On convergence of Dirichlet-to-Neumann operators

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Outline

Sobolev space and trace

Let $\Omega \subset \mathbf{R}^d$ be open, bounded and with C^1-boundary $\partial\Omega$. Set

$$H^1(\Omega) = \{u \in L_2(\Omega) : \partial_j u \in L_2(\Omega) \text{ for all } j \in \{1, \ldots, d\}\}.$$

Give $\partial\Omega$ the surface measure.
There exists a unique continuous linear operator

$$\mathrm{Tr} : H^1(\Omega) \to L_2(\partial\Omega)$$

such that $\mathrm{Tr}\, u = u|_{\partial\Omega}$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$.
Let $H^1_0(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

Theorem
$H^1_0(\Omega) = \{u \in H^1(\Omega) : \mathrm{Tr}\, u = 0\}$. This is a closed subspace of $H^1(\Omega)$.

The normal derivative

Definition

Let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$.
We say that u has a normal derivative in $L_2(\partial\Omega)$ if there exists an $h \in L_2(\partial\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_{\partial\Omega} h \, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$.
Then write $\partial_\nu u = h$.
So

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_{\partial\Omega} (\partial_\nu u) \, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$ (Green).

The operator $-\Delta^D + m$

Let $m \in L_\infty(\Omega, \mathbf{R})$.
Define the operator $-\Delta^D + m$ in $L_2(\Omega)$ by

$$D(-\Delta^D + m) = \{u \in H_0^1(\Omega) : \Delta u \in L_2(\Omega)\}$$

and

$$(-\Delta^D + m)u = -\Delta u + m\, u.$$

Theorem
The operator $-\Delta^D + m$ is self-adjoint and bounded below.
Proof
Choose $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $j(u) = u$ and

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega m\, u\, \overline{v}.$$

The Dirichlet-to-Neumann operator D_m

Let $m \in L_\infty(\Omega, \mathbf{R})$.
Suppose $0 \notin \sigma(-\Delta^D + m)$. Define the graph D_m in $L_2(\partial\Omega) \times L_2(\partial\Omega)$ by

$$D_m = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega) : \text{there exists a } u \in H^1(\Omega) \text{ such that}$$
$$(-\Delta + m)u = 0 \text{ weakly}, \ \operatorname{Tr} u = g \text{ and } \partial_\nu u = h\}.$$

Theorem 8.14
D_m is a self-adjoint operator.
Proof
Choose $H = L_2(\partial\Omega)$, $V = H^1(\Omega)$, $j = \operatorname{Tr} \in \mathcal{L}(V, H)$ and

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega m \, u \, \overline{v}.$$

Hidden compactness The inclusion $V = H^1(\Omega) \to L_2(\Omega)$ is compact.
So $\mathfrak{a}$ is compactly elliptic.

Convergence of DtN operators

Theorem

Let $m, m_1, m_2, \ldots \in L_\infty(\Omega, \mathbf{R})$.
Suppose that $0 \notin \sigma(-\Delta^D + m_n)$ for all $n \in \mathbf{N}$ and $0 \notin \sigma(-\Delta^D + m)$.
Suppose $\lim\limits_{n\to\infty} m_n = m$ weak* in $L_\infty(\Omega)$. Then

$$\lim_{n\to\infty} \|(i\,s\,I + D_{m_n})^{-1} - (i\,s\,I + D_m)^{-1}\|_{\mathcal{L}(L_2(\partial\Omega))} = 0$$

for all $s \in \mathbf{R} \setminus \{0\}$.
Moreover,

$$\lim_{n\to\infty} \|e^{-tD_{m_n}} - e^{-tD_m}\|_{\mathcal{L}(L_2(\partial\Omega))} = 0$$

for all $t > 0$.

Proof

Proof For all $n \in \mathbf{N}$ define $\mathfrak{a}_n(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega m_n \, u \, \overline{v}$. Define similarly $\mathfrak{a}$.
Let $h \in L_2(\Gamma)$. Set $g_n = (i \, s \, I + D_{m_n})^{-1} h$ for all $n \in \mathbf{N}$. Let $n \in \mathbf{N}$. Then $D_{m_n} g_n = h - i \, s \, g_n$. There exists a $u_n \in H^1(\Omega)$ such that $\operatorname{Tr} u_n = g_n$ and

$$\mathfrak{a}_n(u_n, v) = (h - i \, s \, g_n, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$.

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for all $v \in H^1(\Omega)$.

Claim. The sequence $(u_n)_{n \in \mathbf{N}}$ is bounded in $H^1(\Omega)$.

