Functional Calculus based on the Numerical Range

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The context

Let us consider a closed linear operator $A \in \mathcal{L}(D(A), H)$ on a Hilbert space H. We assume $D(A) \subset H$, with dense imbedding, and the spectrum satisfies $\sigma(A) \neq \mathbb{C}$.

Example 1. A square matrix. $H = D(A) = \mathbb{C}^d$, $A \in \mathbb{C}^{d,d}$.

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Example 2. Dirichlet Laplace operator.

$$H = L^2(\Omega),\ V = H_0^1(\Omega),$$

$$a(u,v) = \int_\Omega \nabla u . \nabla \bar{v}\, dx,$$

$$\begin{aligned} D(A) &= \{u \in V\, ;\, \exists Au \in H, (Au, v)_H = a(u,v), \forall v \in V\} \\ &= H^2(\Omega) \cap H_0^1(\Omega), \end{aligned}$$

$$Au = -\Delta u.$$

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Let us consider a closed linear operator $A \in \mathcal{L}(D(A), H)$ on a Hilbert space H. We assume $D(A) \subset H$, with dense imbedding, and the spectrum satisfies $\sigma(A) \neq \mathbb{C}$.

Example 3. Robin Laplace operator.

$$H = L^2(\Omega),\ V = H^1(\Omega),\ a \in C(\partial\Omega; \mathbb{R}).$$

$$a(u,v) = \int_\Omega \nabla u.\nabla \bar{v}\, dx + \int_{\partial\Omega} a\, u\, \bar{v}\, d\sigma,$$

$$\begin{aligned} D(A) &= \{u \in V\,; \exists Au \in H, (Au,v)_H = a(u,v), \forall v \in V\} \\ &= \{u \in H^2(\Omega)\,; \partial_n u + au = 0,\ \text{on } \partial\Omega\}, \end{aligned}$$

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Example 4. Complex Robin-Laplace operator.

$$H = L^2(\Omega),\ V = H^1(\Omega),\ a, b \in C(\partial\Omega; \mathbb{R}).$$

$$a(u,v) = \int_\Omega \nabla u . \nabla \bar{v}\, dx + \int_{\partial\Omega} (a{+}ib)\, u\, \bar{v}\, d\sigma,$$

$$\begin{aligned} D(A) &= \{u \in V\, ; \exists Au \in H, (Au, v)_H = a(u,v), \forall v \in V\} \\ &= \{u \in H^2(\Omega)\, ; \partial_n u + (a{+}ib)u = 0,\ \text{on } \partial\Omega\}, \end{aligned}$$

$$Au = -\Delta u.$$

The self-adjoint case

Let us consider a closed linear operator $A \in \mathcal{L}(D(A), H)$ on a Hilbert space H. If A is a self-adjoint operator, spectral theory is very efficient. We know that the spectrum $\sigma(A)$ is real and we have

$$\|r(A)\| = \sup_{x \in \sigma(A)} |r(x)|,$$

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if the rational function r is bounded on $\sigma(A)$.

By a density argument, this (in)equality allows to define $f(A) \in \mathcal{L}(H)$, for any function f that is continuous and bounded on $\sigma(A)$. The previous inequality also holds with f instead of r.

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$$\|f(A)\| = \sup_{x \in \sigma(A)} |f(x)|,$$

for all continuous functions f that are bounded on $\sigma(A)$.

Furthermore the map $f \mapsto f(A)$, is a homomorphism from the algebra $C_b(\sigma(A))$ into the algebra $\mathcal{L}(H)$.

Example of application

Assume that we have a self-adjoint positive definite operator $A \in \mathcal{L}(D(A), H)$. We can define $C(t) = \cos(t\sqrt{A})$, and we will show that

$$u(t) = C(t)\, u_0, \qquad \text{with } u_0 \in D(A),$$

is the unique solution in $C^2(\mathbb{R}, H) \cap C^0(\mathbb{R}, D(A))$ of

$$\begin{cases} u''(t) + A\, u(t) = 0, \quad t \in \mathbb{R}, \\ u(0) = u_0, \quad u'(0) = 0. \end{cases}$$

Continuity proof. Hyp : $u_0 \in D(A)$, $u(t) = C(t)u_0$, $A \geq \varepsilon > 0$.

With $f_1(z) = \cos(t\sqrt{z})$, $f_2(z) = (\cos((t+h)\sqrt{z}) - \cos((t-h)\sqrt{z}))/z$, we have, for $z > 0$, $|f_1(z)| \leq 1$ and $|f_2(z)| \leq 2\,|ht|$.
This allows to set $C(t) = f_1(A)$, and $B(t,h) = f_2(A)$, and we have

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$$B(t,h) = (C(t+h) - C(t-h))A^{-1},$$

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$B(t,h)Au_0 = u(t{+}h){-}u(t{-}h)$, and $\|u(t{+}h){-}u(t{-}h)\| \leq 2|ht|\,\|Au_0\|$.

