Lecture 2, Solutions

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Exercise 2.1. Show the second equality (2.2).

Proof. We need proof the following equality:

$$(-1)^n \frac{1}{(n-1)!}\left(\frac{d}{d\lambda}\right)^{n-1}\int_0^\infty \mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t. \tag{1}$$

We will use the method of mathematical induction. For $n = 1$ the equality (1) is trivial.

Let the equality

$$(-1)^{n-1} \frac{1}{(n-2)!}\left(\frac{d}{d\lambda}\right)^{n-2}\int_0^\infty \mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t = \frac{1}{(n-2)!}\int_0^\infty t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t$$

holds. Then we will prove (1). We get

$$\begin{aligned}(-1)^n &\frac{1}{(n-1)!}\left(\frac{d}{d\lambda}\right)^{n-1}\int_0^\infty \mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t \\ &= -\frac{1}{n-1}\frac{d}{d\lambda}\left((-1)^{n-1}\frac{1}{(n-2)!}\left(\frac{d}{d\lambda}\right)^{n-2}\int_0^\infty \mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t\right) \\ &= -\frac{1}{n-1}\frac{d}{d\lambda}\left(\frac{1}{(n-2)!}\int_0^\infty t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t\right) = -\frac{1}{(n-1)!}\frac{d}{d\lambda}\int_0^\infty t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t.\end{aligned}$$

For $t \geqslant 0$ we have $\lim\limits_{h\to 0}\frac{\mathrm{e}^{-th}-1}{h} = -t$, where the convergence is uniform on compact subsets of $[0,\infty)$, since

$$\left|\frac{1}{h}(\mathrm{e}^{-th}-1)+t\right| \leqslant t^2|h|\mathrm{e}^{|h|t}.$$

Thus, for $|h| < \operatorname{Re}\lambda - \omega$ the following equality holds

$$\lim_{N\to\infty}\int_0^N \frac{1}{h}(\mathrm{e}^{-ht}-1)\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t = \int_0^\infty \frac{1}{h}(\mathrm{e}^{-ht}-1)\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t,$$

where the convergence is uniform (since $t \mapsto t^2\mathrm{e}^{(|h|+\omega-\operatorname{Re}\lambda)t}$ is integrable).

Therefore, we obtain

$$\begin{aligned}-&\frac{1}{(n-1)!}\frac{d}{d\lambda}\int_0^\infty t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t \\ &= -\frac{1}{(n-1)!}\lim_{h\to 0}\frac{1}{h}\left(\int_0^\infty t^{n-2}\mathrm{e}^{-(\lambda+h)t}T(t)\,\mathrm{d}t - \int_0^\infty t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t\right) \\ &\qquad = -\frac{1}{(n-1)!}\lim_{N\to\infty}\lim_{h\to 0}\int_0^N \frac{1}{h}(\mathrm{e}^{-ht}-1)t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t\end{aligned}$$

$$= -\frac{1}{(n-1)!}\lim_{N\to\infty}\int_0^N \lim_{h\to 0}\frac{1}{h}(\mathrm{e}^{-ht}-1)t^{n-2}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t$$

$$= \frac{1}{(n-1)!}\lim_{N\to\infty}\int_0^N t^{n-1}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathrm{e}^{-\lambda t}T(t)\,\mathrm{d}t.$$

Hence, we have been obtained (1).

Exercise 2.2. Let T be a C_0-semigroup on the Banach space X, with generator A. Let $\lambda \in \mathbb{K}$. Show that

$$T_\lambda(t) := \mathrm{e}^{-\lambda t}T(t), \quad (t \geqslant 0),$$

defines a C_0-semigroup (the rescaled semigroup), and that the generator of T_λ is given by $A - \lambda I$.

Proof. First of all, we will prove properties of C_0-semigroup for T_λ. For $t, s \geqslant 0$ we have

$$T_\lambda(t+s) = \mathrm{e}^{-\lambda(t+s)}T(t+s) = (\mathrm{e}^{-\lambda t}T(t))(\mathrm{e}^{-\lambda s}T(s)) = T_\lambda(t)T_\lambda(s).$$

If $t = 0$, then

$$T_\lambda(0) = \mathrm{e}^{-\lambda\cdot 0}T(0) = I.$$

For all $x \in X$ we obtain

$$\lim_{t\to 0+} T_\lambda(t)x = \lim_{t\to 0+} \mathrm{e}^{-\lambda t}T(t)x = \lim_{t\to 0+} T(t)x = x.$$

Therefore, T_λ is C_0-semigroup.

