

# Characterisation of $L^p$ -contractivity by invariance criteria

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For a  $C_0$ -semigroup  $T$  and a closed, convex subset  $C$  of the underlying Banach space it is often important to find criteria which ensure that  $C$  is invariant with respect to the semigroup. If  $T$  is defined on a Hilbert space and its generator  $A$  is associated with a form, this can be done by means of the invariance criteria discussed in Lecture 10.

In concrete applications  $A$  might be a differential operator, e.g. an elliptic operator as discussed in Lecture 11. In this case, one can often use the invariance criteria to prove  $L^\infty$ -contractivity of the semigroup since the projection onto the  $L^\infty$ -unit ball can easily be computed. Similarly, one can often prove  $L^1$ -contractivity of the semigroup by a duality argument.

It is natural to ask whether one can also expect  $L^p$ -contractivity for some  $p \in (1, \infty)$ ,  $p \neq 2$ . However, this question is much more difficult since the projection from  $L^2$  onto the  $L^p$ -unit ball is not known explicitly. A possible solution is to derive an implicit formula for this projection, instead. This was done in the paper [1], and the implicit formula given there indeed allows to apply the invariance criteria.

The goals of this project are as follows:

- (i) To derive the implicit formula for the projection onto the  $L^p$ -unit ball given in [1, Theorem 3.3].
- (ii) To explain the underlying geometric idea.
- (iii) To combine the formula with the invariance criteria for  $C_0$ -semigroups in order to characterize  $L^p$ -contractivity.
- (iv) To use (iii) in order to prove  $L^p$ -(quasi)-contractivity of semigroups generated by elliptic operators.

Possibly, the following additional topic could also be discussed in the project: An important property of  $C_0$ -semigroups is so-called *maximal regularity*, which has applications for non-linear evolution equations. An important criterion asserts that a  $C_0$ -semigroup on an  $L^p$ -space has maximal regularity if it is positive and contractive. Thus, one can conclude from the above results that the  $C_0$ -semigroup generated by an elliptic operator on an  $L^p$ -space has maximal regularity.

## REFERENCES

- [1] R. Nittka: Projections onto convex sets and  $L^p$ -quasi-contractivity of semigroups. Arch. Math. **98**, 341–353 (2012).