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$$B(t,h)Au_0 = u(t+h) - u(t-h), \quad \text{and} \quad \|u(t+h) - u(t-h)\| \leq 2|ht|\,\|Au_0\|.$$

This shows that $u \in C(\mathbb{R}; H)$, if $u_0 \in D(A)$, but also, by density, if $u_0 \in H$.

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$$B(t,h)Au_0 = u(t+h) - u(t-h), \quad \text{and} \quad \|u(t+h) - u(t-h)\| \leq 2|ht|\,\|Au_0\|.$$

A similar proof gives that $u \in C(\mathbb{R}; D(A))$, if $u_0 \in D(A)$.

Differentiability proof. Hyp : $u_0 \in D(A)$ and $u(t) = C(t)u_0$.

We have

$$\cos(t\sqrt{z})/z = 1/z - \int_0^t (t-s)\cos(s\sqrt{z})\,ds.$$

Hence (homomorphism of algebra)

$$C(t)\,A^{-1} = A^{-1} - \int_0^t (t-s)C(s)\,ds.$$

and

$$u(t) = u_0 - \int_0^t (t-s)Au(s)\,ds.$$

Differentiability proof. Hyp : $u_0 \in D(A)$ and $u(t) = C(t)u_0$.

We have

$$u(t) = u_0 - \int_0^t (t-s) Au(s)\, ds.$$

It follows

$$u(0) = u_0, \quad u'(t) = -\int_0^t Au(s)\, ds, \quad u'(0) = 0,$$

$$u''(t) = -Au(t), \quad \text{and} \quad u \in C^2(\mathbb{R}; H).$$

The numerical range.

Recall that the numerical range of a matrix $A \in \mathbb{C}^{d,d}$ is defined by

$$W(A) = \{(Av, v)_H = v^*Av\,; v \in \mathbb{C}^d,\ \|v\| = 1\}.$$

It is a closed convex subset of $\mathbb{C}$, (Toeplitz & Hausdorff) that contains the spectrum $\sigma(A)$.

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Let us consider a bounded convex domain Ω. We assume that $W(A) \subset \Omega$. If ν denotes the unit outward normal at the point $\sigma \in \partial\Omega$, then

$$z \in \Omega \quad \text{is equivalent to} \quad \mathrm{Re}\,\frac{\nu}{\sigma - z} > 0,\ \forall \sigma \in \partial\Omega.$$

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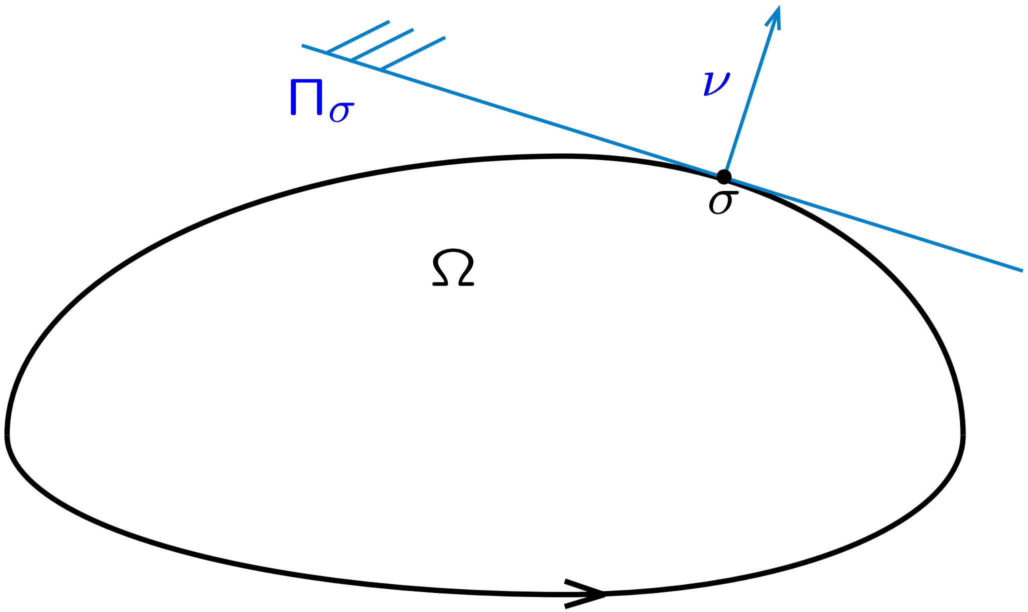

$$d\sigma = i\,\nu\,ds$$

$z \in \Pi_\sigma$ is equivalent to $\mathrm{Re}\,\dfrac{\nu}{\sigma - z} > 0$.

