Elliptic operators with first order degeneracy at the boundary

Roland Schnaubelt*

In this project we study the impact of degenerate diffusion coefficients at the boundary of the domain. We establish generation properties and describe the domain of the generator. To avoid rather tedious localization arguments, we focus on model problems on the halfspace $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} \,|\, y > 0\}$ and treat in $L^p(\mathbb{R}^{n+1}_+)$, $p \in (1,\infty)$, the differential operator

$$A = -y(\Delta_x + \partial_{yy}) + a \cdot \nabla_x + b\partial_y \tag{1}$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The diffusion part degenerates at $\partial\mathbb{R}^{n+1}$ in normal direction of first order and hence it is 'of the same size' as the drift part.

The behavior of the corresponding elliptic and parabolic problems depends very much on the size and sign of b. The case $b > -1/p$ has been treated in [1] for all $p \in (1,\infty)$. Here the operator A is endowed with the domain

$$D^\circ_{p,\mathrm{reg}} = \{u \in W^{1,p}_0(\mathbb{R}^{n+1}_+) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+) : \ y|D^2u|, \sqrt{y}|\nabla u| \in L^p(\mathbb{R}^{n+1}_+)\}$$

and generates an analytic semigroup of positive contractions. The domain $D^\circ_{p,\mathrm{reg}}$ incorporates a Dirichlet boundary condition and full regularity (each summand of Au belongs to $L^p(\mathbb{R}^{n+1}_+)$ if $u \in D^\circ_{p,\mathrm{reg}}$). Replacing here $W^{1,p}_0$ by $W^{1,p}$, we obtain the larger space $D_{p,\mathrm{reg}}$ without boundary conditions.

For $b < -1/p$, one has a strong outward pointing drift term, and rather unexpected phenomena occur. The paper [2] deals with the one-dimensional case (i.e., $A = -y\partial_{yy} + b\partial_y$ and $n = 0$). It is shown that A induces two different generators of positive analytic semigroups if $b \in (-1, -1/p)$: The first one has full regularity, but no boundary conditions; whereas the second one has Dirichlet boundary conditions, but its domain is not contained in $W^{1,p}$. For $b \le -1$, there is just one generator with full regularity and without boundary conditions. For $n \ge 1$, the case $b < -1/p$ is not fully analyzed yet. For $p = 2$, the operator A with the domain $D_{2,\mathrm{reg}}$ generates a positive analytic semigroup on $L^2(\mathbb{R}^{n+1}_+)$. It is contractive for $b \le -1$, but only bounded and not quasi-contrcative if $p \in (-1, -1/2)$, see [3].

The proofs are based on semigroup theory (known from the lectures), approximation arguments, explicit apriori estimates and standard results for partial differential equations with nondegenerate coefficients. The latter ones will be either used without proof or shown by means of form methods.

References

[1] S. Fornaro, G. Metafune, D. Pallara and J. Prüss, *L^p–theory for some elliptic and parabolic problems with first order degeneracy at the boundary.* J. Math. Pures Appl. **87** (2007), 367–393.

[2] S. Fornaro, G. Metafune, D. Pallara and R. Schnaubelt, *One-dimensional degenerate operators in L^p–spaces.* J. Math. Anal. Appl. **402** (2013), 308-318.

[3] S. Fornaro, G. Metafune, D. Pallara and R. Schnaubelt, *Second order elliptic operators in L^2 with first order degeneration at the boundary and outward pointing drift.* Commun. Pure Appl. Anal. **14** (2015), 407–419.

*Karlsruhe Institute of Technology, Dept. of Mathematics, schnaubelt@kit.edu