Internet Seminar 18 - Exercises of Lecture 4

Università del Salento

Exercise 4.1

(a) Let $f \in L_{1,loc}(\mathbb{R}^n)$, $\varphi \in C_c(\mathbb{R}^n)$. Show that $\operatorname{spt}(\varphi * f) \subseteq \operatorname{spt} f + \operatorname{spt} \varphi$.

(b) Let $K \subseteq U \subseteq \mathbb{R}^n$, K compact, U open. Show that exists $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \psi \subseteq U$, $\psi|_K = 1$ and $0 \le \psi \le 1$. (Hint: Find ψ as the convolution of a suitable function $\rho \in C_c^\infty(\mathbb{R}^n)$ with a suitable indicator function.)

Solution

(a) Let $f \in L_{1,loc}(\mathbb{R}^n)$, $\varphi \in C_c(\mathbb{R}^n)$. Since $\operatorname{spt} \varphi$ is compact and $\operatorname{spt} f$ is closed, then $\operatorname{spt} f + \operatorname{spt} \varphi$ is closed; thus $(\operatorname{spt} f + \operatorname{spt} \varphi)^c$ is open. Let $x \in (\operatorname{spt} f + \operatorname{spt} \varphi)^c$; we observe that the following equality holds

$$(x - \operatorname{spt} \varphi) \cap \operatorname{spt} f = \varnothing.$$

Indeed, supposing by absurd that there exists $y \in (x - \operatorname{spt} \varphi) \cap \operatorname{spt} f$, then we would have that there exists $z \in \operatorname{spt} \varphi$ such that $y = x - z \in \operatorname{spt} f$ which immediately gives the contradiction.

Now, considered $x \in (\operatorname{spt} f + \operatorname{spt} \varphi)^c$, we notice that:

$$\big(\varphi * f\big)(x) = \int_{\mathbb{R}^n} \varphi(x-y) f(y) dy = \int_{(x - \operatorname{spt} \varphi) \cap \operatorname{spt} f} \varphi(x-y) f(y) dy = 0.$$

Therefore, one has

$$(\operatorname{spt} f + \operatorname{spt} \varphi)^c \subseteq \bigcup \{V | V \text{ is open and } (f * \varphi)|_V = 0\} = \big(\operatorname{spt}(\varphi * f)\big)^c.$$

The thesis follows.

(b) By hypothesis, one has that $K \subset U$. Let

$$d = \operatorname{dist}(K, U^c) = \inf\{||x-y|| : \ x \in K, \ y \in U^c\} > 0.$$

Let $\varepsilon = \frac{d}{2}$. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ s.t. $\operatorname{spt} \rho \subseteq B(0, \varepsilon)$ and $\int_{\mathbb{R}^n} \rho = 1$.

Defined $K_\varepsilon = \{x \in \mathbb{R}^n : \ \operatorname{dist}(x, K) \le \varepsilon\}$, let $\psi(x) = \chi_{K_\varepsilon} * \rho(x)$. By construction $\psi \ge 0$; moreover

$$\psi(x) = \int_{\mathbb{R}^n} \chi_{K_\varepsilon}(x-y) \rho(y) dy \le \int_{\mathbb{R}^n} \rho(y) dy = 1.$$

So

$$0 \leq \psi(x) \leq 1.$$

Finally, if $x \in K$ and $y \in B(0,\varepsilon)$, then $x-y \in K_\varepsilon$ and

$$\psi(x) = \int_{\mathbb{R}^n} \chi_{K_\varepsilon}(x-y)\rho(y)dy = \int_{B(0,\varepsilon)} \chi_{K_\varepsilon}(x-y)\rho(y)dy = \int_{B(0,\varepsilon)} \rho(y)dy = 1.$$

Now

$$\operatorname{spt}(\chi_{K_\varepsilon} * \psi) \subseteq \operatorname{spt}\, \chi_{K_\varepsilon} + \operatorname{spt}\, \psi = K_\varepsilon + B(0,\varepsilon) \subset U.$$

This concludes the proof.