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Therefore

$$W(A) \subset \Omega \ \text{ is equivalent to } \ \nu(\sigma - A)^{-1} + \bar{\nu}(\bar{\sigma} - A^*)^{-1} > 0, \forall \sigma \in \partial\Omega.$$

The numerical range.

The numerical range of an operator $A \in \mathcal{L}(D(A), H)$ is defined by

$$W(A) = \{(Av, v)_H = a(v, v)\,; v \in D(A),\ \|v\|_H = 1\}.$$

It is a convex subset of $\mathbb{C}$, its closure contains the spectrum $\sigma(A)$.

If Ω is a convex domain, then

$$\overline{W(A)} \subset \Omega \quad \text{implies} \quad \nu(\sigma - A)^{-1} + \bar{\nu}(\bar{\sigma} - A^*)^{-1} \geq 0, \forall \sigma \in \partial\Omega.$$

Example : Complex Robin-Laplace operator.

$$H = L^2(\Omega),\ V = H^1(\Omega),\ a \geq 0, b \in C(\partial\Omega;\mathbb{R}).$$

$$a(u,v) = \int_\Omega \nabla u.\nabla \bar{v}\, dx + \int_{\partial\Omega} (a{+}ib)\, u\, \bar{v}\, d\sigma,$$

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$$|\operatorname{Im} a(v,v)| \leq \|b\|_\infty \|v\|^2_{L^2(\partial\Omega)} \leq N\|v\|_H\|v\|_V.$$

Theorem 7.9

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Hence, if $\|v\|_H = 1$, then

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Therefore $W(A) \subset \mathcal{P}$, (the numerical range is contained in the horizontal parabola $\mathcal{P}$).

Numerical Range and functional calculus

We will use now the following theorem

Theorem. If Ω is a closed convex subset of the complex plane with a conic boundary and if A is a closed operator with $W(A) \subset \Omega$, then, for all rational functions r bounded in Ω,

$$\|r(A)\| \leq 3.2 \sup_{z \in \Omega} |r(z)|.$$

Numerical Range and functional calculus

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Remark. Then $f(A)$ is well defined for all functions that are holomorphic in the interior of Ω, bounded and continuous in Ω. Furthermore

$$\|f(A)\| \leq 3.2 \sup_{z \in \Omega} |f(z)|.$$

Corollary. If there exist a Hilbert space $V \subset H$, V dense in H, and constants $\alpha > 0$, $\lambda, M, N \in \mathbb{R}$, such that

$$\alpha \|v\|_V^2 \leq \mathrm{Re}(Av, v)_H + \lambda \|v\|_H^2 \leq M \, \|v\|_V^2$$
$$|\,\mathrm{Im}(Av, v)_H| \leq N \, \|v\|_H \, \|v\|_V$$

then the problem

$$\begin{cases} u''(t) + A\,u(t) = 0, & t \in \mathbb{R}, \\ u(0) = u_0 \in D(A), & u'(0) = v_0 \in D(A^{1/2}). \end{cases}$$

has a unique solution in $C^2(\mathbb{R}, H) \cap C^0(\mathbb{R}, D(A))$.

Proof.

Numerical Range and functional calculus Application

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Proof. Similarly as for the Robin case, it holds

$$W(A) \subset \mathcal{P} := \{x{+}iy \,;\, x{+}\lambda \ge \frac{\alpha}{N^2} y^2\}.$$

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Exercise : $z \in \mathcal{P}$ implies $|\operatorname{Im} z^{1/2}| \leq \omega$, with $\omega = \max(\sqrt{\lambda_+}, N/2\sqrt{\alpha})$.

Thus $\cos(t\sqrt{z})$ is holomorphic in $\mathcal{P}$ with bound $|\cos(t\sqrt{z})| \leq e^{\omega|t|}$....

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Thus $\cos(t\sqrt{z})$ is holomorphic in $\mathcal{P}$ with bound $|\cos(t\sqrt{z})| \leq e^{\omega|t|}$.
We can define $C(t) = \cos(t\sqrt{A})$ and

$$\|C(t)u_0\| \leq 3.2e^{\omega t}\|u_0\|, \qquad |C(t+h)u_0 - C(t-h)u_0\| \leq 3.2|ht|e^{\omega t}\|Au_0\|.$$

The same proof as in the self-adjoint context works.

Numerical Range and functional calculus

Theorem. If Ω is a closed convex subset of the complex plane with a conic boundary and if A is a closed operator with $W(A) \subset \Omega$, then Ω is a 3.2-spectral set for the operator A.

