Solutions to the Exercises of Lecture 5 of the Internet Seminar 2014/15

Exercise 5.1

Claim: Let V be a normed vector space, and let $a\colon V \times V \to \mathbb{K}$ be a sesquilinear form. Then a is bounded if and only if a is continuous.
(Note that this is slightly more general than the statement given in the exercise, since V needs not be a Hilbert space.)

Proof. First assume that a is bounded, i.e., there exists a constant $M \geq 0$ such that $|a(u,v)| \leq M \|u\|_V \|v\|_V$ for all $u, v \in V$. Let $((u_n, v_n))_{n\in\mathbb{N}} \in (V \times V)^{\mathbb{N}}$ and $u, v \in V$ such that $(u_n, v_n) \to (u, v)$ in $V \times V$ as $n \to \infty$ (which is equivalent to $u_n \to u, v_n \to v$ in V as $n \to \infty$). Since $(u_n)_{n\in\mathbb{N}}$ is convergent there exists $C \geq 0$ such that $\|u_n\|_V \leq C$ for all $n \in \mathbb{N}$. Hence we compute

$$\begin{aligned}
|a(u,v) - a(u_n,v_n)| &\leq |a(u,v) - a(u_n,v)| + |a(u_n,v) - a(u_n,v_n)| \\
&= |a(u-u_n,v)| + |a(u_n, v - v_n)| \\
&\leq M \|u-u_n\|_V \|v\|_V + M \|u_n\|_V \|v-v_n\|_V \\
&\leq M \|u-u_n\|_V \|v\|_V + M\, C \|v-v_n\|_V
\end{aligned}$$

for all $n \in \mathbb{N}$, which tends to 0 as $n \to \infty$. Thus we have shown that a is continuous.
In order to show the converse, assume that a is not bounded. Then there exist sequences $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}}$ such that

$$|a(u_n,v_n)| > n \|u_n\|_V \|v_n\|_V$$

for all $n \in \mathbb{N}$. Now let $\widetilde{u}_n := \frac{1}{\sqrt{n}} \cdot \frac{u_n}{\|u_n\|_V}$ and $\widetilde{v}_n := \frac{1}{\sqrt{n}} \cdot \frac{v_n}{\|v_n\|_V}$ for $n \in \mathbb{N}$ (note that $u_n, v_n \neq 0$ for every $n \in \mathbb{N}$). Then $\widetilde{u}_n, \widetilde{v}_n \to 0$ in V as $n \to \infty$, hence $(\widetilde{u}_n, \widetilde{v}_n) \to (0,0)$ in $V \times V$ as $n \to \infty$. Furthermore,

$$|a(\widetilde{u}_n, \widetilde{v}_n)| = \frac{|a(u_n,v_n)|}{n \|u_n\|_V \|v_n\|_V} > 1 \tag{1}$$

for all $n \in \mathbb{N}$. Thus $(\widetilde{u}_n, \widetilde{v}_n) \to (0,0)$ as $n \to \infty$, but $a(\widetilde{u}_n, \widetilde{v}_n)$ does not converge to 0 as $n \to \infty$ because of (1). Hence a is not continuous. □

Remark: Note that if V is a Banach space, then a sesquilinear form on V is bounded if and only if it is componentwise continuous, as we shall show now.
Claim: Let V be a Banach space and let $a\colon V \times V \to \mathbb{K}$ be a sesquilinear form. Then the following statements are equivalent

(i) a is continuous;

(ii) a is continuous at $(0,0)$;

(iii) a is bounded;

(iv) every component of a is continuous, i.e., the mapping $a_u : V \to \mathbb{K}, v \mapsto a(u,v)$, is continuous for every $u \in V$ and the mapping $a^v : V \to \mathbb{K}, u \mapsto a(u,v)$, is continuous for every $v \in V$;