Exercise 4.2

Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Let $f \in H^1_c(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$.

(b) Let $f \in H^1_0(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$.

Solution

Firstly, consider $f \in H^1_c(\Omega)$ and let $\tilde{f}$ be its extension by zero to $\mathbb{R}^n$. Trivially, $\tilde{f} \in L_2(\mathbb{R}^n)$. Indeed:

$$\|\tilde{f}\|^2_{L_2} = \int_{\mathbb{R}^n} |\tilde{f}|^2 dx = \int_\Omega |\tilde{f}|^2 dx = \int_\Omega |f|^2 dx = \|f\|^2_{L_2}$$

then, in order to conclude point (a), it suffices to prove that $\partial_j \tilde{f}$ is the extension to $\mathbb{R}^n$ by zero of $\partial_j f$ for any $j = 1, \dots, n$. To this aim, we fix $\psi \in C^\infty_c(\Omega)$ such that $\psi = 1$ on the support of f and observe that for any $j = 1, \dots, n$ and for any $\phi \in C^\infty_c(\mathbb{R}^n)$ it holds

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_j \tilde{f}\phi &= \int_{\mathbb{R}^n} \partial_j \tilde{f}\phi\psi = \int_\Omega \partial_j \tilde{f}\phi\psi = \int_\Omega \partial_j f\phi\psi = -\int_\Omega f\partial_j(\phi\psi) \\ &= -\int_\Omega f\partial_j\phi = -\int_\Omega \tilde{f}\partial_j\phi = -\int_{\mathbb{R}^n} \tilde{f}\partial_j\phi. \end{aligned}$$

Since $\partial_j \tilde{f}$ coincides with $\partial_j f$ in Ω and is constantly zero outside Ω, we immediately obtain

$$\|\partial_j \tilde{f}\|_{L_2(\mathbb{R}^n)} = \|\partial_j f\|_{L_2(\Omega)}$$

which implies $\tilde{f} \in H^1(\mathbb{R}^n)$.

As regards point (b), consider a function $f \in H^1_0(\Omega)$ and let $\tilde{f}$ be its extension to $\mathbb{R}^n$ by zero. We recall that by definition $H^1_0(\Omega) := \overline{H^1_c(\Omega)}^{H^1(\Omega)}$; then, consider a sequence $(f_k)_{k\in\mathbb{N}}$ of functions of $H^1_c(\Omega)$ which converges to f in $H^1(\Omega)$ and pointwise a.e. (this can always be done possibly passing to a subsequence). For any $k \in \mathbb{N}$, let $\tilde{f}_k$ be the extension to $\mathbb{R}^n$ by zero of f_k. From point (a) we deduce that $(\tilde{f}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence of functions in $H^1(\mathbb{R}^n)$. Moreover, $(\tilde{f}_k)_{k\in\mathbb{N}}$ converges to $\tilde{f}$ pointwise, hence the $H^1(\mathbb{R}^n)$ limit is again $\tilde{f}$.

Exercise 4.3

Let $H \subseteq \mathbb{R}^2$ be the half-plane $H := \{(x_1, x_2); x_1 \geq 0\}$, and let $f \in L_{1,loc}(\mathbb{R}^2)$ be defined by $f := \mathbf{1}_H$.

(a) Show that $\int \partial_1 \varphi f = -\int_{x_2 \in \mathbb{R}} \varphi(0, x_2)\, dx_2$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and that there is no $g \in L_{1,loc}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f = \int \varphi g$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$.

(b) Decide which of the partial derivatives $\partial_1 f$, $\partial_2 f$, $\partial_1 \partial_2 f$ belong to $L_{1,loc}(\mathbb{R}^2)$.

Solution.

