SCHRÖDINGER OPERATORS WITH INVERSE SQUARE POTENTIALS

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ABSTRACT. Let $A_V = -\Delta - V(\cdot)$ be a Schrödinger operator on $L^2(\mathbb{R}^N)$ with a potential satisfying $0 \le V \in L^\infty(\{x \in \mathbb{R}^N : |x| \ge \varepsilon\})$ for every $\varepsilon > 0$, and $\lim_{x\to 0} V(x) = \infty$. If V is "too singular" at the origin, can this prevent the existence of positive solution to the heat equation $\partial_t u + A_V u = 0$ (on $\mathbb{R}^N \times (0,\infty)$)?

It has long been known that the corresponding initial value problem is wellposed (in $L^2(\mathbb{R}^N)$ and other spaces) if $V(x) \le \frac{C}{|x|^{2-\varepsilon}}$ for some positive constants C and ε.
H. Brezis and J.L. Lions conjectured that no positive solution will exist if $V(x) \ge \frac{C}{|x|^{2-\varepsilon}}$. The problem was settled by P. Baras and J.A. Goldstein [1].
Let $C^*(N) = \left(\frac{N-2}{2}\right)^2$ and $V_c(x) = \frac{c}{|x|^2}$ be the inverse square potential with $c \in \mathbb{R}$. Let us consider the Cauchy problem

$$\begin{cases} \partial_t u + A_{V_c} u = 0, & \text{on } \Omega \times (0,\infty), \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{0.1}$$

with $\Omega = \mathbb{R}^N$ if $N \ge 2$ and for $N = 1$ one has to take for example $\Omega = (0,\infty)$ so as to have a connected spatial domain and add a Dirichlet boundary condition at 0. Baras and Goldstein [1] proved

Theorem 0.1. *The Cauchy problem (0.1) has a unique positive solution for each $0 \le f \in L^2(\Omega)$ if $c \le C^*(N)$ and no positive solutions at all if $c > C^*(N)$.*

Let $W_n(x) = \inf\{V_c(x), n\}$ be the cutoff potential, with $c > C^*(N)$. Let u_n solve

$$\begin{cases} \partial_t u_n - \Delta u_n - W_n u_n = 0, & \text{on } \mathbb{R}^N \times (0,\infty), \\ u_n(x,0) = f(x) \ge 0. \end{cases}$$

Here $0 \ne f \in L^2(\mathbb{R}^N)$ or, more generally, f grows no faster than $e^{|x|^{2-\varepsilon}}$ at infinity. Since W_n is bounded, u_n exists. If a positive solution u to (0.1) were to exist, then $0 < u_n \le u$ which is a contradiction, since $u_n(x,t)$ tends to infinity at all spatial points and at all positive times (see [1, Theorem 2.2.(ii)]). This is called *instantaneous blowup*.

In this project we propose to prove Theorem 0.1 for $\Omega = \mathbb{R}^N$ with $N \ge 3$ by following the approach of Cabré and Martel [2]. This approach is based on the following Hardy inequality with optimal constant $C^*(N)$:

$$C^*(N) \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx, \quad \forall \varphi \in C_c^1(\mathbb{R}^N).$$

We propose also to study the bilinear form associated to A_{V_c}.

REFERENCES

[1] P. Baras and J.A. Goldstein, *The heat equation with singular potential*, Trans. Amer. Math. Soc. 284 (1984), pp. 121–139.

[2] X. Cabré and Y. Martel, *Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier*, C.R. Acad. Sci. Paris 329 (1999), pp. 973–978.

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