Continuity of the Bogovskiĭ operator via Calderón-Zygmund theory

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In this project we want to give a proof of Theorem 13.8 of the Internet Seminar on continuity of the Bogovskiĭ operator $B : L_2^0(\Omega) \to H_0^1(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set that is star-shaped with respect to every point of an open ball and where $L_2^0(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f = 0\}$. Recall that the Bogovskiĭ operator B, originally defined on C^∞-functions with compact support, has the property $\operatorname{div} Bf = f$ for all $f \in L_2^0(\Omega)$. This tells us in particular that any $f \in L_2^0(\Omega)$ is the divergence of a vector field in $H_0^1(\Omega)$. We refer to Lecture 13 of the Internet Seminar for the application of the continuity result to the theory of the Stokes operator.

Continuity of the Bogovskiĭ operator is proved by an application of the Calderón-Zygmund theory of singular integral operators. The nice thing about the Bogovskiĭ operator is that its kernel is explicitly given. Calderón-Zygmund theory belongs to the real variable methods in harmonic analysis and gives sufficient conditions on the kernels of singular integral operators to induce bounded operators on L^q-spaces for $1 < q < \infty$. The basic example is the Hilbert transform $Hf(x) = \text{p.v.} - \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$ on the real line (p.v. means "principle value" and we see that already the definition of the operator needs some thought). Here we are interested in operators $\partial_j B$ for $q = 2$, but we need a result for kernels that are not of convolution type. We shall follow the proof in [2, Lem.III.3.1] using [2, Thm.II.11.4]. We can also take a look at the original paper [1] (depending on the interest of the participants).

References

[1] A.P. Calderón and A. Zygmund, On singular integrals, Amer. J. Math. 78, 289-309 (1956).

[2] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems*, Second edition, Springer, New York, 2011.