Proof. In the parabolic case $\Omega = \mathcal{P} = \{x+iy\,;x \geq y^2\}$, and if A is a bounded operator on H. (The unbounded case follows by replacing A by $A_\varepsilon = A(I+\varepsilon A)^{-1}$, $\varepsilon > 0$, and taking the limit as $\varepsilon \to 0$.)

We will need three lemmata.

Lemma 1. Assume that

$dm(t)$ is a complex-valued measure, bounded on E,

$M(t) \in \mathcal{L}(H), \quad M(t) = M^*(t) \geq 0,$ in E,

r is a rational function bounded by 1 on E.

Then

$$\left\| \int_E r(t) M(t)\, dm(t) \right\| \leq \left\| \int_E M(t)\, |dm(t)| \right\|.$$

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Key of the proof. We have

$$|(M(t)u, v)_H| \leq (M(t)u, u)_H^{1/2} (M(t)v, v)_H^{1/2}$$

Lemma 2. Assume that

$dm(t)$ is a complex-valued measure, bounded on E,

$M(t), N(t) \in \mathcal{L}(H), \quad N(t) = N^*(t), \quad$ in $E, \quad \alpha > 0$

$\text{Re}\, M(t) = \frac{1}{2}(M(t) + M(t)^*) \geq N(t) \geq \alpha, \quad$ in E,

r is a rational function bounded by 1 on E.

Then

$$\left\| \int_E r(t)(M(t))^{-1}\, dm(t) \right\| \leq \left\| \int_E (N(t))^{-1}\, |dm(t)| \right\|.$$

Lemma 2. Assume that r is bounded by 1, $\alpha > 0$,

$$\operatorname{Re} M(t) = \tfrac{1}{2}(M(t) + M(t)^*) \geq N(t) \geq \alpha, \quad \text{in } E.$$

Then

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Key of the proof.

$$|(M(t)^{-1}u, v)_H| \leq (N(t)^{-1}u, u)_H^{1/2} (N(t)^{-1}v, v)_H^{1/2}.$$

Lemma 3. Assume that

$$M \in \mathcal{L}(H), \qquad \overline{W(M)} \subset S_\theta := \{z \in \mathbb{C}, z \neq 0\,;\, |\arg(z)| < \theta\}.$$

Then

$$\mathrm{Re}(M^{-1}) \geq \cos^2\theta\,(\mathrm{Re}\,M)^{-1}.$$

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Proof. We write $M = B(I+iC)B$, with $B = (\mathrm{Re}\, M)^{1/2}$.

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Therefore

$$\mathrm{Re}(I+iC)^{-1} \geq \inf_{y\in\sigma(C)} \mathrm{Re}\,\frac{1}{1+iy} \geq \frac{1}{1+\tan^2\theta} = \cos^2\theta,$$

whence $\mathrm{Re}(M^{-1}) \geq B^{-1}\cos^2\theta\, B^{-1} = \cos^2\theta\,(\mathrm{Re}\,M)^{-1}.$

Theorem. If Ω is a closed convex subset of the complex plane with a conic boundary and if A is a closed operator with $W(A) \subset \Omega$, then Ω is a 3.2-spectral set for the operator A.

Proof. We assume $|r(z)| \leq 1$ for $z \in \mathcal{P} = \{x+iy\,;\, x \geq y^2\}$, and we want to show that $\|r(A)\| \leq 3.2$.

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$$R_1 = \int_{\partial\mathcal{P}} r(\sigma)\,\frac{1}{2\pi i}\Big((\sigma - A)^{-1}d\sigma - (\bar{\sigma} - A^*)^{-1}d\bar{\sigma}\Big),$$

$$R_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)\,(\bar{\sigma} - A^*)^{-1}d\bar{\sigma},$$

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$$R_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)\,(\bar{\sigma}-A^*)^{-1}d\bar{\sigma},$$

and note that, from the Cauchy formula,

$$r(A) = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)\,(\sigma-A)^{-1}d\sigma = R_1 + R_2.$$

Theorem. If Ω is a closed convex subset of the complex plane with a conic boundary and if A is a closed operator with $W(A) \subset \Omega$, then Ω is a 3.2-spectral set for the operator A.

Proof. We assume $|r(z)| \leq 1$ for $z \in \mathcal{P} = \{x+iy\,; x \geq y^2\}$, and we want to show that $\|r(A)\| \leq 3.2$. We set

$$R_1 = \int_{\partial\mathcal{P}} r(\sigma)\,\frac{1}{2\pi i}\Big((\sigma-A)^{-1}d\sigma - (\bar{\sigma}-A^*)^{-1}d\bar{\sigma}\Big),$$

$$R_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)\,(\bar{\sigma}-A^*)^{-1}d\bar{\sigma},$$

and note that, from the Cauchy formula,

$$r(A) = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)\,(\sigma-A)^{-1}d\sigma = R_1 + R_2.$$

Therefore, it suffices to show that $\|R_1\| \leq 2$ and $\|R_2\| \leq 1.2$.