(a) Let us verify the first statement. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We obtain, using Fubini's Theorem:

$$\begin{aligned} \int_{\mathbb{R}^2} f \partial_1 \varphi \, dx_1 dx_2 = \int_H \partial_1 \varphi \, dx_1 dx_2 = \int_{x_2 \in \mathbb{R}} dx_2 \int_{x_1 \geq 0} \partial_1 \varphi(x_1, x_2)\, dx_1 = \\ = -\int_{x_2 \in \mathbb{R}} \varphi(0, x_2)\, dx_2, \end{aligned} \tag{1}$$

where the last equality holds since $\varphi(\cdot, x_2) \in C_c^\infty(\mathbb{R})$ for any $x_2 \in \mathbb{R}$. This proves the first assertion.

To prove the second statement, let us suppose that there exists $g \in L_{1,loc}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f = \int \varphi g$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$.

Let $\rho \in C_c^\infty(\mathbb{R}^2 \setminus [x_1 = 0])$. Therefore $\tilde{\rho}$, the extension of ρ to $\mathbb{R}^2$ by zero, belongs to $C_c^\infty(\mathbb{R}^2)$. Since $\tilde{\rho}(0, x_2) = 0$ for all $x_2 \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^2} \tilde{\rho} g \, dx_1 dx_2 = -\int_{x_2 \in \mathbb{R}} \tilde{\rho}(0, x_2)\, dx_2 = 0.$$

Since $[x_1 = 0]$ is $\mathcal{L}^2$-negligible and $\rho = \tilde{\rho}$ in $\mathbb{R}^2 \setminus [x_1 = 0]$, we also have

$$\int_{\mathbb{R}^2 \setminus [x_1 = 0]} \rho g \, dx_1 dx_2 = \int_{\mathbb{R}^2} \tilde{\rho} g \, dx_1 dx_2 = 0.$$

From the fundamental lemma of the calculus of variation applied to $g \in L_{1,loc}(\mathbb{R}^2 \setminus [x_1 = 0])$, we obtain $g = 0$ a.e. in $\mathbb{R}^2 \setminus [x_1 = 0]$. Therefore $g = 0$ a.e. in $\mathbb{R}^2$, because g is supposed to be in $L_{1,loc}(\mathbb{R}^2)$ and $[x_1 = 0]$ is $\mathcal{L}^2$-negligible.

On the other hand, we observe that a simple counterxample proves that the integral in (1) is in general different from zero. Indeed, let $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} \psi \subseteq B(0, 2)$, $\psi|_{B(0,1)} = 1$ and $0 \leq \psi \leq 1$ (such a function exists, as proved in Exercise 1). In particular $\psi(0, x_2) > 0$ for all x_2 in an interval containing $[0, 1]$; then, from the equality (1) we obtain:

$$\int_{\mathbb{R}^2} f \partial_1 \psi \, dx_1 dx_2 = -\int_{x_2 \in \mathbb{R}} \psi(0, x_2)\, dx_2 \leq -\int_{-1}^{1} \psi(0, x_2)\, dx_2 \leq -2.$$

Therefore, there is no $g \in L_{1,loc}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f = \int \varphi g$ for all $\varphi \in C_c^\infty(\mathbb{R}^2)$.

(b) From the step above, the distributional derivative $\partial_1 f$ does not belong to $L_{1,loc}(\mathbb{R}^2)$.

Let us show that $\partial_2 f \in L_{1,loc}(\mathbb{R}^2)$. For all $\varphi \in C_c^\infty(\mathbb{R}^2)$ we obtain:

$$\begin{aligned}(\partial_2 f|\varphi) &= -\,(f|\partial_2\varphi) = -\int_H \partial_2\varphi\, dx_1 dx_2 = \\ &= -\int_{x_1\geq 0} dx_1 \int_{x_2\in\mathbb{R}} \partial_2\varphi(x_1,x_2)\, dx_2 = 0,\end{aligned}$$

because the inner integral equals zero. This proves that $\partial_2 f = 0 \in L_{1,loc}(\mathbb{R})$.