Proof. The implications *(i)* to *(ii)* and *(i)* to *(iv)* are easy to see and the implication *(iii)* to *(i)* has already been shown above. Moreover, a closer look at the proof of Exercise 5.1 reveals that in order to show that a is bounded, one only needs that a is continous at $(0,0)$. Thus we already proved the implication *(ii)* to *(iii)*. Hence we only have to show the implication *(iv)* to *(iii)*. To this end assume that a_u and a^v are continuous for all $u, v \in V$. Since a is sesquilinear, we conclude that $a^v \in V'$ for all $v \in V$ and $a_u \in V^*$ for all $u \in V$. Thus for every $u \in V$ there exists a constant $M_u \geq 0$ such that $|a_u(v)| \leq M_u \|v\|_V$ for all $v \in V$. Let $B = \{v \in V : \|v\|_V \leq 1\}$. Then

$$\sup_{v \in B} |a^v(u)| = \sup_{v \in B} |a(u,v)| = \sup_{v \in B} |a_u(v)| \leq \sup_{v \in B} M_u \|v\|_V = M_u$$

for all $u \in V$. Applying the uniform boundedness principle (Banach-Steinhaus) (see for instance Theorem 2.2 in [Bre11]) we conclude that $\sup_{v \in B} \|a^v\|_{V'}$ is finite, i.e., there exists a constant $M \geq 0$ such that

$$|a^v(u)| \leq M \|u\|_V$$

for all $u \in V$ and all $v \in V$ with $\|v\|_V \leq 1$. Hence

$$|a(u,v)| = \|v\|_V \left| a(u, \frac{v}{\|v\|_V}) \right| = \|v\|_V \left| a^{\frac{v}{\|v\|_V}}(u) \right| \leq M \|v\|_V \|u\|_V$$

for all $u, v \in V, v \neq 0$, which finishes the proof. □

Note that the same result holds if we consider Banach spaces X and Y and a sesquilinear or bilinear form $b : X \times Y \to \mathbb{K}$, the proof being the same as above.

Exercise 5.2

a) Let V, H be complex Hilbert spaces, $j \in \mathcal{L}(V,H)$ with dense range and $a : V \times V \to \mathbb{C}$ a bounded and coercive form, i.e. there are constants $M > 0$ and $\alpha > 0$ with

$$|a(u,v)| \leq M \|u\|_V \|v\|_V$$

and

$$\operatorname{Re} a(u) \geq \alpha \|u\|_V^2$$

for $u, v \in V$. Let A denote the operator associated to the form a and T the semigroup generated by $-A$.

Claim: For all $\varepsilon \in (0, \frac{\alpha}{\|j\|^2}]$ one can estimate

$$\|T(t)\| \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$.

Proof. Let $\varepsilon \in (0, \frac{\alpha}{\|j\|^2})$, then $C_\varepsilon := \alpha - \varepsilon\|j\|^2 > 0$. Define a sesquilinear form $b\colon V \times V \to \mathbb{C}$ by

$$b(u,v) := a(u,v) - \varepsilon(j(u) \,|\, j(u))_H$$

for $u, v \in V$. By Cauchy-Schwarz and $j \in \mathcal{L}(V, H)$ we obtain the boundedness estimate

$$\begin{aligned} |b(u,v)| &\leq |a(u,v)| + \varepsilon|(j(u) \,|\, j(u))_H| \\ &\leq M\|u\|_V\|v\|_V + \varepsilon\|j\|^2\|u\|_V\|v\|_V = (M + \varepsilon\|j\|^2)\|u\|_V\|v\|_V \end{aligned}$$

and the coercivity estimate

$$\operatorname{Re} b(u) = \operatorname{Re} a(u) - \varepsilon\|j(u)\|_H^2 \geq \alpha\|u\|_V^2 - \varepsilon\|j\|^2\|u\|_V^2 = C_\varepsilon\|u\|_V^2$$

for $u, v \in V$. Let B denote the operator associated to the form b. By Theorem 5.9 $-B$ generates a holomorphic contraction semigroup S and Remark 5.6 yields the representation $-A = -B - \varepsilon$ and we obtain

