Internet Seminar 18 · Lecture 1 · Solutions
Wuppertal Team

Solution of Exercise 1 (Julia Kleinhans). Let X be a Banach space and let $A \in \mathcal{L}(X)$. Then $\sum_{j=0}^{\infty} \frac{A^j}{j!}$ is absolutely convergent in $\mathcal{L}(X)$ since

$$\sum_{j=0}^{\infty} \left\| \frac{A^j}{j!} \right\| \leq \sum_{j=0}^{\infty} \frac{\|A\|^j}{j!} = \mathrm{e}^{\|A\|}$$

holds. Suppose $A, B \in \mathcal{L}(X)$ commute. To verify the identity

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B} = \mathrm{e}^B \mathrm{e}^A$$

we use the Cauchy product for absolute convergent sums:

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big).$$

Therefore we have, using that $AB = BA$,

$$\begin{aligned}\mathrm{e}^A \mathrm{e}^B &= \left(\sum_{j=0}^{\infty} \frac{A^j}{j!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right) = \sum_{j=0}^{\infty} \sum_{n=0}^{j} \frac{A^n}{n!} \frac{B^{j-n}}{(j-n)!} = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{n=0}^{j} \binom{j}{n} A^n B^{j-n} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} (A+B)^j = \mathrm{e}^{A+B}.\end{aligned}$$

Furthermore, it follows that, $t \mapsto \mathrm{e}^{tA}$ is a one-parameter group. Indeed, $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$ $\forall t, s \in \mathbb{R}$, because $tA \cdot sA = tsA^2 = sA \cdot tA$ and we can apply the previous result.
Next we show that the one-parameter group is continuously differentiable and that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ $(t \in \mathbb{R})$: Let $t \in \mathbb{R}$ be fixed and set $c := \|\mathrm{e}^{tA}\|$. If $|h| > 0$, then

$$\begin{aligned}\left\| \frac{\mathrm{e}^{(t+h)A} - \mathrm{e}^{tA}}{h} - A\mathrm{e}^{tA} \right\| &= \left\| \frac{\mathrm{e}^{tA}\mathrm{e}^{hA} - \mathrm{e}^{tA}}{h} - A\mathrm{e}^{tA} \right\| \leq \left\| \frac{\mathrm{e}^{hA} - I}{h} - A \right\| \|\mathrm{e}^{tA}\| \\ &= c \left\| \frac{1}{h} \Big(\sum_{j=0}^{\infty} \frac{h^j A^j}{j!} - I \Big) - A \right\| = c \left\| \sum_{j=1}^{\infty} \frac{h^{j-1} A^j}{j!} - A \right\| \\ &\leq c \sum_{j=2}^{\infty} \frac{h^{j-1} \|A\|^j}{j!} = c \sum_{j=0}^{\infty} \frac{h^{j+1} \|A\|^{j+2}}{(j+2)!} \leq c \cdot h \|A\|^2 \cdot \mathrm{e}^{h\|A\|} \overset{h \to 0}{\to} 0.\end{aligned}$$

Solution of Exercise 2 (Robert Nabiullin, Jens Wintermayr). Let us accept that the set $\mathrm{UC}_b(G)$ of all bounded uniformly continuous $\mathbb{K}$-valued functions on G is a Banach space with the sup-norm and with the pointwise operations.

(a) We show that $C_0(G)$ is a closed subspace of the Banach space $\mathrm{UC}_b(G)$. We have $C_0(G) \subseteq \mathrm{UC}_b(G)$. Indeed, let $\varepsilon > 0$ and $f \in C_0(G)$ be given. Then $K = [|f| \geq \varepsilon/2]$ is compact. Since f is continuous on the compact set K, we can cover K by finitely many balls $B(x_1, r_1), \ldots, B(x_N, r_N)$ such that $y \in B(x_j, r_j)$ implies $|f(x_j) - f(y)| \leq \varepsilon/2$. Let $\delta = \min\{r_1/2, \ldots, r_N/2\}$. If $|x - y| < \delta$, then either both of x, y lie outside the finitely many balls, or both of them belongs to the same ball $B(x_j, r_j)$ for some j. In the first case $|f(x) - f(y)| \leq 2\varepsilon/2 = \varepsilon$ by the choice of the compact set K; in the second case $|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| \leq 2\varepsilon/2 = \varepsilon$ by the choice of $\delta > 0$.

