Exercise 3.1

We have to show that $||\cdot||_E$, defined by

$$(x_n)_{n\in\mathbb{N}} \mapsto \sup\left\{\left|\sum_{n=1}^{\infty} y_n x_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \leq 1\right\},$$

is a norm, which is equivalent to the norm $||\cdot||_{c_0}$, given by

$$(x_n)_{n\in\mathbb{N}} \mapsto \sup_{n\in\mathbb{N}} |x_n|.$$

At first, we want to show that $||\cdot||_E$ really defines a norm. Of course, it holds $||0||_E = 0$. Otherwise, if $0 \neq x = (x_n)_{n\in\mathbb{N}} \in c_0$, there must be an index $m \in \mathbb{N}$ with $x_m \neq 0$. Furthermore, there is another index $k \in \mathbb{N}$ with $x_m \neq x_k$. Considering $y = (y_n)_{n\in\mathbb{N}} \in E$ with

$$y_n := \begin{cases} \frac{1}{2}, & n = m, \\ -\frac{1}{2}, & n = k, \\ 0, & \text{else}, \end{cases}$$

it holds

$$\left|\sum_{n=1}^{\infty} y_n x_n\right| = \frac{1}{2}|x_m - x_k| > 0$$

and therefore $||x||_E > 0$. Moreover, for $\alpha \in \mathbb{C}$ and $x = (x_n)_{n\in\mathbb{N}}$ the identity

$$\begin{aligned} ||\alpha x||_E &= \sup\left\{\left|\sum_{n=1}^{\infty} y_n\, \alpha x_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \leq 1\right\} \\ &= |\alpha| \sup\left\{\left|\sum_{n=1}^{\infty} y_n x_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \leq 1\right\} \\ &= |\alpha| \cdot ||x||_E \end{aligned}$$

is valid. In order to show the triangle inequality, let $x, z \in c_0$ with $x = (x_n)_{n\in\mathbb{N}}$ and $z = (z_n)_{n\in\mathbb{N}}$. Then

$$\left|\sum_{n=1}^{\infty} y_n(x_n + z_n)\right| \leq \left|\sum_{n=1}^{\infty} y_n x_n\right| + \left|\sum_{n=1}^{\infty} y_n z_n\right|$$

holds for all $y = (y_n)_{n\in\mathbb{N}} \in \ell_1$ and hence

$$\begin{aligned}
||x+z||_E &\le \sup\left\{\left|\sum_{n=1}^{\infty} y_n x_n\right| + \left|\sum_{n=1}^{\infty} y_n z_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \le 1\right\} \\
&\le \sup\left\{\left|\sum_{n=1}^{\infty} y_n x_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \le 1\right\} \\
&\quad + \sup\left\{\left|\sum_{n=1}^{\infty} y_n z_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \le 1\right\} \\
&= ||x||_E + ||z||_E.
\end{aligned}$$

Consequently, $||\cdot||_E$ defines a norm on c_0.
Now we want to show the equivalence of the norms $||\cdot||_E$ and $||\cdot||_{c_0}$. On the one hand, the estimate

$$\begin{aligned}
||x||_E &= \sup\left\{\left|\sum_{n=1}^{\infty} y_n x_n\right| : y = (y_n)_{n\in\mathbb{N}} \in E, ||y||_{\ell_1} \le 1\right\} \\
&\le \sup\left\{||y||_{\ell_1} \cdot ||x||_{c_0} : y \in E, ||y||_{\ell_1} \le 1\right\} \\
&\le ||x||_{c_0}
\end{aligned}$$

holds. On the other hand, for each $x = (x_n)_{n\in\mathbb{N}} \in c_0$ there exists an $m \in \mathbb{N}$ such that $|x_m| = \sup_{n\in\mathbb{N}} |x_n|$. Now we define for each $k \in \mathbb{N}$ a sequence $y_k := (y_{n,k})_{n\in\mathbb{N}}$ by

$$y_{n,k} := \begin{cases} \frac{1}{2}, & m = n \ne k, \\ -\frac{1}{2}, & m \ne n = k, \\ 0, & \text{else.} \end{cases}$$