$$\|T(t)\| = \mathrm{e}^{-\varepsilon t}\|S(t)\| \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$. The estimate for $\varepsilon = \frac{\alpha}{\|j\|^2}$ can be obtained by choosing $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, \frac{\alpha}{\|j\|^2})$ with $\varepsilon_n \to \frac{\alpha}{\|j\|^2}$ for $n \to \infty$ and using the estimate above for each ε_n. □

b) *Claim:* Let $\Omega \subset \mathbb{R}^d$ be an open set which lies in a strip of width $d > 0$. Let $\varepsilon \in (0, \frac{1}{d^2}]$. Then

$$\|\mathrm{e}^{t\Delta_D}\|_{\mathcal{L}(L_2(\Omega))} \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$.

Proof. Let $a\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ be the classical Dirichlet form from the lecture given by

$$a(u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v} \mathrm{d}x.$$

Since $\Omega \subset \mathbb{R}^d$ is an open set lying in a strip, we obtain by Poincaré's inequality (Theorem 5.12.)

$$a(u) = \int_\Omega |\nabla u|^2 \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |u|^2 \mathrm{d}x \leq (1 + c_p)\int_\Omega |\nabla u|^2 \mathrm{d}x = (1 + c_p)a(u)$$

for $u \in H_0^1(\Omega)$, where $c_p \in (0, d^2)$ denotes the Poincaré constant. So $\|\cdot\|_{H_0^1(\Omega)} := a(\cdot)^{\frac{1}{2}}$ defines an equivalent norm on $H_0^1(\Omega)$ (one can check the norm properties, the definiteness follows directly from the estimate above).
With respect to this norm, a is coercive with constant $\alpha = 1$ and bounded because of the estimate

$$|a(u,v)| = \Big|\int_\Omega \nabla u \cdot \overline{\nabla v}\mathrm{d}x\Big| \leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \|u\|_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)}$$

for all $u, v \in H_0^1(\Omega)$. Furthermore, Poincaré's inequality yields the bounded embedding $j\colon H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ with operator norm $\|j\| = \sqrt{c_p} \leq d$. Finally, we can apply part a) of this exercise and obtain

$$\|\mathrm{e}^{t\Delta_D}\|_{\mathcal{L}(L_2(\Omega))} \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$, if we choose $\varepsilon \in (0, \frac{1}{d^2}]$. □

Exercise 5.3

Let $-\infty < a < b < \infty$. By Theorem 4.9 we know that $H^1(a,b) \subseteq C[a,b]$ and so we always use the continuous representative for a function in $H^1(a,b)$.

Claims:

(a) Every function $u \in H^1(a,b)$ is Hölder continuous of index $1/2$, i.e., there is a constant $c \in (0,\infty)$ such that $|u(t) - u(s)| \leq c|t-s|^{1/2}$ for every $t, s \in (a,b)$.

(b) $H^1(a,b)$ is compactly embedded in $C[a,b]$, i.e., every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(a,b)$ has a uniformly convergent subsequence w.r.t. the norm $\|\cdot\|_{C[a,b]}$.

(c) Let $H^2(a,b) := \{u \in H^1(a,b)\colon u' \in H^1(a,b)\}$, then $u'' := (u')'$ is defined for all $u \in H^2(a,b)$. Then we have that $H^2(a,b) \subseteq C^1[a,b]$ if $C^1[a,b]$ carries the norm

$$\|u\|_{C^1[a,b]} = \|u\|_{C[a,b]} + \|u'\|_{C[a,b]}.$$

Proof. (a) Let $u \in H^1(a,b),\ s,t \in (a,b),\ s \leq t$. By Hölder's inequality we have

$$\begin{aligned} |u(t) - u(s)| &= \left|\int_s^t u'(x)\mathrm{d}x\right| \leq \int_s^t |u'(x)|\mathrm{d}x \\ &\leq \left(\int_s^t |u'(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_s^t 1\mathrm{d}x\right)^{\frac{1}{2}} \leq \|u\|_{H^1(a,b)}|t-s|^{\frac{1}{2}}. \end{aligned}$$

