Form methods for evolution equations on graphs

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The starting point of our project is the classical Dirichlet form introduced in the Lecture 5. One can replace the second derivative operator

$$\Delta : f \mapsto \lim_{h\to 0}\lim_{k\to 0} \frac{f(\cdot+h+k)-f(\cdot+h)-f(\cdot+k)+f(\cdot)}{hk} ,$$

which is well-defined on smooth functions $f : \mathbb{R} \to \mathbb{R}$, by its finite differences counterpart

$$\mathscr{L} : f \mapsto f(\cdot+h) - 2f(\cdot) + f(\cdot-h) .$$

Because $\mathscr{L} f$ is a well-defined object even if f is only defined in countably many points, $\mathscr{L}$ can be considered as a discretisation of the Laplacian.
A similar construction can be actually performed for general functions $f : \mathsf{V} \to \mathbb{R}$, where V is a general (finite or countable) set of points. In order to make sense of the terms like $f(x+h)$, let us turn this unordered set of points into a *graph* by assigning an adjacency structure $\sim$ and replace

$$f(x+h)+f(x-h) \qquad \text{by} \qquad \sum_{\substack{y\in\mathsf{V}\\ y\sim x}} f(y) .$$

In this way, differential operators on intervals or domains can be effectively turned, at least formally, into difference operators on graphs – an idea that goes back to Gustav Kirchhoff, who studied the *discrete Laplacian* $\mathscr{L}$ in [Kir47] due to its interplay with the theory of electric circuits. Kirchhoff also found the quadratic form a associated with $\mathscr{L}$; Beurling and Deny then proved in [BD59] that this a is a Dirichlet form (in the sense of Remark 10.14 in the ISem lecture notes) – in fact, the earliest example of Dirichlet form in the literature!
The aim of this project is to present in detail the theory of discrete Laplacians on graphs. We will first introduce in detail the form a in the easy case of finite graphs (the one considered in [BD59]) and re-prove the result by Beurling and Deny, showing that space-a discrete version of the heat equation is governed by a Markovian semigroup. We will then pass to the more delicate case of infinite graphs, which requires the introduction of discrete version h^1 of the common Sobolev space H^1, and show how certain combinatorial properties of the graph influence some functional analytic properties of h^1 following [KL10, Mug13]; we will in particular identify the Friedrichs extension of $\mathscr{L}$.
Finally, we will consider non-autonomous heat equations on graphs. That is, we will discuss the case of discrete Laplacians $\mathscr{L}$ (or equivalently, quadratic forms a) that are not constant with respect to the time variable. Time-dependent forms were the subject of Lecture 14 in the ISem lecture notes, where a result on maximal regularity by Lions [Lio61] was presented. As mentioned in the Notes of Lecture 14, recently several improvements of Lions' result have been achieved. In this part of the project we attempt to apply some results on maximal regularity and stability obtained in [ADLO14] and [ADK] to the case of non-autonomous discrete Laplacian.
Apart from the ISem lecture notes, this project is self-contained. Even if we borrow some notations and concepts from graph theory, no deep combinatorial result will be used; in fact, one of the main scopes of this project is to show that already the innocent looking ℓ^2-spaces allow for a rich theory of quadratic forms.

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