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Subsequence: $(u_n)_{n \in \mathbf{N}}$ converges weakly in $H^1(\Omega)$ to u. Then $\lim u_n = u$ in $L_2(\Omega)$ and $\lim g_n = \mathrm{Tr} \, u =: g$ in $L_2(\Gamma)$.

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$$\mathfrak{a}_n(u_n, v) = (h - i\, s\, g_n, \operatorname{Tr} v)_{L_2(\Gamma)}$$

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Subsequence: $(u_n)_{n\in\mathbf{N}}$ converges weakly in $H^1(\Omega)$ to u. Then $\lim u_n = u$ in $L_2(\Omega)$ and $\lim g_n = \operatorname{Tr} u =: g$ in $L_2(\Gamma)$. Fix $v \in H^1(\Omega)$. Then $\lim_{n\to\infty} \mathfrak{a}_n(u_n, v) = \mathfrak{a}(u,v)$ and $\lim_{n\to\infty} (h - i\, s\, g_n, \operatorname{Tr} v)_{L_2(\Gamma)}$. So

$$\mathfrak{a}(u, v) = (h - i\, s\, g, \operatorname{Tr} v)_{L_2(\Gamma)}$$

and $g = (i\, s\, I + D_m)^{-1} h$. This proves strong resolvent convergence.
Uniform convergence requires an additional technical argument.

Aim

Include the situation that $0 \notin \sigma(-\Delta^D + m)$.

Self-adjoint operator

Let H be a (complex) Hilbert space.
An operator A on H is called self-adjoint if

- $(Ag, g)_H \in \mathbf{R}$ for all $g \in D(A)$ (symmetry) and
- $\forall_{s \in \mathbf{R} \setminus \{0\}} \forall_{h \in H} \exists_{g \in D(A)} \Big[(i\, s\, I + A) g = h \Big]$ (range condition).

This is equivalent with $D(A)$ is dense in H and $A^* = A$.

Note that the second condition reads

$$\forall_{s \in \mathbf{R} \setminus \{0\}} \forall_{h \in H} \exists_{g \in D(A)} \Big[Ag = h - i\, s\, g \Big].$$

Self-adjoint graph

Definition A self-adjoint graph is a subspace A of $H \times H$ such that

- $(g,h)_H \in \mathbf{R}$ for all $(g,h) \in A$ (symmetry) and
- $\forall_{s \in \mathbf{R} \setminus \{0\}} \forall_{h \in H} \exists_{g \in H} \Big[(g, h - i\, s\, g) \in A \Big]$ (range condition).

Conequences

The element g in the range condition is unique.
The operator $(i\, s\, I + A)^{-1} \colon h \mapsto g$ is bounded in H.

Define

$$A(0) = \{ y \in H : (0,y) \in A \} \quad \text{(degenerate space).}$$

Then there exists a unique self-adjoint operator A° in $A(0)^\perp$ such that

$$A = \{ (g, A^\circ g + h) : g \in D(A^\circ) \text{ and } h \in A(0) \}.$$

The operator A° is called the single-valued part of A.

Let $s \in \mathbf{R} \setminus \{0\}$. Then

$$(i\,s\,I + A)^{-1} \;=\; (i\,s\,I + A^\circ)^{-1} \;\oplus\; 0,$$

$$H \;=\; A(0)^\perp \;\oplus\; A(0).$$

Lower boundedness and semigroup

A self-adjoint graph A is called lower bounded if

$$\exists_{\omega \in \mathbf{R}} \forall_{(g,h) \in A} \Big[(g,h)_H \geq \omega \, \|g\|_H^2 \Big].$$

This is equivalent with the self-adjoint operator A° is lower bounded in $A(0)^\perp$.

Suppose the self-adjoint graph A is lower bounded. For all $t > 0$ define

$$e^{-tA} \quad = \quad e^{-tA^\circ} \quad \oplus \quad 0,$$

$$H \quad = \quad A(0)^\perp \quad \oplus \quad A(0).$$

The semigroup generated by $-A$ is $(e^{-tA})_{t>0}$.

The Fredholm-Lax–Milgram lemma

Let V be a Hilbert space and $\mathfrak{a} \colon V \times V \to \mathbf{C}$ a continuous sesquilinear form.

Definition The form $\mathfrak{a}$ is called compactly elliptic if there exist a Hilbert space $\widetilde{H}$, a compact $\tilde{j} \colon V \to \widetilde{H}$ and $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \|\tilde{j} u\|_{\widetilde{H}}^2 \geq \mu \, \|u\|_V^2 \quad \forall_{u \in V}.$$

Theorem Assume

- $\mathfrak{a}$ is compactly elliptic and
- $\mathfrak{a}(u,v) = 0$ for all $v \in V \quad \Rightarrow \quad u = 0$ (injectivity).