Let B is generator of T_λ. Suppose that $x \in D(A)$. By using definition of generator semigroup (see section 1.4) we obtain

$$\lim_{h\to 0+}\frac{T_\lambda(h)x - x}{h} = \lim_{h\to 0+}\frac{e^{-\lambda h}T(h)x - x}{h} = \lim_{h\to 0+}\frac{e^{-\lambda h}T(h)x - e^{-\lambda h}x + e^{-\lambda h}x - x}{h}$$

$$= \lim_{h\to 0+}\left(\frac{e^{-\lambda h}(T(h)x - x)}{h} + \frac{(e^{-\lambda h}-1)x}{h}\right) = Ax - \lambda x,$$

from which it follows that $x \in D(B)$ and that $Bx = Ax - \lambda x$. Thus, $D(A) \subset D(B)$ and since $T(t) = \mathrm{e}^{\lambda t}T_\lambda(t)$ the same method applies to prove that $D(B) \subset D(A)$. Hence, $D(A) = D(B)$ and $B = A - \lambda I$ is generator of T_λ.

Exercise 2.3. Let T be a bounded C_0-semigroup on the Banach space X, $M := \sup\limits_{t\geqslant 0}\|T(t)\|$, and let A its generator.

(a) Show that

$$|||x||| := \sup_{t\geqslant 0}\|T(t)x\| \quad (x \in X)$$

defines a norm $|||\cdot|||$ on X which is equivalent to $\|\cdot\|$, and that T is a C_0-semigroup of contractions on $(X, |||\cdot|||)$.

(b) For any $\alpha_1, \dots, \alpha_n > 0$, show that

$$\|(I - \alpha_1 A)^{-1}\dots(I - \alpha_n A)^{-1}\| \leqslant M.$$

Proof. (a) Obviously, the first and second norm properties hold. For $x, y \in X$ we consider the following estimates

$$|||x+y||| = \sup_{t\geqslant 0}\|T(t)(x+y)\| = \sup_{t\geqslant 0}\|T(t)x + T(t)y\|$$

$$\leqslant \sup_{t\geqslant 0}\|T(t)x\| + \sup_{t\geqslant 0}\|T(t)y\| = |||x||| + |||y|||.$$

Therefore, $|||\cdot|||$ is norm.

Now we prove the equivalence of the norms $\|\cdot\|$ and $|||\cdot|||$, i.e.

$$m_1\|x\| \leqslant |||x||| \leqslant m_2\|x\|, \quad x \in X,$$

where m_1, m_2 are constants.

Indeed, for $x \in X$ we consider the following inequality

$$\|x\| = \|T(0)x\| \leqslant \sup_{t\geqslant 0}\|T(t)x\| = |||x||| \leqslant \sup_{t\geqslant 0}\|T(t)\|\|x\| \leqslant M\|x\|, \tag{2}$$

with $m_1 = 1$ and $m_2 = M$. Thus, norms are equivalent.

Since norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent, then all properties C_0-semigroup hold. Thus, T is C_0-semigroup on $(X, |||\cdot|||)$.

Prove that T is semigroup of contractions. For all $x \in X$ we have

$$|||T(t)x||| = \sup_{s\geqslant 0}\|T(s)T(t)x\| = \sup_{s\geqslant 0}\|T(s+t)x\| \leqslant \sup_{s\geqslant 0}\|T(s)x\| = |||x|||.$$

Hence,

$$|||T(t)||| \leqslant 1.$$

Thus, T is semigroup of contractions.

(b) Using part (a) we have $|||T(t)||| < 1$. Now we consider the operator $(I-\alpha_1 A)^{-1}$. From Theorem 2.7 we obtain

$$|||(I-\alpha_1 A)^{-1}||| = |||\frac{1}{\alpha_1}\left(\frac{1}{\alpha_1}I - A\right)^{-1}||| \leqslant \frac{1}{\alpha_1}\int_0^\infty \mathrm{e}^{-\frac{1}{\alpha_1}t}|||T(t)|||\,\mathrm{d}t \leqslant 1.$$

Analogical we have the same estimate for every operators $(I-\alpha_2 A)^{-1}, \dots, (I-\alpha_n A)^{-1}$. Thus,

$$|||(I-\alpha_1 A)^{-1}\dots(I-\alpha_n A)^{-1}||| \leqslant |||(I-\alpha_1 A)^{-1}|||\dots|||(I-\alpha_n A)^{-1}||| \leqslant 1.$$

Since the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent, then for $x \in X$ the following inequalities hold (see (2))

$$\begin{aligned}\|(I-\alpha_1 A)^{-1}\dots(I-\alpha_n A)^{-1}x\| &\leqslant |||(I-\alpha_1 A)^{-1}\dots(I-\alpha_n A)^{-1}x|||\\ &\leqslant |||(I-\alpha_1 A)^{-1}\dots(I-\alpha_n A)^{-1}|||\cdot|||x||| \leqslant |||x||| \leqslant M\|x\|.\end{aligned}$$

Hence, we showed that $\|(I-\alpha_1 A)^{-1}\dots(I-\alpha_n A)^{-1}\| \leqslant M$.

