Lecture 14

Non-autonomous equations

So far we have considered forms that do not depend on time. The aim of this lecture is to prove an elegant theorem of Lions that establishes well-posedness of the non-autonomous equation

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \qquad u(0) = u_0$$

where each operator $\mathcal{A}(t)$ is associated with a form $a(t, \cdot, \cdot)$. In contrast to previous results, the existence of solutions will be investigated in a larger space V^*. In fact, the situation is slightly more delicate and interesting: the solution will take its values in V, but the equation will be solved only in V^*.

We start by introducing the setup for V^* in the first section of the lecture. The next sections are devoted to the Bochner integral for Hilbert space valued functions and to Hilbert space valued Sobolev spaces. Of special interest are the maximal regularity spaces – as we call them –, certain mixed Sobolev spaces which have interesting properties. Lions' representation theorem extends the Lax-Milgram lemma and yields an elegant proof of the final results.

14.1 Gelfand triples

Let V, H be Hilbert spaces over $\mathbb{K}$ such that $V \overset{\mathrm{d}}{\hookrightarrow} H$. We have frequently encountered this situation in previous lectures; typical examples are $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ or $H^1(\Omega) \hookrightarrow L_2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open. In this section we want to add the antidual V^* to our considerations. For $\eta \in V^*$, $u \in V$ we let

$$\langle \eta, u \rangle := \eta(u).$$

Note that V^* is a Hilbert space (being isomorphic to V). Given $f \in H$ we define $\eta_f \in V^*$ by

$$\langle \eta_f, u \rangle := (f \,|\, u)_H \qquad (u \in V).$$

It is not difficult to see that the mapping $H \ni f \mapsto \eta_f \in V^*$ is linear, injective, continuous and has dense range. We will identify η_f and f and thus consider H as a subspace of V^*. This yields the following consistency identity:

$$\langle f, u \rangle = (f \,|\, u)_H \qquad (f \in H,\ u \in V).$$

Thus we now have a triple of injected spaces

$$V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*,$$

called a **Gelfand triple**. We have encountered this situation already in Lecture 13 – see Remark 13.1 – and illustrate it with another example.

14.1 Example. Let (Ω, μ) be a measure space and let $m\colon \Omega \to [\delta, \infty)$ be measurable, where $\delta > 0$. Let

$$H := L_2(\Omega, \mu), \quad V := L_2(\Omega, m\mu) = \big\{u\colon \Omega \to \mathbb{K} \text{ measurable}; \int |u|^2 m \,\mathrm{d}\mu < \infty\big\}.$$

Then $V \overset{\mathrm{d}}{\hookrightarrow} H$. We can identify V^* with $L_2(\Omega, \frac{1}{m}\mu)$, where the duality is given by

$$\langle f, u\rangle = \int_\Omega f(x)\overline{u(x)}\,\mathrm{d}\mu(x) \qquad \big(f \in L_2(\Omega, \tfrac{1}{m}\mu),\ u \in V\big).$$

One can easily see this by using that $\Phi f := mf$ defines an isometric isomorphism $\Phi\colon L_2(\Omega, m\mu) \to L_2(\Omega, \frac{1}{m}\mu)$.

14.2 Remark. Example 14.1 is generic. Whenever $V \overset{\mathrm{d}}{\hookrightarrow} H$, there exist a measure space (Ω, μ), a unitary operator $U\colon H \to L_2(\Omega, \mu)$ and a measurable function $m\colon \Omega \to [\delta, \infty)$ (with $\delta > 0$) such that $U|_V$ is a unitary mapping from V onto $L_2(\Omega, m\mu)$. This can be proved with the help of the spectral theorem, applied to the accretive self-adjoint operator A in H associated with the form $a\colon V \times V \to \mathbb{K}$, $a(u, v) := (u \,|\, v)_V$; see also Exercise 14.1.

Now let $a\colon V \times V \to \mathbb{K}$ be a continuous form. Denote by $\mathcal{A} \in \mathcal{L}(V, V^*)$ the operator given by $\langle \mathcal{A}u, v\rangle = a(u, v)$. Let A be the operator in H associated with a. It is easy to see that A is the part of $\mathcal{A}$ in H, i.e.,

$$A = \mathcal{A} \cap (V \times H),$$

where we identify $V \hookrightarrow H \hookrightarrow V^*$ as before. We will illustrate this situation below in Example 14.3. If a is H-elliptic, then we know that $-A$ generates a holomorphic C_0-semigroup on H. One can show that $-\mathcal{A}$ generates a holomorphic C_0-semigroup on V^*; we will not pursue this here. However, our final aim will be to study the inhomogeneous Cauchy problem

$$\begin{aligned} u' + \mathcal{A}u &= f, \\ u(0) &= u_0 \end{aligned}$$

when $\mathcal{A}$ *depends on time*; see Section 14.5.