Proof of $\|R_1\| \leq 2$.

We have

$$R_1 = \int_{\partial\mathcal{P}} r(\sigma)\, \frac{1}{2\pi i}\Big((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}\Big),$$

and we have seen that $W(A) \subset \mathcal{P}$ implies

$$\frac{1}{2\pi i}\Big((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}\Big) \geq 0.$$

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Thus it follows from Lemma 1 that

$$\|R_1\| \leq \left\| \int_{\partial\mathcal{P}} \frac{1}{2\pi i}\Big((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma}\Big) \right\| = 2.$$

Indeed $\displaystyle\int_{\partial\mathcal{P}} \frac{1}{2\pi i}(\sigma - A)^{-1} d\sigma = Id.$

It remains to show

$$\|R_2\| \leq 1.2 \quad \text{with} \quad R_2 := \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)(\bar{\sigma}-A^*)^{-1}d\bar{\sigma}.$$

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On the boundary $\partial\mathcal{P}$, it holds $y^2 = x$, i.e. $\bar{\sigma} = \sigma-1+\sqrt{1-4\sigma}$.
(We choose the square root which takes the value 1 if $\sigma = 0$ and is analytic in $\mathbb{C}\setminus\Gamma$, with $\Gamma := \{x\in\mathbb{R}\,;\,x\geq 1/4\}$).