(b) Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(a,b)$, i.e., there is a constant $C \in (0,\infty)$ such that $\|u_n\|_{H^1(a,b)} \leq C$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary and define $\delta := \frac{\varepsilon^2}{C^2} > 0$. Then it holds by part (a) for every $s, t \in (a,b)$ with $|t-s| \leq \delta$

$$|u_n(t) - u_n(s)| \leq \|u_n\|_{H^1(a,b)}|t-s|^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}} = \varepsilon \ \forall n \in \mathbb{N},$$

i.e., the sequence $(u_n)_{n\in\mathbb{N}}$ is equi-continuous.

By the proof of Theorem 4.9 it holds

$$\|u\|_{C[a,b]} \leq ((b-a)^{-\frac{1}{2}} + (b-a)^{\frac{1}{2}})\|u\|_{H^1(a,b)} \ \forall u \in H^1(a,b).$$

This implies that $(u_n)_{n\in\mathbb{N}}$ is bounded by $((b-a)^{-\frac{1}{2}} + (b-a)^{\frac{1}{2}})C$ in $C[a,b]$. Now the Arzelá-Ascoli Theorem yields the assertion (b).

(c) Let $u \in H^2(a,b)$. Then $u' \in H^1(a,b)$ and it follows again by the inequality in the proof of Theorem 4.9 (see also (b))

$$\begin{aligned}\|u\|_{C^1[a,b]} &= \|u\|_{C[a,b]} + \|u'\|_{C[a,b]} \\ &\le ((b-a)^{-\frac{1}{2}} + (b-a)^{\frac{1}{2}})\|u\|_{H^1(a,b)} + ((b-a)^{-\frac{1}{2}} + (b-a)^{\frac{1}{2}})\|u'\|_{H^1(a,b)} \\ &\le 2((b-a)^{-\frac{1}{2}} + (b-a)^{\frac{1}{2}})\|u\|_{H^2(a,b)} < \infty,\end{aligned}$$

i.e., $H^2(a,b) \subseteq C^1[a,b]$. □

Exercise 5.4
Let $-\infty < a < b < \infty$ and α, $\beta > 0$. Define the operator A in $L_2(a,b)$ by

$$\begin{aligned}\operatorname{dom}(A) &= \{u \in H^2(a,b);\ -u'(a) + \alpha u(a) = 0, u'(b) + \beta u(b) = 0\}, \\ Au &= -u''.\end{aligned}$$

Claim: The following assertions hold.

(a) The operator A is m-sectorial.

(b) There exists some $\varepsilon > 0$ such that $\|e^{-tA}\|_{\mathcal{L}(L_2(a,b))} \le e^{-\varepsilon t}$ for all $t \ge 0$.

Proof. (a) In the following we will always use the continuous representative for a function in $H^1(a,b)$ (see Proposition 4.6).

We consider the form $a\colon H^1(a,b) \times H^1(a,b) \to \mathbb{K}$ given by

$$a(u,v) = \int_a^b u'\overline{v'}\,dx + \alpha u(a)\overline{v(a)} + \beta u(b)\overline{v(b)}.$$

First, we show that the operator A is associated with the form a and the embedding of $H^1(a,b)$ into $L_2(a,b)$. By the proof of Theorem 4.9 we have $\|u\|_{C[a,b]} \le C\|u\|_{H^1(a,b)}$ with $C := ((b-a)^{-1/2} + (b-a)^{1/2})$ for $u \in H^1(a,b)$. Together with the Cauchy-Schwarz inequality this implies

$$\begin{aligned}|a(u,v)| &\le \|u'\|_{L_2(a,b)}\|v'\|_{L_2(a,b)} + \alpha\|u\|_{C[a,b]}\|v\|_{C[a,b]} + \beta\|u\|_{C[a,b]}\|v\|_{C[a,b]} \\ &\le \|u'\|_{L_2(a,b)}\|v'\|_{L_2(a,b)} + \alpha C^2\|u\|_{H^1(a,b)}\|v\|_{H^1(a,b)} + \beta C^2\|u\|_{H^1(a,b)}\|v\|_{H^1(a,b)} \\ &\le \max\{1, \alpha C^2, \beta C^2\}\|u\|_{H^1(a,b)}\|v\|_{H^1(a,b)}\end{aligned}$$

for all u, $v \in H^1(a,b)$. As a consequence, the form a is bounded.