Then for all $f \in V'$ there exists a unique $u \in V$ such that

$$\mathfrak{a}(u,v) = f(v) \quad \forall_{v \in V}.$$

Proof There exists a unique $T_0 \in \mathcal{L}(V)$ such that

$$\mathfrak{a}(u,v) = (T_0 u, v)_V \qquad (u, v \in V).$$

Then T_0 is injective by assumption.
Define $T = T_0 + K$, where $K = \tilde{j}^* \tilde{j}$ is compact. Then

$$\mathrm{Re}(Tu, u)_V \geq \mu \|u\|_V^2 \text{ and } \|Tu\|_V \geq \mu \|u\|_V$$

for all $u \in V$.
Hence T is injective and T has closed range. Similarly T^* is injective.
Therefore T is invertible.
Since $T_0 = T(I - T^{-1} K)$ is injective, also $(I - T^{-1} K)$ is injective.
Moreover $T^{-1} K$ is compact, hence $(I - T^{-1} K)$ is invertible by the Fredholm alternative.
Thus T_0 is invertible and in particular surjective.

Self-adjoint graphs associated with forms

Let $\mathfrak{a}\colon V \times V \to \mathbf{C}$ be a symmetric continuous form.
Let $j \in \mathcal{L}(V, H)$.
Define the graph A in $H \times H$ by

$$A = \{(g, h) \in H \times H : \text{there exists a } u \in V \text{ such that}$$
$$j(u) = g \text{ and } \mathfrak{a}(u, v) = (h, j(v))_H \text{ for all } v \in V\}.$$

Then A is called the graph associated with (a, j) and we write $A \sim (a, j)$.

Theorem
If $\mathfrak{a}$ is compactly elliptic, then A is a self-adjoint graph and A is bounded below.

Space of non-uniqueness

Note:

$$(g, h) \in A \Leftrightarrow \exists_{u \in V} \Big[j(u) = g \text{ and } \forall_{v \in V} \big[\mathfrak{a}(u, v) = (h, j(v))_H \big] \Big].$$

The element u is not unique in general.

Definition

$$W(\mathfrak{a}) = \{ u \in \ker j : \mathfrak{a}(u, v) = 0 \text{ for all } v \in V \}$$

is called the space of non-uniqueness.
Always $\dim W(\mathfrak{a}) < \infty$ if $\mathfrak{a}$ is compactly elliptic.

Proof of self-adjointness of A if $W(\mathfrak{a}) = \{0\}$

Let $s \in \mathbf{R} \setminus \{0\}$ and $h \in H$.
Claim: There exists a $g \in H$ such that $(g, h - i\,s\,g) \in A$.
Define $\mathfrak{b} \colon V \times V \to \mathbf{C}$ by

$$\mathfrak{b}(u, v) = \mathfrak{a}(u, v) + i\,s\,(j(u), j(v))_H.$$

Then $\mathfrak{b}$ is continuous and compactly elliptic.
Let $u \in V$ and suppose that $\mathfrak{b}(u, v) = 0$ for all $v \in V$. Then

$$0 = \mathfrak{b}(u, u) = \mathfrak{a}(u, u) + i\,s\,\|j(u)\|_H^2.$$

So $j(u) = 0$. Hence $u \in W(\mathfrak{a}) = \{0\}$.

Define $f \in V'$ by $f(v) = (h, j(v))_H$.
The Fredholm-Lax–Milgram lemma gives: there exists a $u \in V$ such that

$$\mathfrak{b}(u, v) = f(v) = (h, j(v))_H$$

for all $v \in V$.
Then $(g, h - i\,s\,g) \in A$.

General case

Now suppose that $W(\mathfrak{a}) \neq \{0\}$. Define

$$V_1 = W(\mathfrak{a})^\perp,$$

$$\mathfrak{a}_1 = \mathfrak{a}|_{V_1 \times V_1} \text{ and}$$

$$j_1 = j|_{V_1}.$$

Then $W(\mathfrak{a}_1) = \{0\}$ and $\mathfrak{a}_1$ is compactly elliptic.
Let A_1 be the self-adjoint graph associated with $(\mathfrak{a}_1, j_1)$.
An easy calculation gives $A = A_1$.