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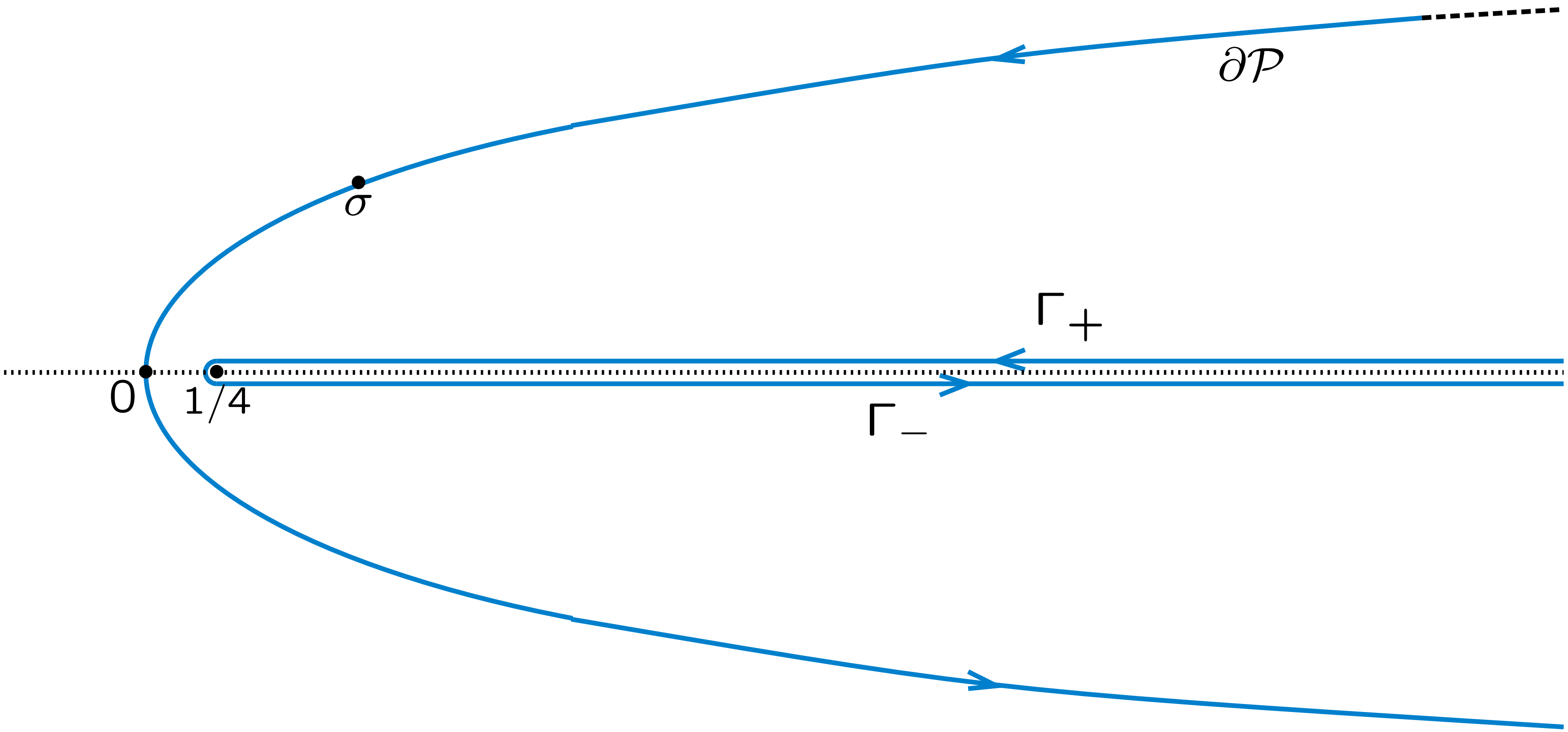
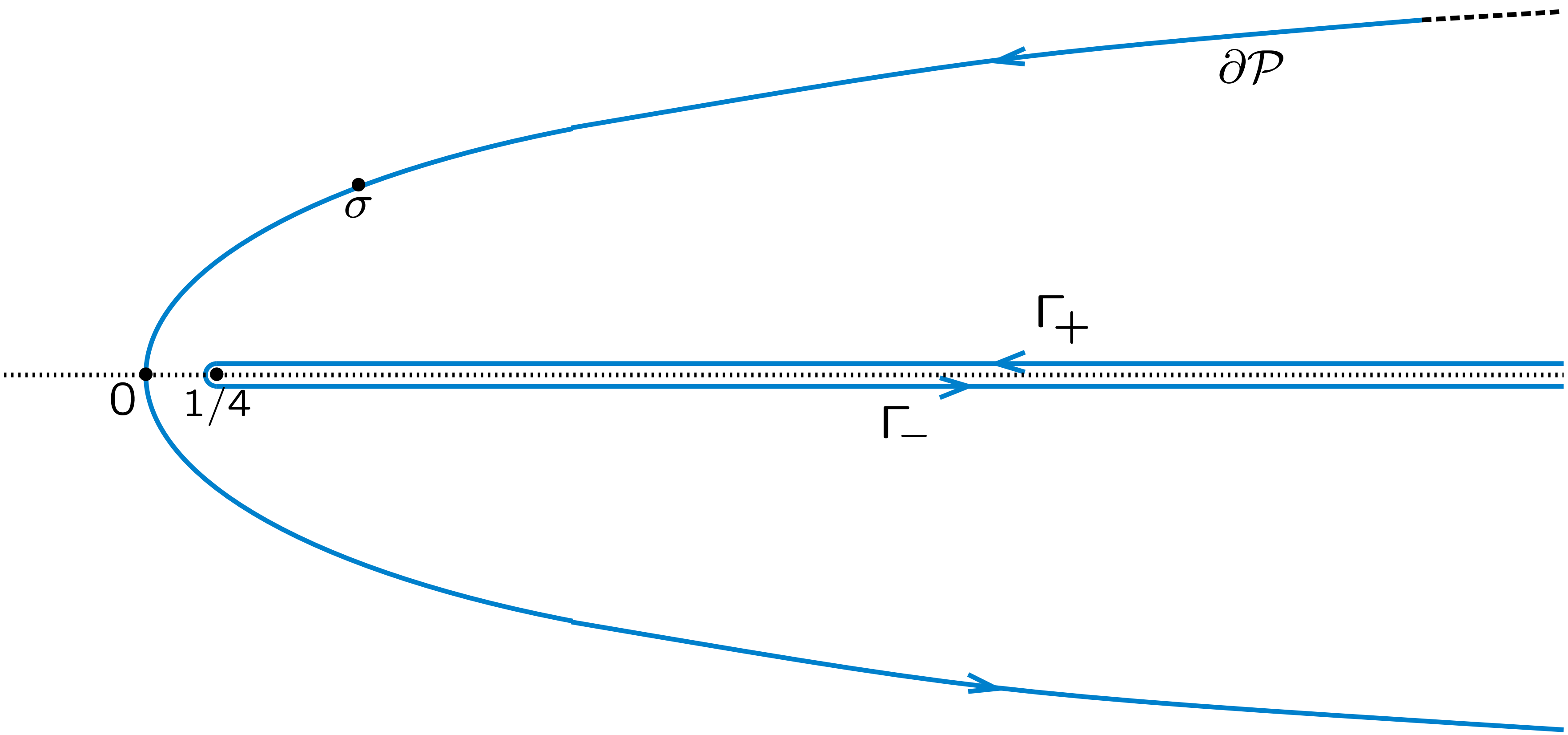
$$R_2 = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)(\sigma-1+\sqrt{1-4\sigma}-A^*)^{-1}\Big(1-\frac{2}{\sqrt{1-4\sigma}}\Big)d\sigma,$$

The integrand is analytic w.r.t. $\sigma \in \mathcal{P}\setminus\Gamma$.



