Fractional powers and Kato's conjecture

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Let $a : V \times V \to \mathbb{C}$ be a closed, densely defined, sectorial form in a complex Hilbert space H, and let A be the associated sectorial operator. In between the operators $A^0 := \mathrm{Id}$ and $A^1 := A$ a continuous scale of so-called fractional powers of A can be defined via Balakrishnan's formula

$$A^\alpha u := \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda + A)^{-1} u \, \mathrm{d}\lambda \qquad (0 < \alpha < 1).$$

Related to these operators is a famous conjecture first formulated by T. Kato in 1961 [2]:

> *We do not know whether or not* $\mathrm{dom}(A^{1/2}) = \mathrm{dom}(A^{*1/2})$ *[...]. This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both* $\mathrm{dom}(A^{1/2})$ *and* $\mathrm{dom}(A^{*1/2})$ *coincide with* $\mathrm{dom}(a)$*[...].*

As a first main result we will consider the so-called second representation theorem[1] to the effect that Kato's conjecture holds provided the sesquilinear form a is symmetric, i.e. A is self-adjoint. We will see that square roots are the borderline case for this type of question: On the one hand

$$\mathrm{dom}(A^\alpha) = \mathrm{dom}(A^{*\alpha}) \qquad (0 \leq \alpha < \frac{1}{2}).$$

On the other hand

$$\mathrm{dom}(A^\alpha), \, \mathrm{dom}(A^{*\alpha}) \subseteq V \qquad (\frac{1}{2} < \alpha \leq 1)$$

but for every such α counterexamples can be given for which $\mathrm{dom}(A^\alpha)$ and $\mathrm{dom}(A^{*\alpha})$ are not the same subset of V.

In the critical case $\alpha = \frac{1}{2}$ Kato's conjecture is false for general sectorial operators but it does hold true for differential operators of formal type $Au = -\mathrm{div}(\mu \nabla u)$. This was a long-standing open problem that was eventually solved in 2001 using harmonic analysis' most delicate methods [1]. These techniques lie far beyond the scope of this project but we aim to discuss in which sense this result is surprising and how it affects other topics in analysis.

References

[1] P. Auscher, S. Hofmann, M. Lacey, A. M$^\mathrm{c}$Intosh, and P. Tchamitchian. *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$. Ann. of Math. (2) **156** (2002), no. 2, 633–654.

[2] T. Kato. *Fractional powers of dissipative operators*. J. Math. Soc. Japan **13** (1961), 246–274.

[3] T. Kato. Perturbation Theory for Linear Operators. Classics in Mathematics, Springer, Berlin, 1995.

[4] A. M$^\mathrm{c}$Intosh and M. Schmalmack. *Kato's Square Rooot Problem – Background and recent results*. Unpublished, available at `http://maths-people.anu.edu.au/~alan/lectures/Blau.pdf`.

[1] In the terminology of Kato [3] the first representation theorem is Thm. 12.4 of the lecture notes.