The Dirichlet-to-Neumann graph D_m

Let $m \in L_\infty(\Omega, \mathbf{R})$.
Choose $H = L_2(\partial\Omega)$, $V = H^1(\Omega)$, $j = \mathrm{Tr} \in \mathcal{L}(V, H)$ and

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega m\, u\, \overline{v}.$$

Let D_m be the self-adjoint graph associated with $(\mathfrak{a}, j)$. Then

$$D_m = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega) : \text{there exists a } u \in H^1(\Omega) \text{ such that}$$
$$(-\Delta + m)u = 0 \text{ weakly, } \mathrm{Tr}\, u = g \text{ and } \partial_\nu u = h\}.$$

Space of non-uniqueness for DtN operator D_m

Note that

$$W(\mathfrak{a}) = \{u \in H_0^1(\Omega) : -\Delta u + m\, u = 0 \text{ and } \partial_\nu u = 0\}.$$

Theorem $W(\mathfrak{a}) = \{0\}$.

Proof Define $\tilde{u} \colon \mathbf{R}^d \to \mathbf{C}$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbf{R}^d \setminus \Omega. \end{cases}$$

Since $\partial_\nu u = 0$ it follows that

$$-\Delta \tilde{u} + \tilde{m}\, \tilde{u} = 0.$$

The **unique continuation property** implies that $\tilde{u} = 0$.

Graph of an operator

Theorem (Behrndt)
D_m is the graph of an operator in and only if

$$0 \notin \sigma(-\Delta^D + m).$$

Convergence of DtN **graphs**

Theorem

Let $m, m_1, m_2, \ldots \in L_\infty(\Omega, \mathbf{R})$.

~~Suppose that $0 \notin \sigma(-\Delta^D + m_n)$ for all $n \in \mathbf{N}$ and $0 \notin \sigma(-\Delta^D + m)$.~~

Suppose $\lim_{n\to\infty} m_n = m$ weak* in $L_\infty(\Omega)$. Then

$$\lim_{n\to\infty} \|(\lambda I + D_{m_n})^{-1} - (\lambda I + D_m)^{-1}\|_{\mathcal{L}(L_2(\partial\Omega))} = 0$$

for all $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

Weak convergence of forms

Let $\mathfrak{a}\colon V \times V \to \mathbf{C}$. Moreover, for all $n \in \mathbf{N}$ let $\mathfrak{a}_n\colon V \times V \to \mathbf{C}$ be symmetric, continuous, **uniformly** compactly elliptic. Assume

$\lim_{n\to\infty} u_n = u$ in $V \Rightarrow \lim_{n\to\infty} \mathfrak{a}_n(u_n, v) = \mathfrak{a}(u, v)$ for all $v \in V$. (weak converge

Let $j \in \mathcal{L}(V, H)$, let $A_n \sim (\mathfrak{a}_n, j)$ and $A \sim (\mathfrak{a}, j)$.

Theorem If $W(\mathfrak{a}) = \{0\}$, then

$$\lim_{n\to\infty} (i\, s\, I + A_n)^{-1} = (i\, s\, I + A)^{-1}$$

in the strong operator topology for all $s \in \mathbf{R} \setminus \{0\}$.
Moreover, if j is compact, then $\lim_{n\to\infty} (i\, s\, I + A_n)^{-1} = (i\, s\, I + A)^{-1}$ uniformly.
Remark It suffices that $\lim_{n\to\infty} \dim W(\mathfrak{a}_n) = \dim W(\mathfrak{a})$.
Always $\limsup_{n\to\infty} \dim W(\mathfrak{a}_n) \le \dim W(\mathfrak{a})$.

Symmetric operators in divergence form

For all $j, k \in \{1, \ldots, d\}$ let $a_{jk} \in L_\infty(\Omega, \mathbf{R})$ with $a_{jk} = a_{kj}$.
Suppose there exists a $\mu > 0$ such that

$$\sum_{j,k=1}^{d} a_{jk}(x)\, \xi_j\, \xi_k \geq \mu\, |\xi|^2$$

for all $\xi \in \mathbf{R}^d$ and a.e. $x \in \Omega$.
Let $m \in L_\infty(\Omega, \mathbf{R})$. Define the form $\mathfrak{a} \colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{C}$ by

$$\mathfrak{a}(u, v) = \sum_{j,k=1}^{d} \int_\Omega a_{jk}\, (\partial_j u)\, \overline{\partial_k v} + \int_\Omega m\, u\, \overline{v}.$$

Then the graph associated with $(\mathfrak{a}, \mathrm{Tr}\,)$ is a self-adjoint graph.

Space of non-uniqueness

Theorem If $d = 2$, or if $a_{jk} \in W^{1,\infty}(\Omega)$ for all $j, k \in \{1, \ldots, d\}$, then $W(\mathfrak{a}) = \{0\}$.

There is an example of Filonov with $W(\mathfrak{a}) \neq \{0\}$ and all a_{jk} Hölder continuous of order ν for all $\nu \in (0, 1)$.

Under weak conditions the previous resolvent convergence is still valid, as long as for the limit form one has $W(\mathfrak{a}) = \{0\}$.

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