We can replace $\int_{\partial\mathcal{P}}\cdots$ by $\int_{\Gamma_+\cup\Gamma_-}\cdots.$



On Γ_+, $\sqrt{1-4x} = -i\sqrt{4x-1}$, on Γ_-, $\sqrt{1-4x} = i\sqrt{4x-1}$.

$$
\begin{aligned}
R_2 &= \frac{1}{2\pi i}\int_{\partial\mathcal{P}} r(\sigma)(\sigma-1+\sqrt{1-4\sigma}-A^*)^{-1}\Big(1-\frac{2}{\sqrt{1-4\sigma}}\Big)d\sigma\\
&= \frac{1}{2\pi i}\int_{\Gamma_+} r(x)(x-1-i\sqrt{4x-1}-A^*)^{-1}\Big(1-\frac{2\,i}{\sqrt{4x-1}}\Big)dx\\
&\quad+\frac{1}{2\pi i}\int_{\Gamma_-} r(x)(x-1+i\sqrt{4x-1}-A^*)^{-1}\Big(1+\frac{2\,i}{\sqrt{4x-1}}\Big)dx,
\end{aligned}
$$

By setting $x = y^2 + 1/4$, we get

$$
\begin{aligned}
R_2 = -\frac{1}{\pi i}\int_0^\infty r(y^2+\tfrac{1}{4})(y^2-\tfrac{3}{4}-A^*-2iy)^{-1}(y-i)\,dy\\
+\frac{1}{\pi i}\int_0^\infty r(y^2+\tfrac{1}{4})(y^2-\tfrac{3}{4}-A^*+2iy)^{-1}(y+i)\,dy
\end{aligned}
$$

This can also be written

$$R_2 = -\frac{2}{\pi}\int_0^\infty r(y^2+\tfrac{1}{4})(M(y,A^*))^{-1}dy, \qquad \text{with}$$

$$M(y,A^*) = A^*-3y^2+\tfrac{3}{4} + 4y^2(1+y^2)(A^*+y^2+\tfrac{3}{4})^{-1}.$$

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We write $A = B+iC$, and we admit for a time

$$W(A) \subset \mathcal{P} \implies \mathrm{Re}\Big((A^*+y^2+\tfrac{3}{4})^{-1}\Big) \geq \frac{4y^2+3}{4(y^2+1)}(B+y^2+\tfrac{3}{4})^{-1}.$$

This result implies that

$$\mathrm{Re}\, M(y,A^*) \geq N(y,B), \qquad \text{with}$$
$$N(y,B) := B-3y^2+\tfrac{3}{4} + y^2(4y^2+3)(B+y^2+\tfrac{3}{4})^{-1}.$$

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This result implies that

$$\begin{aligned} \mathrm{Re}\, M(y,A^*) &\geq N(y,B), \qquad \text{with} \\ N(y,B) &:= B-3y^2+\tfrac{3}{4} + y^2(4y^2+3)(B+y^2+\tfrac{3}{4})^{-1} \\ &\geq \tfrac{3}{4}. \end{aligned}$$

Now, we can use Lemma 2.