To prove that $\partial_1\partial_2 f \in L_{1,loc}(\mathbb{R}^2)$, we recall that for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, the partial derivatives $\partial_1\varphi$, $\partial_2\varphi$ belong to $C_c^\infty(\mathbb{R}^2)$, too. Let us compute:

$$(\partial_1\partial_2 f|\varphi) = -\,(\partial_2 f|\partial_1\varphi) = -\,(0|\partial_1\varphi) = 0.$$

Then $\partial_1\partial_2 f = 0 \in L_{1,loc}(\mathbb{R}^2)$.

Exercise 4.4

Let $n \geq 3$. Show that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n \setminus \{0\})$. For more ambitious participants: show this also for $n = 2$.

Solution.

Case $n \geq 3$.

Let $J : H_0^1(\mathbb{R}^n \setminus \{0\}) \to H^1(\mathbb{R}^n)$, $f \mapsto \tilde{f}$, where $\tilde{f}$ is the extension of f to $\mathbb{R}^n$ by zero.
From Exercise 4.3, J is an isometric immersion between Banach spaces. In order to prove that J is surjective, we will show that $H_0^1(\mathbb{R}^n \setminus \{0\}) \equiv J(H_0^1(\mathbb{R}^n \setminus \{0\}))$ is dense in $H^1(\mathbb{R}^n)$.
Recalling that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{H^1(\mathbb{R}^n)}$, it is sufficient to show that for any $f \in C_c^\infty(\mathbb{R}^n)$ there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $H_0^1(\mathbb{R}^n \setminus \{0\})$ converging to f in $H^1(\mathbb{R}^n)$.
Let us fix $f \in C_c^\infty(\mathbb{R}^n)$. Applying Exercise 4.1, let us consider $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $u(x) = 1$ if $|x| \leq 1$ and $u(x) = 0$ if $|x| \geq 2$ and let $u := 1 - \varphi$. Obviously $u \in C_b^\infty(\mathbb{R}^n)$ and $u(x) = 0$ if $|x| \leq 1$ and $u(x) = 1$ if $|x| \geq 2$. It follows that for $i = 1, \ldots, n$, $\operatorname{spt}(\partial_i u) \subseteq \overline{B(0,2)} \setminus B(0,1)$ and $\partial_i u \in C_c^\infty(\mathbb{R}^n)$.
Let $u_\varepsilon := u(\frac{\cdot}{\varepsilon}) \in C_b^\infty(\mathbb{R}^n \setminus \{0\})$.
Let us observe that $u_\varepsilon f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ for all $\varepsilon > 0$ and clearly

$$\partial_i(u_\varepsilon f) = u_\varepsilon \partial_i f + f\partial_i u_\varepsilon, \qquad i = 1, \ldots, n.$$

Now $u_\varepsilon \to 1$ for $\varepsilon \to 0^+$ pointwise in $\mathbb{R}^n \setminus \{0\}$ and an application of the Lebesgue Theorem proves $u_\varepsilon f \to f$, $u_\varepsilon \partial_i f \to \partial_i f$ in $L_2(\mathbb{R}^n)$.

In order to prove that $u_\varepsilon f \to f$ in $H^1(\mathbb{R}^n)$, it remains to show that $f\partial_i u_\varepsilon \to 0$ in $L_2(\mathbb{R}^n)$.

From

$$\partial_i u_\varepsilon = \frac{1}{\varepsilon}\,\partial_i u\left(\frac{\cdot}{\varepsilon}\right)$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\partial_i u_\varepsilon(x) f(x)|^2\,dx \;&\le\; \frac{\|f\|_\infty^2}{\varepsilon^2}\int_{spt(\partial_i u_\varepsilon)} \left|\partial_i u\left(\frac{x}{\varepsilon}\right)\right|^2 dx \qquad \left(y=\frac{x}{\varepsilon}\right)\\ &=\; \frac{\|f\|_\infty^2}{\varepsilon^2}\,\varepsilon^n\int_{spt(\partial_i u)} |\partial_i u\,(y)|^2\,dy \;=\; \|f\|_\infty^2\,\varepsilon^{n-2}\,\|\partial_i u\|_2 \to 0.\end{aligned}$$

This proves the desired convergence of $u_\varepsilon f \to f$ in $H^1(\mathbb{R}^n)$ and the required statement.

