FORMS AND SPECTRAL THEORY: PROPERTIES OF EIGENVALUES AND EIGENVECTORS VIA FORMS AND SEMIGROUPS

JAMES KENNEDY (UNIVERSITY OF STUTTGART)

According to the spectral theorem, any positive, self-adjoint operator A with compact resolvent on a separable Hilbert space H is a diagonal operator, i.e., it has a sequence of eigenvalues $\lambda_n \geq 0$ and corresponding eigenvectors $e_n \in H$ which form an orthonormal basis of H (see Lecture 6).

Such an operator is also associated with a form a and λ_n and e_n can be characterised purely in terms of the behaviour of a on subsets of its form domain $D(a) =: V$ via the *min-max* and *max-min principles* for the eigenvalues and *Glazman's Lemma* (see [4, 5, 7] and cf. also Exercise 6.3). These formulae are the starting point for a large part of spectral analysis and geometry.

For example, they can be used to establish the *Weyl asymptotics* describing the eigenvalues of the Dirichlet or Neumann Laplacian on $L^2(\Omega)$ for a (bounded, sufficiently regular) domain $\Omega \subset \mathbb{R}^d$:

$$\lambda_n \sim c_d \left(|\Omega|^{-1} n\right)^{\frac{2}{d}} \qquad \text{as } n \to \infty,$$

where λ_n is the nth smallest eigenvalue (repeated according to multiplicities), $|\Omega|$ is the volume of Ω and c_d is a dimensional constant. (See [7]; an expository proof of this in two dimensions can also be found in [5, Ch. VI].)

There is another proof using ideas from semigroup theory. If an operator A (as above) acts on $H = L^2(\Omega)$ and we define

$$k_t(x,y) := \sum_{n=1}^{\infty} e^{-\lambda_n t} e_n(x)\overline{e_n(y)}, \qquad x, y \in \Omega$$

for all $t > 0$ (k_t is called a heat kernel), then according to *Mercer's Theorem*

$$e^{-tA} f(x) = \int_\Omega k_t(x,y) f(y)\,\mathrm{d}y$$

for all $f \in L^2(\Omega)$. It is not hard to prove the following formula for the *trace* of the semigroup

$$\mathrm{Tr}(e^{-tA}) := \sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_\Omega k_t(x,x)\,\mathrm{d}x.$$

By estimating k_t from above and below for $t \to 0$ and using *Karamata's Theorem*, which relates the behaviour of a positive Borel measure on $[0, \infty)$ near ∞ to that of its Laplace transform near 0, one can recover the Weyl asymptotics. (There is a nice presentation of this in a former Internet Seminar [1, Ch. 6].)

A consequence of the Weyl asymptotics is that one can determine the volume of Ω if one knows the entire sequence λ_n of (Dirichlet or Neumann) eigenvalues on Ω. This led

Mark Kac in the 1960s [6] to ask a now-famous question: does knowledge of the sequence (λ_n) allow one to determine Ω completely, at least up to isometries?

This question can be formulated precisely as follows: given two domains Ω_1, Ω_2 and Dirichlet Laplacians Δ_1, Δ_2, say, on Ω_1 and Ω_2, respectively, with normalised eigenfunctions e_n, f_n, we suppose there exists an operator $U : L^2(\Omega_1) \to L^2(\Omega_2)$ with $Ue_n = f_n$ for all $n \geq 1$. U is then unitary and *intertwines* the semigroups, $U^{-1}e^{-t\Delta_2}U = e^{-t\Delta_1}$ for all $t > 0$, with similar statements for the forms and operators. Does it then follow that Ω_1 and Ω_2 are congruent? Although the answer to this question is no in general, if one replaces "unitary intertwining operators" with intertwining operators with other properties, the answer becomes yes. See [2, 3].

In this project, we intend to explore examples such as (a subset of) the ones described above, which demonstrate the interplay between forms, semigroups and the behaviour of eigenvalues and eigenfunctions. Depending on the number and interests of participants, we could additionally or alternatively consider other consequences of the min-max principles such as *Courant's Theorem* for the number of nodal domains of the eigenfunctions (connected components of the set where a function is nonzero;see [5, Ch. VI] or [4]).

REFERENCES

[1] W. Arendt, *Heat Kernels*, 9th Internet Seminar on Evolution Equations, 2005-06.

[2] W. Arendt, *Does diffusion determine the body?* J. Reine Angew. Math. 550 (2002), 97–123.

[3] W. Arendt, A.F.M. ter Elst and J.B. Kennedy, *Analytical aspects of isospectral drums*, Oper. Matrices 8 (2014), 255–277.

[4] C. Bandle, *Isoperimetric Inequalities and applications*, Monographs and Studies in Mathematics Vol. 7, Pitman, Boston, 1980.

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience Publishers, New York, 1953.

[6] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly 102 (1966), 1–23.

[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 4: Analysis of Operators, Academic Press, New York, 1978.