Exercise 2.4. Show the Hille-Yosida theorem for general case:

Let A be a closed, densely defined operator in the Banach space X. Assume that there exists $M \geqslant 1$ and $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and

$$\|R(\lambda, A)^n\| \leqslant \frac{M}{(\lambda - w)^n}, \qquad (\lambda \in (w,\infty), n \in \mathbb{N}).$$

Then A is generator of a C_0-semigroup T satisfying the estimate

$$\|T(t)\| \leqslant M\mathrm{e}^{w}t \qquad (t \geqslant 0).$$

Proof. Note that without loss of generality we can only treat case $w = 0$. In this case we have the following estimate

$$\|\lambda^n R(\lambda, A)^n\| \leqslant M.$$

Define Yosida approximations

$$A_n := A(I - \frac{1}{n}A)^{-1} = nAR(n, A) = n^2 R(n, A) - nI \in \mathcal{L}(X) \qquad (n \in \mathbb{N}).$$

Now let obtain estimates for norm of exponent $\|\mathrm{e}^{tA_n}\|$ for any $t > 0$ and $n \in \mathbb{N}$

$$\begin{aligned} \|\mathrm{e}^{tA_n}\| =& \|\mathrm{e}^{t(n^2R(n,A)-nI)}\| = \mathrm{e}^{-tn}\|\sum_{k=0}^{\infty} \frac{(tn^2 R(n, A))^k}{k!}\| \\ \leqslant& \mathrm{e}^{-tn} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!}\|n^k R(n, A)^k\| \leqslant \mathrm{e}^{-tn} \sum_{k=0}^{\infty} \frac{(tn)^k}{k!} M = M\mathrm{e}^{-tn}\mathrm{e}^{tn} = M. \end{aligned} \tag{3}$$

For $x \in X$, $t > 0$, and $m, n \in \mathbb{N}$ we compute

$$\begin{aligned} \mathrm{e}^{tA_m} - \mathrm{e}^{tA_n} =& \int_0^t \frac{d}{ds}(\mathrm{e}^{(t-s)A_n}\mathrm{e}^{sA_m}x)\,\mathrm{d}s = \int_0^t \mathrm{e}^{(t-s)A_n}(A_m - A_n)\mathrm{e}^{sA_m}x\,\mathrm{d}s \\ =& \int_0^t \mathrm{e}^{(t-s)A_n}\mathrm{e}^{sA_m}(A_m - A_n)x\,\mathrm{d}s, \end{aligned}$$

(where in the last equality we have used that A_m, A_n as well as the generated semigroups commute). Recalling (3) we obtain the estimate

$$\|\mathrm{e}^{tA_m}x - \mathrm{e}^{tA_n}x\| \leqslant M^2 t\|(A_m - A_n)x\|. \tag{4}$$

Let $c > 0$. For $n \in \mathbb{N}$ we define the operator $\mathcal{T}_n^c : X \to C([0, c], X)$ (where $C([0, c], X)$ denotes the Banach space of continuous X- valued functions, equipped with the supremum norm) by

$$\mathcal{T}_n^c x := [t \mapsto \mathrm{e}^{tA_n}x] \qquad (x \in X).$$

The proof of estimate (3) shows that $\|\mathcal{T}_n^c\| \leqslant M$ and inequality (4) shows that

$$\|\mathcal{T}_m^c x - \mathcal{T}_n^c x\| \leqslant cM^2\|A_m x - A_n x\| \qquad (m, n \in \mathbb{N})$$

for all $x \in \mathrm{dom}(A)$, which implies that $(\mathcal{T}_n^c x)_{n\in\mathbb{N}}$ is a Cauchy sequence, because $(A_n x)_{n\in\mathbb{N}}$ is convergent to Ax. Applying proposition 1.6 we conclude that there exists $\mathcal{T}^c \in \mathcal{L}(X, C([0, c]; X))$ such that $\mathcal{T}_n^c \to \mathcal{T}^c$ $(n \to \infty)$ strongly.