These sequences are elements of E because they are in ℓ_1 with $\sum_{n=1}^{\infty} y_{n,k} = 0$ for each $k \in \mathbb{N}$. We obtain the estimate

$$\left|\sum_{n=1}^{\infty} y_{n,k}\, x_n\right| = \left|\frac{x_m}{2} - \frac{x_k}{2}\right| \ge \left|\frac{x_m}{2}\right| - \left|\frac{x_k}{2}\right|$$

and, by considering the limit $k \to \infty$, we gain the lower bound $\left|\frac{x_m}{2}\right|$ for $||x||_E$. This implies $||x||_{c_0} \le 2||x||_E$. Since $x \in c_0$ has been chosen arbitrarily, the norms are finally equivalent and therefore E is almost norming for c_0.
If we assume, that E is norming, it holds $||x||_{c_0} = ||x||_E$ for all $x \in c_0$. Now we consider $x = (x_n)_{n\in\mathbb{N}}$, defined by

$$x_n := \begin{cases} 1, & n = 1, \\ 0, & \text{else,} \end{cases}$$

and conclude, that $||x||_E = ||x||_{c_0} = 1$ holds by assumption. So there must be a sequence $y = (y_n)_{n\in\mathbb{N}} \in E$ with

$$|y_1| = \left|\sum_{n=1}^{\infty} y_n x_n\right| > \frac{1}{2}$$

and $||y||_{\ell_1} \leq 1$. Then we obtain, that

$$\left|\sum_{n=2}^{\infty} y_n\right| > \frac{1}{2},$$

but this yields

$$1 \geq ||y||_{\ell_1} \geq |y_1| + \left|\sum_{n=2}^{\infty} y_n\right| > 1,$$

what provides a contradiction. In summary, E is an almost norming subset for c_0, which is not norming.

Exercise 3.2

(a) We want to apply Theorem 3.2 (iv). Define for $n \in \mathbb{N}$ the functional $\phi_n \in (\ell_\infty)'$ for each $x = (x_k) \in \ell_\infty$ by $\phi_n(x) = x_n$. Note that all of these functionals have a norm equal to one. Therefore $E := \bigcup_{n\in\mathbb{N}}\{\phi_n\}$ is a norming subset for ℓ_∞ as

$$\|x\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}\{|x_k|\} = \sup\{|\phi(x)|\colon \phi \in E\}.$$

For any given $\phi_n \in E$ the function $\phi_n \circ f = f_n$ is holomorphic, because f_n is holomorphic by assumption. Moreover, f is locally (even globally) bounded in Ω due to

$$\sup_{z\in\Omega} \|f(z)\|_{\ell_\infty} = \sup_{z\in\Omega, n\in\mathbb{N}} |f_n(z)| < \infty.$$

Hence, all requirements of Theorem 3.2 (iv) are fulfilled and f is holomorphic.

(b) Due to the assumption that $\lim_{n\to\infty} f_n(z)$ exists for every $z \in \Omega$, it is clear that f indeed maps Ω into c. Moreover, as c is a subspace of ℓ^∞, we know that $f : \Omega \to c$ is holomorphic by part (a). By the hint $\psi : c \to \mathbb{C}\colon (x_n) \mapsto \lim x_n$ is a bounded linear functional, and hence invoking Theorem 3.2 (ii) $f_\infty = \psi \circ f$ is holomorphic, too.

