Capacities

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The classical Wiener criterion asserts that on a bounded, open subset $\Omega \subseteq \mathbb{R}^n$ the Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega, \end{aligned} \tag{0.1}$$

admits for every $g \in C(\partial\Omega)$ a unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ if and only if Ω is regular in capacity, that is, for every boundary point $x \in \partial\Omega$ the series

$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}(\Omega \cap B(x, 2^{-k})$$

diverges [4, (2.37)]. Here, the capacity of a set $A \subseteq \mathbb{R}^n$ is defined by

$$\operatorname{cap}(A) := \inf\{\int_\Omega |\nabla u|^2 : u \in H^1 \cap C(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } A\}.$$

The capacity also appears in the study of regularity of Sobolev functions. For instance, every function $u \in H^1(\mathbb{R}^n)$ admits a representative which is continuous quasi everywhere, that is, up to an exceptional set of capacity 0.

In the literature, the notion of capacity has been generalised in the context of Dirichlet forms on L^2, and the aim of this project is to study some properties of these general capacities. For example, the Poincaré inequality, the logarithmic Sobolev inequality, the Nash inequality can be characterised by conditions involving the capacity, and the above mentioned regularity theorem for Sobolev functions can be generalized to a more general context. For example, the project can be based on [1, Chapter 8].

References

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