Project: Form inequalities for symmetric contraction semigroups

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Let $-A$ be the generator of a strongly continuous semigroup $(S_t)_{t\geq 0}$ of symmetric operators on $\mathrm{L}^2(\mathrm{X})$, where $\mathrm{X} = (X, \Sigma, \mu)$ is an arbitrary measure space. The semigroup is called a *symmetric contraction semigroup* if each operator S_t is L^1- and L^∞-contractive, i.e.

$$\|S_t f\|_1 \leq \|f\|_1 \quad \text{and} \quad \|S_t f\|_\infty \leq \|f\|_\infty$$

for all $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$.

In Lecture 10 of the ISem it has been shown by virtue of complex interpolation techniques that the semigroup $(S_t)_t$ extends to a contractive analytic semigroup on $\mathrm{L}^p(\mathrm{X})$ of angle $\theta_p = \frac{\pi}{2}\left(1 - \left|1 - \frac{2}{p}\right|\right)$ for $1 < p < \infty$. However, the optimal angle is $\varphi_p = \arccos\left|1 - \frac{2}{p}\right|$. In the sub-Markovian case this has been proved first in [4], see also [5]. The general case is due to Kriegler [3].

The aim of this project is to give a proof based on the recent article [2]. Note that the statement can be equivalently reformulated as

$$\mathrm{Re} \int_{\mathrm{X}} \mathrm{e}^{\pm \mathrm{i}\varphi_p} \, (Af) \cdot \overline{f} \, |f|^{p-2} \geq 0 \tag{1}$$

for all $f \in \mathrm{dom}(A_p)$, where $-A_p$ is the generator in L^p of the semigroup. In the project we look more generally at inequalities of the form

$$\mathrm{Re} \sum_j \int_{\mathrm{X}} AF_j(\mathbf{f}) \cdot G_j(\mathbf{f}) \; \geq 0$$

where $\mathbf{f}$ is a $\mathbb{C}^d$-valued measurable function on X and the F_j, G_j are measurable functions on $\mathbb{C}^d$. It is shown that such an inequality holds in general, i.e., for *every* symmetric contraction semigroup, iff it holds for a couple of simple two-dimensional test cases. In this way, (1) is reduced to a scalar inequality which can then be established by elementary methods.

The main reduction step involves a clever idea from a recent work of Carbonaro and Dragičević from [1] and some abstract operator theory, interesting in its own right,

References

[1] Andrea Carbonaro and Oliver Dragicevic. Functional calculus for generators of symmetric contraction semigroups. arXiv:1308.1338v2.

[2] Markus Haase. Form inequalities for symmetric contraction semigroups. Submitted. Available at: `http://fa.its.tudelft.nl/~haase/`, 2015.

[3] Ch. Kriegler. Analyticity angle for non-commutative diffusion semigroups. *J. Lond. Math. Soc. (2)*, 83(1):168–186, 2011.

[4] V. A. Liskevich and M. A. Perel′muter. Analyticity of sub-Markovian semigroups. *Proc. Amer. Math. Soc.*, 123(4):1097–1104, 1995.

[5] El Maati Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.