Exercise 3.3

(a) Fix $\theta' \in (0, \theta)$. If there is no $\delta > 0$ with $\sup_{z \in \Sigma_{\theta'}, \mathrm{Re} z \leq \delta} \|T(z)\| < \infty$, there would be a sequence $(z_n)_{n \in \mathbb{N}} \subset \Sigma_{\theta'}$ with $z_n \to 0$ and $\|T(z_n)\| \to \infty$. But then, according to the uniform boundedness principle, there would also exist some $x \in X$ with $\|T(z_n)x\| \to \infty$ as $n \to \infty$, contradicting the strong continuity of the semigroup T.

(b) Take $\delta > 0$ with $k = \sup_{z \in \Sigma_{\theta'}, \mathrm{Re} z \leq \delta} \|T(z)\| < \infty$. For every $z \in \Sigma_{\theta'}$ there exist $z' \in \Sigma_{\theta'}$ with $\mathrm{Re} z' = \delta, n \in \mathbb{N}$ and $0 \leq \varepsilon < 1$ such that $z = (n + \varepsilon) z'$. Due to $k \geq \|T(0)\| = 1$ and $n \leq \frac{\mathrm{Re} z}{\delta}$, this implies that

$$\|T(z)\| \leq \|T(z')\|^n \|T(\varepsilon z')\| \leq k^{n+1} \leq k \cdot k^{\mathrm{Re} z / \delta}.$$

With $M' = k$ and $\omega' = (\log k)/\delta$, this yields the assertion.

(c) Since $|\frac{\mathrm{Im} z}{\mathrm{Re} z}| \leq \tan \theta'$ for each $z \in \Sigma_\theta'$, the assertion follows from the estimate

$$\mathrm{Re} z \leq |z| \leq (1 + \tan \theta') \mathrm{Re} z,$$

which is valid on each admissible sector.

Exercise 3.4

(a) Firstly, we introduce the extension $\tilde{T}(z_0) : \bigcup_{z \in \Sigma_\theta} \mathrm{ran}(T(z)) \to X$ for $z_0 \in \bar{\Sigma_\theta}$ as

$$\tilde{T}(z_0)x = \lim_{n \to \infty} T(z_n)x \quad (\text{with } \Sigma_\theta \ni z_n \to z_0). \tag{1}$$

We will now deduce that this extension is well-defined because the limit exists and does not depend on the choice of $(z_n)_n$: For all $x \in \bigcup_{z \in \Sigma_\theta} \mathrm{ran}(T(z))$, there exist $x' \in X, z' \in \Sigma_\theta$ such that $x = T(z')x'$. Now, consider $\Sigma_\theta \ni z_n \to z_0 \in \bar{\Sigma_\theta}$ and hence, $z_n + z' \to z_0 + z' \in \Sigma_\theta$ because $\theta < \frac{\pi}{2}$. Then, the following holds (using the semigroup property of T to obtain the second equality):

$$T(z_n)x = T(z_n)T(z')x' = T(z_n + z')x' \to T(z_0 + z')x' \tag{2}$$

where the existence of the limit is due to the holomorphy of T which implies the continuity of T at $z_0 + z' \in \Sigma_\theta$. Observe that $\tilde{T}(z_0)$ is linear because the limit of linear maps is still linear.

The operator norms of $\tilde{T}(\cdot)$ are uniformly bounded by the same bound as the operator norm of $T(\cdot)$ because for $x \in \bigcup_{z\in\Sigma_\theta} \operatorname{ran}(T(z))$ the estimate $\|\tilde{T}(z)x\| \leq \lim_n \|T(z_n)\|\|x\| \leq M\|x\|$ with $M = \sup_{z\in\Sigma_\theta} \|T(z)\|$ holds for all $z \in \bar{\Sigma}_\theta$.