We have $0 \in C_0(G)$: For all $c > 0$ the set $[0 \geq c] = \emptyset$ is compact. Now let $\alpha \in \mathbb{K}$ and $f, g \in C_0(G)$. We have to show that $\alpha f \in C_0(G)$ and $f + g \in C_0(G)$ holds. If $\alpha = 0$, then $\alpha f = 0 \in C_0(G)$ as seen above. Suppose that $\alpha \neq 0$. In this case $[|\alpha f| \geq c] = [|f| \geq c/|\alpha|]$, and the latter set is compact by assumption. For any continuous function h on G the set $[h \geq c] = h^{-1}([c, \infty))$ is closed. Now we have

$$[|f + g| \geq c] \subseteq [|f| + |g| \geq c] \subseteq \Big[|f| \geq \frac{c}{2}\Big] \cup \Big[|g| \geq \frac{c}{2}\Big]$$

implying that the set $[|f+g|\geq c]$, as a closed subset of a compact set, is itself compact. This shows that $C_0(G)$ is a linear subspace of the vector space $\mathrm{UC}_b(G)$.
Let (f_n) be a sequence in $C_0(G)$ which converges to some $f\in \mathrm{UC}_b(G)$ with respect to supremum norm. For each $c>0$ there is an $N\in\mathbb{N}$ such that $\|f-f_n\|_\infty\leq c/2$ for all $n\geq N$. Hence we have for any $n\geq N$

$$[|f|\geq c]\subseteq[|f-f_n|+|f_n|\geq c]\subseteq[|f_n|\geq c/2].$$

The last set is compact by assumption and hence the set $[|f|\geq c]$ is compact as a closed subset of a compact set, and so $f\in C_0(G)$.
We now show that $C_c(G)$ is a dense subspace of $C_0(G)$. Let $f\in C_0(G)$ and $\varepsilon>0$. Since the set $[|f|\geq\varepsilon]$ is compact by Urysohn's lemma there is a continuous function $h\colon G\to[0,1]$ such that $h(x)=1$ for all $x\in[|f|\geq\varepsilon]$ and $h(x)=0$ for all $x\in[|f|\leq\varepsilon/2]$. We define $g:=hf$. Then g is a continuous function with compact support, since $\operatorname{spt} g\subset[|f|\geq\varepsilon/2]$. Further we have $\|g-f\|_\infty=\|fh-f\|_\infty<\varepsilon$.
(b) Let T be the one-parameter semigroup of right translations on $C_0(\mathbb{R})$ with generator $\tilde{A}$. We also define

$$Af=-f' \qquad\text{with}\qquad \operatorname{dom}(A)=C_0^1(\mathbb{R}):=\{f\in C^1(\mathbb{R}); f,f'\in C_0(\mathbb{R})\}.$$

We have to show $A=\tilde{A}$.
Since uniform convergence implies pointwise convergence, for each $f\in\operatorname{dom}(\tilde{A})$, $x\in\mathbb{R}$ we have

$$\tilde{A}f(x)=\lim_{h\searrow 0}\frac{(T(h)f)(x)-f(x)}{h}=\lim_{h\searrow 0}\frac{f(x-h)-f(x)}{h}=-\lim_{h\searrow 0}\frac{f(x)-f(x-h)}{h}=-f'(x).$$

(Only the right differentiability of f at x follows this way, but one can use Theorem 1.10 and the fact that $T(t)$ are invertible to obtain that f is indeed differentiable at x.)

Let $f\in\operatorname{dom}(A)$, then f is differentiable, $f'\in C_0(\mathbb{R})$ and from part (a) we know that f' is uniformly continuous. For $\varepsilon>0$ take $\delta>0$ according to the uniform continuity of f'. For every $x\in\mathbb{R}$ we have

$$\begin{aligned}\left|\frac{(T(h)f)(x)-f(x)}{h}-Af(x)\right|&=\left|\frac{f(x-h)-f(x)}{h}-(-f'(x))\right|\\&=\left|f'(x)-\frac{f(x)-f(x-h)}{h}\right|=\left|\frac{1}{h}\int_{x-h}^{x}f'(y)\mathrm{d}y-f'(x)\right|\\&\leq\frac{1}{h}\int_{x-h}^{x}|f'(y)-f'(x)|\,\mathrm{d}y\leq\varepsilon \qquad (1)\end{aligned}$$

whenever $h\in(0,\delta)$. This shows $f\in\operatorname{dom}(\tilde{A})$ and $\tilde{A}f=-f'=Af$ for all $f\in\operatorname{dom}(A)$. We obtain $\tilde{A}=A$.