Furthermore, let j be the embedding of $H^1(a,b)$ into $L_2(a,b)$. Since $C_c^\infty(a,b)$ is dense in $L_2(a,b)$ and $C_c^\infty(a,b) \subseteq H^1(a,b)$, the embedding j has dense range. Moreover, we have $j \in \mathcal{L}(H^1(a,b), L_2(a,b))$ with $\|j\| = 1$.

Now let $\widetilde{A}$ be the operator associated with a. For $u, y \in L_2(a,b)$ we then have $u \in \operatorname{dom}(\widetilde{A})$ and $\widetilde{A}u = y$ if and only if there exists $u \in H^1(a,b)$ and $a(u,v) = (y \,|\, v)$

for all $v \in H^1(a,b)$. If $u \in \mathrm{dom}(A)$ and $Au = -u'' = y$, integration by parts yields (see Corollary 8.10 in [Bre11])

$$\begin{aligned} a(u,v) &= \int_a^b u'\overline{v'}\mathrm{d}x + \alpha u(a)\overline{v(a)} + \beta u(b)\overline{v(b)} \\ &= u'(b)\overline{v(b)} - u'(a)\overline{v(a)} - \int_a^b u''\overline{v}\mathrm{d}x + \alpha u(a)\overline{v(a)} + \beta u(b)\overline{v(b)} \\ &= -\int_a^b u''\overline{v}\mathrm{d}x + (-u'(a) + \alpha u(a))\overline{v(a)} + (u'(b) + \beta u(b))\overline{v(b)} \\ &= -\int_a^b u''\overline{v}\mathrm{d}x. \end{aligned}$$

Hence, it follows $a(u,v) = (y \,|\, v)$ for all $v \in H^1(a,b)$. Conversely, if there exists $u \in H^1(a,b)$ satisfying $a(u,v) = (y \,|\, v)$ for all $v \in H^1(a,b)$, it follows

$$\int_a^b u'\overline{v'}\mathrm{d}x = -\alpha u(a)\overline{v(a)} - \beta u(b)\overline{v(b)} + \int_a^b y\,\overline{v}\mathrm{d}x \tag{2}$$

for all $v \in H^1(a,b)$ by the definition of a. Further, for $v \in C_c^\infty(a,b)$ this equality yields $\int_a^b u'\overline{v'}\mathrm{d}x = \int_a^b y\,\overline{v}\mathrm{d}x$. As a consequence, we obtain $u'' = -y$ and $u \in H^2(a,b)$. Now choose $v \in H^1(a,b)$ with $v(b) = 0$ and $v(a) \neq 0$. Then equality (2) and integration by parts yield

$$\alpha u(a)\overline{v(a)} = \int_a^b -u'\overline{v'}\mathrm{d}x - \int_a^b u''\overline{v}\mathrm{d}x = -u'(b)\overline{v(b)} + u'(a)\overline{v(a)} = u'(a)\overline{v(a)},$$

which is equivalent to $-u'(a) + \alpha u(a) = 0$. Analogously we obtain $u'(b) + \beta u(b) = 0$. Altogether we have $u \in \mathrm{dom(A)}$ and $\widetilde{A}u = -u''$. Finally, it follows $A = \widetilde{A} \sim a$.