Note that T and $\tilde{T}$ coincide on Σ_θ. This extension conserves the semigroup property: Consider $z, z' \in \bar{\Sigma}_\theta$ and sequences $\Sigma_\theta \ni z_n \to z$, $\Sigma_\theta \ni z'_n \to z'$ and we obtain

$$(3)\quad \tilde{T}(z+z')x = \lim_{n\to\infty} T(z_n + z'_n)x = \lim_{n\to\infty} T(z_n)T(z'_n)x = \tilde{T}(z)\tilde{T}(z')x$$

where the last equality is obtained with the help of Lemma 1.11 and the assumption that T is a bounded C_0-semigroup. The strong continuity of $\tilde{T}$ is obtained as follows. Consider $\bar{\Sigma}_\theta \ni z_n \to 0$ and $x \in \bigcup_{z\in\Sigma_\theta} \operatorname{ran}(T(z))$ implying again the existence of $x' \in X, z' \in \Sigma_\theta$ such that $x = T(z')x'$. We obtain

$$\begin{aligned}\tilde{T}(z_n)x &= \tilde{T}(z_n)T(z')x' = \tilde{T}(z_n)\tilde{T}(z')x' \\ &= \tilde{T}(z_n + z')x' = T(z_n + z')x' \to T(z')x' = x\end{aligned}$$

which proves the strong continuity.

In the second step, we introduce the extension $\hat{T}(z_0) : X \to X$ for $z_0 \in \bar{\Sigma}_\theta$. Note that $\bigcup_{z\in\Sigma_\theta} \operatorname{ran}(T(z))$ is dense in X by Remark 3.12. Thus, define

$$(4)\qquad \hat{T}(z_0)x = \lim_{n\to\infty} \tilde{T}(z_0)x_n \ \Big(\text{with} \bigcup_{z\in\Sigma_\theta} \operatorname{ran}(T(z)) \ni x_n \to x\Big).$$

Then, the existence of the limit follows from $\|\tilde{T}(z)x_n - \tilde{T}(z)x_m\| \leq \|\tilde{T}(z)\|\|x_n - x_m\| \leq M\|x_n - x_m\| \to 0$ for $n, m \to \infty$. For its uniqueness, let $(x_n)_n$ and $(y_n)_n$ be sequences converging to $x \in X$. We obtain $\|\tilde{T}(z_0)x_n - \tilde{T}(z_0)y_n\| \leq M\|x_n - y_n\| \leq M(\|x_n - x\| + \|x - y_n\|) \to 0$. The semigroup property of $\hat{T}$ follows immediately from the semigroup property of $\tilde{T}$.

Its strong continuity is obtained as follows. Consider $\bar{\Sigma}_\theta \ni z_n \to 0$ and $\varepsilon > 0$. It exists $m \in \mathbb{N}$ with $\|x - x_m\| < \frac{\varepsilon}{3\min(M,1)}$ where $M = \sup_{z\in\bar{\Sigma}_\theta} \|\tilde{T}(z)\|$. For this m, we choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$: $\|\tilde{T}(z_n)x_m - x_m\| < \frac{\varepsilon}{3}$ and finally obtain

$$\begin{aligned}\|\hat{T}(z_n)x - x\| &\leq \|\hat{T}(z_n)x - \tilde{T}(z_n)x_m\| + \|\tilde{T}(z_n)x_m - x\| \\ &\leq \|\hat{T}(z_n)\|\|x - x_m\| + \|\tilde{T}(z_n)x_m - x\| \\ &\leq M\|x - x_m\| + \|\tilde{T}(z_n)x_m - x_m\| + \|x_m - x\| \\ &< \varepsilon\end{aligned}$$

which proves the strong continuity of $\hat{T}$. Therefore, $\hat{T}$ is the desired extension of T.

(b) follows immediately from (a).

(c) The semigroup property is clear from (a). Note that $i\mathbb{R} \subset \bar{\Sigma}_{\frac{\pi}{2}}$. Let $t \in \mathbb{R}$ and $x \in X$. We obtain $T_{\frac{\pi}{2}}(t)T_{\frac{\pi}{2}}(-t)x = T(it)T(-it)x = T(it - it)x = x$ with the semigroup property for the extension of T proven in (a) which implies the desired group property.