(c) Let $\tilde{A}$ denote the generator of the right shift semigroup, and A the operator from the assertion. We carry out a similar calculation as above. Let $f\in\operatorname{dom}(A)$, let $\varepsilon>0$, and take $\delta>0$ such that $|f'(x)|\leq\varepsilon$ for every $x\in(0,\delta)$ and $|f(x)-f(y)|\leq\varepsilon$ for every $x,y\in(0,1]$ with $|x-y|\leq\delta$ (uniform continuity). We have

$$\left|\frac{(T(h)f)(x)-f(x)}{h}-Af(x)\right|=\begin{cases}\left|\frac{1}{h}\int_{x-h}^{x}f'(y)\mathrm{d}y-f'(x)\right| & \text{if } x\in[h,1],\\ \left|\frac{1}{h}\int_{0}^{x}f'(y)\mathrm{d}y+f'(x)\right| & \text{if } x\in(0,h).\end{cases}$$

From this we conclude for $h\in(0,\delta)$

$$\left|\frac{(T(h)f)(x)-f(x)}{h}-Af(x)\right|=\begin{cases}\varepsilon & \text{if } x\in[h,1] \text{ (similarly as in part (b))},\\ 2\varepsilon & \text{if } x\in(0,h).\end{cases}$$

This yields $f\in\operatorname{dom}(\tilde{A})$. Hence $A\subseteq\tilde{A}$. The inclusion $\tilde{A}\subseteq A$ can be proved similarly to part (b).

Solution of Exercise 3 (Björn Augner, Hafida Laasri). Let $(\Omega,\mathcal{A},\mu)$ be a measure space and let $p\in[1,\infty)$. Let $a:\Omega\to\mathbb{K}$ be a measurable function and suppose that $\mathrm{Re}a(x)\leq 0$ $(x\in\Omega)$.
For $t\geq 0$ define $T(t)\in\mathcal{L}(L_p(\mu))$ by

$$T(t)f:=\mathrm{e}^{ta}f \quad (f\in L_p(\mu)).$$

(a) **Claim:** T is a C_0-semigroup on $L_p(\mu)$.
Proof: 1) For all $t \geq 0$ and $f \in L_p(\mu)$ one has

$$\begin{aligned}\|T(t)f\|_{L_p(\mu)} &= \left\|\mathrm{e}^{ta} f\right\|_{L_p(\mu)} = \left\|\left|\mathrm{e}^{ta}\right| f\right\|_{L_p(\mu)} \\ &= \left\|\mathrm{e}^{t\mathrm{Re}a} f\right\|_{L_p(\mu)} \leq \|f\|_{L_p(\mu)},\end{aligned}$$

hence $T(t) \in \mathcal{L}(L_p(\mu))$.
2) For all $s, t \geq 0$ and $f \in L_p(\mu)$ the semigroup property holds:

$$T(t+s)f = \mathrm{e}^{(t+s)a} f = \mathrm{e}^{ta}\mathrm{e}^{sa} f = T(t)T(s)f.$$

3) To obtain strong continuity of T at $t = 0$ first note that for all $t \geq 0$ and $f \in L_p(\mu)$

$$|T(t)f(x)| = \mathrm{e}^{t\mathrm{Re}a(x)} |f(x)| \leq |f(x)|, \quad \mu\text{-a.e. } x \in \Omega$$

and $|f| \in L_p(\mu)$. Moreover for all $f \in L_p(\mu)$ and μ-a.e. $x \in \Omega$ one has

$$(T(t)f - f)(x) = \left(\mathrm{e}^{ta(x)} - 1\right) f(x) \xrightarrow{t \searrow 0} 0.$$

Lebesgue's theorem yields that

$$T(t)f - f \xrightarrow{t \searrow 0} 0 \quad \text{in } L_p(\mu).$$

As a result T is a C_0-semigroup on $L_p(\mu)$.

(b) **Claim:** The maximal multiplication operator M_a defined by

$$M_a = \{(f, g) \in L_p(\mu) \times L_p(\mu);\ g = af\}$$

is closed and densely defined.
Proof: From the definition it immediately follows that M_a is a linear operator.
1) To see that $\mathrm{dom}(M_a)$ is dense in $L_p(\mu)$ fix $f \in L_p(\mu)$. For $k \in \mathbb{N}$ define

$$\Omega_k := \{x \in \Omega : |a(x)| \leq k\} \in \mathcal{A}$$

(since a is measurable), so that $\Omega_k \nearrow \Omega$ $(k \to \infty)$. Set $f_k := 1_{\Omega_k} f$ $(k \in \mathbb{N})$. Then for all $k \in \mathbb{N}$ one has