We have

$$R_2 = -\frac{2}{\pi}\int_0^\infty r(y^2+\tfrac{1}{4})(M(y,A^*))^{-1}dy,$$

and

$$\operatorname{Re} M(y,A^*) \geq N(y,B), \qquad \text{with}$$

$$N(y,B) := B-3y^2+\tfrac{3}{4} + y^2(4y^2+3)(B+y^2+\tfrac{3}{4})^{-1} > 0$$

Lemma 2 gives,

$$\|R_2\| \leq \frac{2}{\pi}\left\| \int_0^\infty (N(y,B))^{-1}dy \right\|$$

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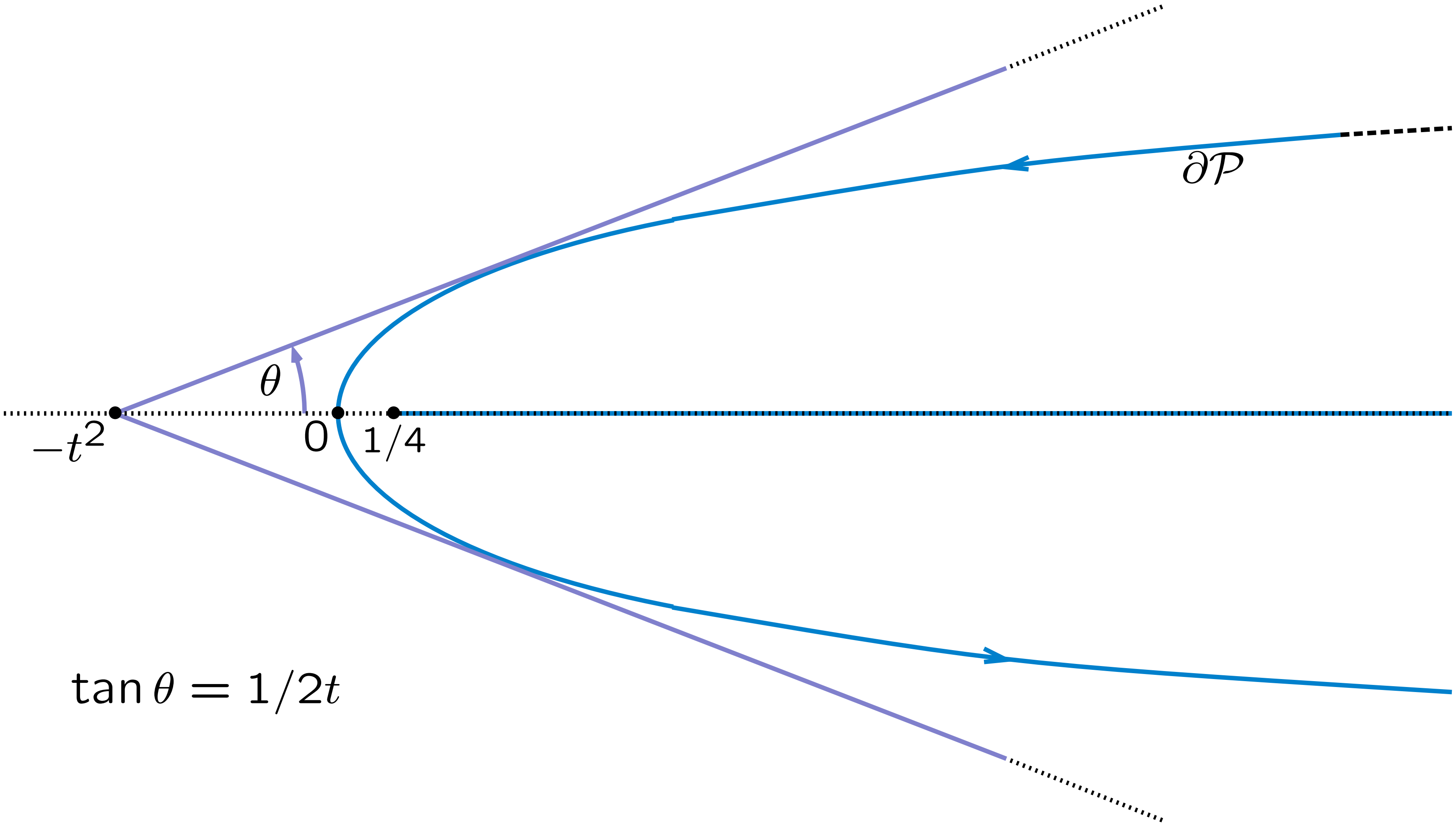
$$N(y,B) := B-3y^2+\tfrac{3}{4} + y^2(4y^2+3)(B+y^2+\tfrac{3}{4})^{-1} > 0.$$

Lemma 2 gives,

$$\begin{aligned}\|R_2\| &\leq \frac{2}{\pi}\left\|\int_0^\infty (N(y,B))^{-1}dy\right\| \\ &= \frac{2}{\pi}\sup_{x\in\sigma(B)}\left|\int_0^\infty (N(y,x)^{-1}dy\right| = \frac{2}{\sqrt{3}}.\end{aligned}$$

It remains to show that

$$W(A) \subset \mathcal{P} \implies \mathrm{Re}\Big((A^*+y^2+\tfrac{3}{4})^{-1}\Big) \geq \frac{4y^2+3}{4(y^2+1)}(B+y^2+\tfrac{3}{4})^{-1}.$$



$\tan\theta = 1/2t$

Remark. $W(A) \subset \mathcal{P} \implies W(A+t^2) \subset S_\theta := \{z\,;\,|\arg z| \le \theta\}$.

It remains to show that

$$W(A) \subset \mathcal{P} \implies \mathrm{Re}\Big((A^*+y^2+\tfrac{3}{4})^{-1}\Big) \geq \frac{4y^2+3}{4(y^2+1)}(B+y^2+\tfrac{3}{4})^{-1}.$$

Setting $t^2 = y^2+\frac{3}{4}$, $M = A^*+t^2$, $\theta = \arctan(1/2t)$,

The result follows from Lemma 3, which reads

$$W(M) \subset S_\theta \implies \mathrm{Re}(M^{-1}) \geq \cos^2\theta\,(\mathrm{Re}\,M)^{-1}.$$

A more general result is :

For all operators A on a Hilbert space and for all rational functions bounded on the numerical range

$$\|r(A)\| \leq 11.08 \sup_{z \in W(A)} |r(z)|.$$

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