Clearly, if $0 < c < c'$, then $\mathcal{T}^{c'}x|_{[0,c]} = \mathcal{T}^c$ for all $x \in X$, and therefore we can define $T : [0, \infty) \to \mathcal{L}(X)$ by

$$T(t)x := \mathcal{T}^c x(t) \qquad (0 \leqslant t < c, x \in X).$$

From $T(\cdot)x_{[0,c]} = \mathcal{T}^c x(c > 0, x \in X)$ we infer that T is strongly continuous. Since $T(t) = s - \lim_{n\to\infty} \mathrm{e}^{tA_n}(t \geqslant 0)$, the semigroup property carries over from the semigroups

$(\mathrm{e}^{tA_n})_{t\geqslant 0}$ to T. As a result T is a C_0 semigroup with estimation for norm $\|T(t)\| \leqslant M$ for all $t > 0$.

The last part of proof showing that A is a generator of T is exactly the same as part (iii) of theorem 2.9 proof so we skip it.

Exercise 2.5. Let T be a C_0-semigroup on the Banach space X. For $h > 0$ we define $A_h := h^{-1}(T(h) - I)$. Show that $\mathrm{e}^{tA_h} \to T(t)x$ for all $x \in X$ as $h \to 0$, uniformly for t in compact set of $[0, \infty)$.

Proof. Let K is arbitrary compact set of $[0, \infty)$. Then exists such $c > 0$ that $K \subset [0, c]$. So it is enough to show that $\mathrm{e}^{tA_h} \to T(t)x$ for all $x \in X$ as $h \to 0$, uniformly for t in $[0, c]$.

Let $\|T(t)\| \leqslant M\mathrm{e}^{wt}$, where $M > 0$, $w \in \mathbb{R}$. First we show that for all $h \in (0, 1)$ there exists $w' > w$ such that exponent e^{tA_h} can be estimated by

$$\|\mathrm{e}^{tA_h}\| \leqslant M\mathrm{e}^{w't} \qquad (t \geqslant 0). \tag{5}$$

For e^{tA_h} we obtain

$$\begin{aligned}\|\mathrm{e}^{tA_h}\| =&\mathrm{e}^{-th}\|\mathrm{e}^{\frac{tT(h)}{h}}\| = \mathrm{e}^{-th}\|\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{tT(h)}{h}\right)^k\| \leqslant \mathrm{e}^{-th}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{h}\right)^k\|T(kh)\| \\ \leqslant&\mathrm{e}^{-th}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{h}\right)^k M\mathrm{e}^{wkh} = M\mathrm{e}^{-th}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\mathrm{e}^{wh}\frac{t}{h}\right)^k = M\exp\left(\frac{t}{h}(\mathrm{e}^{wh}-1)\right).\end{aligned}$$

Since $h^{-1}(\mathrm{e}^{wh} - 1) \to w$ as $h \to 0$ then exists $w' > w$:

$$|h^{-1}(\mathrm{e}^{wh} - 1)| \leqslant w' \qquad (h \in (0, 1)).$$

Hence e^{tA_h} can be estimated by (5) when $h \in (0, 1)$.

For any $x \in \mathrm{dom}(A)$ we can define function $[0, t] \ni s \to \mathrm{e}^{sA_h}T(t-s)x$ that is continuous as well as continuously differentiable on $(0, t)$, with

$$\frac{d}{ds}\mathrm{e}^{sA_h}(T(t-s)x) = A_h\mathrm{e}^{sA_h}T(t-s)x - \mathrm{e}^{sA_h}T(t-s)Ax = \mathrm{e}^{sA_h}T(t-s)(A_h - A)x.$$

By the fundamental theorem of calculus (see Theorem 1.9) it follows that

$$\|(T(t) - \mathrm{e}^{sA_h})x\| = \left\|\int_0^t \mathrm{e}^{sA_h}T(t-s)(A_h - A)x\,\mathrm{d}s\right\|.$$

Using estimation (5) we get the following inequality

$$\begin{aligned}\|(T(t) - \mathrm{e}^{tA_h})x\|_c \leqslant& M^2\max\{1, \mathrm{e}^{wt}\}\max\{1, \mathrm{e}^{w't}\}t\|(A_h - A)x\|_c \\ \leqslant& M^2\max\{1, \mathrm{e}^{wc}\}\max\{1, \mathrm{e}^{w'c}\}c\|(A_h - A)x\|_c \qquad (t, h \in [0, c]).\end{aligned}$$

From last inequality we obtain that $\mathrm{e}^{tA_h}x \to T(t)x$ for all $x \in \mathrm{dom}(A)$ as $h \to 0$, uniformly for t in compact set of $[0, \infty)$. Using that $\mathrm{dom}(A)$ is dense and applying Proposition 1.6 we obtain that $\mathrm{e}^{tA_h}x \to T(t)x$ as $h \to 0$ uniformly for t in compact set of $[0, \infty)$ for all $x \in X$.