Case $n = 2$.

Analogously to the previous case, we will show that for any $f \in C_c^\infty(\mathbb{R}^2)$ there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $H_0^1(\mathbb{R}^2\setminus\{0\})$ converging to f in $H^1(\mathbb{R}^2)$.

Let us fix $f \in C_c^\infty(\mathbb{R}^2)$.

Let us define for all $0 < \varepsilon < 1$

$$u_\varepsilon(x) := \begin{cases} 0 & \text{if } |x| \le \varepsilon,\\ 1 - \frac{\log\frac{|x|}{\sqrt{\varepsilon}}}{\log\sqrt{\varepsilon}} & \text{if } \varepsilon < |x| < \sqrt{\varepsilon},\\ 1 & \text{if } |x| \ge \sqrt{\varepsilon}.\end{cases}$$

u_ε is infinitely differentiable in $\mathbb{R}^2$ except on the null set $\partial B(0,\varepsilon)\cup\partial B(0,\sqrt{\varepsilon})$ but it belongs to $H^1(\mathbb{R}^2)$[1] and its weak derivatives are given by

$$\partial_i u_\varepsilon(x) := \begin{cases} 0 & \text{if } |x|\le\varepsilon,\ |x|\ge\sqrt{\varepsilon},\\ -\frac{1}{\log\sqrt{\varepsilon}}\,\frac{1}{|x|}\,\frac{x_i}{|x|} & \text{if } \varepsilon<|x|<\sqrt{\varepsilon}.\end{cases}$$

The same reasoning above can be applied to the functions $u_\varepsilon f$, obtaining $u_\varepsilon f \in H^1(\mathbb{R}^2)$, $u_\varepsilon f(0) = 0$, with weak derivatives given by

$$\partial_i(u_\varepsilon f) = u_\varepsilon\partial_i f + f\partial_i u_\varepsilon, \qquad i = 1,\dots,n.$$

As above $u_\varepsilon \to 1$ for $\varepsilon \to 0^+$ pointwise in $\mathbb{R}^2\setminus\{0\}$ and an application of the Lebesgue Theorem proves $u_\varepsilon f \to f$, $u_\varepsilon\partial_i f \to \partial_i f$ in $L_2(\mathbb{R}^2)$.

In order to prove that $u_\varepsilon f \to f$ in $H^1(\mathbb{R}^2)$, it remains to show that $f\partial_i u_\varepsilon \to 0$ in $L_2(\mathbb{R}^2)$.

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[1] See D. Gilbarg, N.S Trudinger, *Elliptic Partial Differential Equations of Second Order*, Theorem 7.8, pag. 146.

We obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\partial_i u_\varepsilon(x) f(x)|^2\, dx &\le \frac{\|f\|_\infty^2}{(\log\sqrt{\varepsilon})^2} \int_{\{\varepsilon<|x|<\sqrt{\varepsilon}\}} \frac{1}{|x|^2}\, dx \\ \text{(using polar coordinates)} &= \frac{\|f\|_\infty^2}{(\log\sqrt{\varepsilon})^2}\, 2\pi \int_{\{\varepsilon<\rho<\sqrt{\varepsilon}\}} \frac{1}{\rho}\, d\rho \\ &= \frac{\|f\|_\infty^2}{(\log\sqrt{\varepsilon})^2}\, 2\pi \left|\log\sqrt{\varepsilon}\right| = \frac{\|f\|_\infty^2}{|\log\sqrt{\varepsilon}|}\, 2\pi \to 0 \text{ as } \varepsilon \to 0^+.\end{aligned}$$

This proves the desired convergence of $u_\varepsilon f \to f$ in $H^1(\mathbb{R}^2)$ and the required statement for $n = 2$.