$$\|af_k\|_{L_p(\mu)} \leq k \|f\|_{L_p(\mu)} < +\infty,$$

hence $(f_k, af_k) \in M_a$. Moreover

$$\|f - f_k\|_{L_p(\mu)}^p = \int_{\Omega_k^c} |f|^p \,\mathrm{d}\mu \xrightarrow{k \to \infty} 0$$

since $f \in L_p(\mu)$ and so M_a is densely defined.
2) To see that $M_a \subset L_p(\mu) \times L_p(\mu)$ is closed let $(f_n, g_n)_{n \geq 1} \subset M_a$ such that $(f_n, g_n) \xrightarrow{n \to \infty} (f, g) \in L_p(\mu) \times L_p(\mu)$. Since $p \in [1, \infty)$ there is a subsequence such that $(f_{n_k}, g_{n_k}) \to (f, g)$ pointwise μ-a.e. as $k \to \infty$ (see, e.g. Theorem 4.9 in [1]). Then for μ-a.e. $x \in \Omega$

$$g(x) \leftarrow g_{n_k}(x) = a(x) f_{n_k}(x) \to a(x) f(x) \quad (k \to \infty)$$

and hence $af = g$, so that $(f, g) \in M_a$ and M_a is closed.

(c) **Claim:** M_a is the generator of T.
Proof: Let A be the generator of T. If $(f, g) \in A$, then, e.g. by Theorem 4.9 in [1], there is a sequence $t_n \searrow 0$ such that

$$\frac{1}{t_n}\left(T(t_n)f - f\right)(x) \xrightarrow{n \to \infty} g(x) \quad \mu\text{-a.e. } x \in \Omega$$

and since we also have μ-a.e. pointwise convergence to af (since $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ta(x)}|_{t=0} = a(x)$) it follows $af = g \in L_p(\mu)$, i.e. $(f, g) \in M_a$, so that $A \subset M_a$.

On the other hand, for every $(f,g) \in M_a$ one has

$$\frac{1}{t}(T(t)f - f) = \frac{1}{t}(e^{ta} - 1)f \xrightarrow{t \searrow 0} af = g$$

pointwise μ-a.e. Moreover, for all $x \in \Omega$ with $a(x) \neq 0$ one has

$$\begin{aligned} \left| \frac{\mathrm{e}^{ta(x)} - 1}{ta(x)} \right| &= \left| \frac{1}{t} \int_0^t \mathrm{e}^{sa(x)} \mathrm{d}s \right| \\ &\leq \frac{1}{t} \int_0^t \left| \mathrm{e}^{sa(x)} \right| \mathrm{d}s \leq 1 \end{aligned}$$

and hence

$$\left| \frac{1}{t}(T(t)f - f)(x) \right| \leq |a(x)f(x)|, \quad \mu\text{-a.e. } x \in \Omega,$$

so by Lebesgue's theorem the convergence also holds in $L_p(\mu)$.

References

[1] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York, 2011.

Solution of Exercise 4 (Sebastian Möller).

(a) Because f is continuous we know that

$$\forall \varepsilon > 0 \exists \delta > 0 : \|f(s) - f(0)\| < \varepsilon$$

for $|s| < \delta$. So we get

$$\left\| \frac{1}{h} \int_0^h f(s) \mathrm{d}s - f(0) \right\| = \frac{1}{h} \left\| \int_0^h f(s) - f(0) \mathrm{d}s \right\| \leq \frac{1}{h} \int_0^h \varepsilon \mathrm{d}s = \varepsilon$$

for $|h| < \delta$.

(b) With part (a) it follows that

$$\left\| I - \frac{1}{\tau} \int_0^\tau T(s) \mathrm{d}s \right\| < 1$$

for $\tau > 0$ sufficient small.
So the Neumann-series implies that

$$I - \Big(I - \frac{1}{\tau} \int_0^\tau T(s) \mathrm{d}s \Big) = \frac{1}{\tau} \int_0^\tau T(s) \mathrm{d}s$$

and consequently

$$\int_0^\tau T(s) \mathrm{d}s$$

is invertible.
Using Theorem 1.10.(b) we know that

$$A \int_0^\tau T(s) \mathrm{d}s = T(\tau) - I.$$

Because $\int_0^\tau T(s)\mathrm{d}s$ is invertible we can write A as the bounded operator

$$(T(\tau) - I)\left(\int_0^\tau T(s)\mathrm{d}s\right)^{-1}.$$

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