (a+\frac{2}{n}, b). \end{cases}$$

g_n is continuous on $[a,b]$, $supp(g_n) \subseteq [a+\frac{1}{n}, c]$ is a compact subset of $[a,b]$. The functions $g_n \mid_{[a,a+\frac{1}{n}]}$, $g_n \mid_{[a+\frac{1}{n},a+\frac{2}{n}]}$ and $g_n \mid_{[a+\frac{2}{n},b]}$ are all differentiable and their derivatives belong to $L^2(a,b) \subset L^1(a,b)$. By the same argument as in question (a), we can show that g_n is differentiable in the distributional sense and its derivative is given by

$$g_n'(x) = \begin{cases} 0, & x \in (a, a+\frac{1}{n}), \\ ng(a+\frac{2}{n}), & x \in (a+\frac{1}{n}, a+\frac{2}{n}), \\ g'(x), & x \in (a+\frac{2}{n}, b). \end{cases}$$

Since $g_n' \in L^2(a,b)$, then $g_n \in H_c^1(a,b)$. Moreover, $g = \lim_{n\to\infty} g_n$ and $g' = \lim_{n\to\infty} g_n'$ almost everywhere. By Lebesgue Theorem, this limit takes place in $L^2(a,b)$. We conclude that

$$g = \lim_{n\to\infty} g_n \in H_0^1(a,b).$$

Now we will approximate f by a function sequence $(f_n)_{n\geq N}$ satisfying (5). let $N \in \mathbb{N}$ such that $\frac{2}{N} < b-a$. For all $n \geq N$, we define

$$f_n(x) = \begin{cases} f(x), & x \in (a, b-\frac{2}{n}), \\ nf(b-\frac{2}{n})(b-\frac{1}{n}-x), & x \in (b-\frac{2}{n}, b-\frac{1}{n}), \\ 0, & x \in (b-\frac{1}{n}, b). \end{cases}$$

According to the argument used for the sequence $(g_n)_{n\geq 1}$, one could show that $(f_n)_{n\geq N} \subset H^1(a,b)$ such that $f_n(a) = 0$, $f_n \mid_{|(b-\frac{1}{n},b)} = 0$ and

$$f_n'(x) = \begin{cases} f'(x), & x \in (a, a+\frac{1}{n}) \\ -nf(b-\frac{2}{n}), & x \in (a+\frac{1}{n}, a+\frac{2}{n}) \\ 0, & x \in (a+\frac{2}{n}, b). \end{cases}$$

Moreover, we have

$$\lim_{n\to\infty} \|f_n - f\|_{H^1(a,b)} = 0.$$

Therefore, by (5) (and hint (2)), we conclude that $f \in H_0^1(a,b)$. $\square$

(c) Compute the orthonormal basis of eigenfunctions and the eigenvalues of $-\Delta_D$ for $\Omega = (0,\pi)$.

• It is well known that

$$dom(\Delta_D) = \{u \in H_0^1(0,\pi) : \Delta u = u'' \in L^2(0,\pi)\}.$$

According to question (b),

$$dom(\Delta_D) = \{u \in H^1(0,\pi) : u' \in H^1(0,\pi) \text{ and } u(0) = u(\pi) = 0\}$$

and from Theorem 4.9, if $u, u' \in H^1(0,\pi)$ then $u \in C^1(0,\pi)$.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of $-\Delta_D$ and $u \in dom(\Delta_D)$ such that $-u'' = \lambda u$.

$-\Delta_D$ is a self-adjoint positive operator, and its eigenvalues are all positive. Therefore $\lambda \geq 0$, $u \in C^\infty(0, \pi)$ and

$$u(x) = Asin(\lambda x) + Bcos(\lambda x), \qquad \forall x \in [0, \pi].$$

$u(0) = 0$ implies that $B = 0$ and if $u \neq 0$ then $u(\pi) = 0$ implies $sin(\lambda\pi) = 0$ and therefore $\lambda = n \in \mathbb{N}$. Finally the system of eigenvalues of $-\Delta_D$ is $(\lambda_n = n)_{n\in\mathbb{N}}$ and the associated eigenfunctions are $(e_n(x) = A_n sin(nx))_{n\in\mathbb{N}}$ where A_n can be choosed such that $\|e_n\|_{L^2(0,\pi)} = 1$. □

(d) Determine the optimal value of the Poincaré constant for the open set $(0, \pi)$ (see Section 5.4).

• Let α_0 be this optimal constant of Poincaré inequality then

$$\|u\|_{L^2(0,\pi)} \leq \alpha_0 \|\nabla u\|_{L^2(0,\pi)}, \quad \forall u \in H_0^1(0, \pi) \tag{6}$$

and for any constant α satisfying (6) : $\alpha \geq \alpha_0$. Now let $u \in H_0^1(0, \pi) \cap H^2(0, \pi)$ then, from (c) and by Example (6.19), we get :

$$-\Delta u = -u'' = \sum_{n=0}^{+\infty} n(u|e_n)e_n,$$

where $e_n(x) = A_n sin(nx)$ for almost every $x \in (0, \pi)$, (See question (c)) and

$$(u|e_n) = (u|e_n)_{L^2(0,\pi)} = \int_0^\pi u(x)\bar{e}_n(x)dx.$$

Therefore

$$\begin{aligned}
\|\nabla u\|^2_{L^2(0,\pi)} &= (-\Delta u|u)_{L^2(0,\pi)} \\
&= \Big(\sum_{n=0}^{+\infty} n(u|e_n)e_n \Big| \sum_{n=0}^{+\infty}(u|e_n)e_n\Big)_{L^2(0,\pi)} \\
&= \sum_{n=1}^{+\infty} n|(u|e_n)|^2 \\
&\geq \sum_{n=1}^{+\infty} |(u|e_n)|^2 \\
&\geq \|u\|^2_{L^2(0,\pi)},
\end{aligned}$$

and by density of $H_0^1(0, \pi) \cap H^2(0, \pi)$ in $H_0^1(0, \pi)$ (remark that $C_c^\infty(0, \pi) \subset H_0^1(0, \pi) \cap H^2(0, \pi)$), (6) holds for every u in $H_0^1(0, \pi)$ and so $1 \geq \alpha_0$.

On the other hand, for $u \in L^2(0, \pi)$, $u(x) = e_1(x) = sin(x)$, we have :

$$\|\nabla u\|^2 = \|u'\|^2 = \int_0^\pi cos^2(x)dx = \int_0^\pi sin^2(x)dx = \|u\|^2 \leq \alpha_0^2 \|\nabla u\|^2.$$

then $1 \leq \alpha_0$. Finally the optimal constant of poincar inequality is $\alpha_0 = 1 = \frac{1}{\lambda_1}$ the inverse of the first nonzero eigenvalue of $-\Delta_D$. □