Next, we show that a is coercive. For this purpose let $u \in H^1(a,b)$. By Hölder's inequality we estimate

$$\begin{aligned} \int_a^b |u(x)|^2\mathrm{d}x &= \int_a^b \Big|u(a) + \int_a^x u'(y)\mathrm{d}y\Big|^2\mathrm{d}x \\ &\leq 2\Big(\int_a^b |u(a)|^2\mathrm{d}x + \int_a^b \Big|\int_a^x u'(y)\mathrm{d}y\Big|^2\mathrm{d}x\Big) \\ &\leq 2(b-a)|u(a)|^2 + 2\int_a^b \Big(\int_a^x |u'(y)|^2\mathrm{d}y\Big)\Big(\int_a^x 1^2\ \mathrm{d}y\Big)\mathrm{d}x \\ &\leq 2(b-a)|u(a)|^2 + 2(b-a)^2\int_a^b |u'(y)|^2\mathrm{d}y. \end{aligned}$$

Thus, we obtain

$$\|u'\|^2_{L_2(a,b)} \geq \frac{1}{2(b-a)^2}\|u\|^2_{L_2(a,b)} - \frac{1}{b-a}|u(a)|^2 \tag{3}$$

for $u \in H^1(a,b)$. Now fix $s \in (0,1)$ with $s \leq \alpha(b-a)$. By estimate (3) we have

$$\mathrm{Re}(a(u)) = \|u'\|^2_{L_2(a,b)} + \alpha|u(a)|^2 + \beta|u(b)|^2$$

$$
\begin{aligned}
&\geq s\Big(\frac{1}{2(b-a)^2}\|u\|^2_{L_2(a,b)} - \frac{1}{b-a}|u(a)|^2\Big) \\
&\qquad\qquad\qquad\qquad + (1-s)\|u'\|^2_{L_2(a,b)} + \alpha|u(a)|^2 + \beta|u(b)|^2 \\
&= \frac{s}{2(b-a)^2}\|u\|^2_{L_2(a,b)} + (1-s)\|u'\|^2_{L_2(a,b)} + \Big(\alpha - \frac{s}{b-a}\Big)|u(a)|^2 + \beta|u(b)|^2 \\
&\geq \frac{s}{2(b-a)^2}\|u\|^2_{L_2(a,b)} + (1-s)\|u'\|^2_{L_2(a,b)} \\
&\geq \min\Big\{\frac{s}{2(b-a)^2}, 1-s\Big\}\|u\|^2_{H^1(a,b)},
\end{aligned}
$$

where we used that by choice of s the inequality $\alpha - \frac{s}{b-a} \geq 0$ is satisfied. Therefore, the form a is coercive, i.e., we have $\mathrm{Re}(a(u)) \geq \widetilde{\alpha}\|u\|^2_{H^1(a,b)}$ for all $u \in H^1(a,b)$ where $\widetilde{\alpha} = \min\{\frac{s}{2(b-a)^2}, 1-s\}$. Applying the Generation theorem (see Theorem 5.9) it follows that the Operator A is m-sectorial, i.e. $-A$ generates a holomorphic C_0-semigroup on $L_2(a,b)$ which is contractive on a sector.

(b) In part (a) we have shown that all assumptions of Exercise 2 (a) are satisfied. By applying this exercise we obtain

$$\|\mathrm{e}^{-tA}\|_{\mathcal{L}(L_2(a,b))} \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$ and some $\varepsilon \in (0, \frac{\widetilde{\alpha}}{\|j\|^2}]$. According to part (a) we have $\|j\| = 1$ and $\widetilde{\alpha} = \min\{\frac{s}{2(b-a)^2}, 1-s\}$ for some fixed $s \in (0,1)$ with $s \leq \alpha(b-a)$. So, it follows

$$\|\mathrm{e}^{-tA}\|_{\mathcal{L}(L_2(a,b))} \leq \mathrm{e}^{-\varepsilon t}$$

for all $t \geq 0$ and some $\varepsilon \in (0, \min\{\frac{s}{2(b-a)^2}, 1-s\}]$. □

References

[Bre11] Brezis, H.: *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York, 2011.