Exercise 4.5

Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that there exists a standard exaustion $(\Omega_k)_{k\in\mathbb{N}}$ of Ω, i.e., Ω_k is open, relatively compact in Ω_{k+1} $(k \in \mathbb{N})$, $\bigcup\limits_{k\in\mathbb{N}} \Omega_k = \Omega$.

(b) Let $f \in L_{1,loc}(\Omega)$, and assume that $f = 0$ locally, i.e., for all $x \in \Omega$ there exists $r > 0$ such that $f|_{B(0,r)} = 0$. Then $f = 0$. (All "=0" should be interpreted as a.e.)

Solution.

(a) We distinguish two cases: $\Omega = \mathbb{R}^n$ and $\Omega \neq \mathbb{R}^n$.

Let $\Omega = \mathbb{R}^n$. We set

$$\Omega_k := B(0,k) \quad (k \in \mathbb{N}).$$

Trivially, the open set Ω_k is relatively compact in Ω_{k+1}, for all $k \in \mathbb{N}$. Moreover, we have

$$\bigcup_{k\in\mathbb{N}} \Omega_k = \mathbb{R}^n,$$

so $(\Omega_k)_{k\in\mathbb{N}} = (B(0,k))_{k\in\mathbb{N}}$ is a standard exaustion of $\mathbb{R}^n$.

Now let Ω be an open set strictly contained in $\mathbb{R}^n$. Then $\partial\Omega \neq \emptyset$.

Let us define

$$\Omega_k := B(0,k) \cap \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{k} \right\}, \quad (k \in \mathbb{N}).$$

Ω_k is an open bounded set for all $k \in \mathbb{N}$.

Let $x \in \overline{\Omega}_k$; we have

$$\|x\| \le k < k+1, \quad \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{k} > \frac{1}{k+1},$$

so $x \in \Omega_{k+1}$. Therefore $\overline{\Omega}_k$ is a compact subset of Ω_{k+1}. This proves that Ω_k is relatively compact in Ω_{k+1}.

Let us show that

$$\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

It is sufficient to prove that $\Omega \subseteq \bigcup_{k\in\mathbb{N}} \Omega_k$ (the converse inclusion is trivial). Let $x \in \Omega$. Since Ω is open, there exists $k \in \mathbb{N}$ such that $\mathrm{dist}(x, \partial\Omega) > 1/k$ and $\|x\| < k$. Then $x \in \Omega_k$ and therefore $\Omega \subseteq \bigcup_{k\in\mathbb{N}} \Omega_k$. This proves the equality, so $(\Omega_k)_{k\in\mathbb{N}}$ is a standard exaustion of Ω.

(b) Let $(\Omega_k)_{k\in\mathbb{N}}$ be a standard exaustion of Ω and let us fix $k \in \mathbb{N}$.

Since $f = 0$ a.e. locally in Ω, for each $x \in \overline{\Omega_k} \subseteq \Omega_{k+1}$, there exists $r_x > 0$ such that $f = 0$ a.e. in $B(x, r_x) \subseteq \Omega_{k+1}$. Then

$$\overline{\Omega_k} \subseteq \bigcup_{x\in\Omega_k} B(x, r_x).$$

From the part (a), Ω_k is relatively compact, so there exist $h_k \in \mathbb{N}$ and $x_1, \ldots, x_{h_k} \in \Omega$ such that

$$\overline{\Omega_k} \subseteq \bigcup_{j=1}^{h_k} B(x_j, r_{x_j}),$$

with $f = 0$ a.e. in $B(x_j, r_{x_j})$ $(j = 1, \ldots, h_k)$.

Therefore $f = 0$ a.e. in Ω_k, for all $k \in \mathbb{N}$.

From

$$[f \neq 0] = \bigcup_{k\in\mathbb{N}} ([f \neq 0] \cap \Omega_k)$$

we have $f = 0$ a.e. in Ω.