14.3 Example. Consider the situation described in Example 14.1. Let $a\colon V \times V \to \mathbb{K}$ be given by

$$a(u, v) = \int_\Omega mu\bar{v}\,\mathrm{d}\mu.$$

Then a is continuous and coercive. The operator $\mathcal{A}$ in V^* is given by

$$\mathcal{A}u = mu \text{ with } \operatorname{dom}(\mathcal{A}) = V,$$

and the operator A in $L_2(\Omega, \mu)$ is given by

$$\begin{aligned} \operatorname{dom}(A) &= \big\{u \in L_2(\Omega, \mu);\, mu \in L_2(\Omega, \mu)\big\}, \\ Au &= mu \qquad (u \in \operatorname{dom}(A)). \end{aligned}$$

14.2 Interlude: The Bochner integral for Hilbert space valued functions

There is a general theory extending the Lebesgue integral to Banach space valued functions (the Bochner integral, see [ABHN11; Section 11]). On separable Hilbert spaces one may use a more elementary approach, which we will give here (cf. [AU10; Section 8.5]).

Let H be a separable Hilbert space, and let $-\infty \leqslant a < b \leqslant \infty$. A function $f\colon (a,b) \to H$ is called **measurable** if $(f(\cdot)\,|\,v)$ is measurable for all $v \in H$. If (v_n) is a dense sequence in $B_H(0,1)$ then $\|f(t)\| = \sup_{n\in\mathbb{N}} |(f(t)\,|\,v_n)|$ for all $t \in (a,b)$, and this implies that $\|f(\cdot)\|\colon (a,b) \to \mathbb{R}$ is measurable. We let

$$L_1(a,b;H) := \Big\{ f\colon (a,b) \to H \text{ measurable};\ \int_a^b \|f(t)\|\,\mathrm{d}t < \infty \Big\},$$

where the elements of $L_1(a,b;H)$ are to be understood as equivalence classes of a.e. equal functions.

14.4 Lemma. *Let $f \in L_1(a,b;H)$. Then there exists a unique $w \in H$ such that*

$$\int_a^b (f(t)\,|\,v)_H\,\mathrm{d}t = (w\,|\,v)_H \qquad (v \in H),$$

and we define $\int_a^b f(t)\,\mathrm{d}t := w$. The mapping $L_1(a,b;H) \ni f \mapsto \int_a^b f(t)\,\mathrm{d}t \in H$ is linear and continuous.

Lemma 14.4 is an easy consequence of the theorem of Riesz-Fréchet; see Exercise 14.2. We also introduce the space

$$L_2(a,b;H) := \Big\{ f\colon (a,b) \to H \text{ measurable};\ \int_a^b \|f(t)\|^2\,\mathrm{d}t < \infty \Big\},$$

again identifying a.e. equal functions. Note that $L_2(a,b;H) \subseteq L_1(a,b;H)$ if (a,b) is a bounded interval.

14.5 Proposition. *The space $L_2(a,b;H)$ is a Hilbert space for the scalar product*

$$(f\,|\,g)_{L_2(a,b;H)} := \int_a^b (f(t)\,|\,g(t))_H\,\mathrm{d}t.$$

In order to see that the function $t \mapsto (f(t)\,|\,g(t))$ in Proposition 14.5 is measurable we note the following denseness property.

14.6 Lemma. *For each $f \in L_2(a,b;H)$ there exists a sequence (f_n) in*

$$L_2(a,b) \otimes H := \lin\big\{\varphi(\cdot)v;\ \varphi \in L_2(a,b),\ v \in H\big\}$$

such that $\|f_n(\cdot)\| \leqslant \|f(\cdot)\|$ for all $n \in \mathbb{N}$ and $f_n \to f$ a.e.

In particular, $L_2(a,b) \otimes H$ is dense in $L_2(a,b;H)$.

Proof. This is obvious if $\dim H < \infty$; assume that H is infinite-dimensional. Let $f \in L_2(a,b;H)$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. Then, with $f_n := \sum_{k=1}^n (f(\cdot)\,|\,e_k)\,e_k$, the sequence (f_n) has the required properties. The dominated convergence theorem implies that $f_n \to f$ in $L_2(a,b;H)$. □

Proof of Proposition 14.5. We prove that $t \mapsto (f(t)\,|\,g(t))_H$ is measurable.

If $f \in L_2(a,b)\otimes H$, $f = \sum_{j=1}^n \varphi_j(\cdot)v_j$, then $(f(\cdot)\,|\,g(\cdot)) = \sum_{j=1}^n \varphi_j(\cdot)\,(v_j\,|\,g(\cdot))$ is measurable. Every $f \in L_2(a,b;H)$ can be approximated pointwise a.e. by a sequence in $L_2(a,b)\otimes H$, by Lemma 14.6, and this implies the measurability of $(f(\cdot)\,|\,g(\cdot))$.

The completeness is proved in the same way as the completeness of the scalar valued $L_2(a,b)$. (Show that every absolutely convergent series is convergent.) □

14.3 Vector valued Sobolev spaces

We now define Hilbert space valued Sobolev spaces. As before, let H be a separable Hilbert space over $\mathbb{K}$, and let $-\infty \leqslant a < b \leqslant \infty$.

Given $u \in L_2(a,b;H)$, a function $u' \in L_2(a,b;H)$ is called **weak derivative** of u if

$$-\int_a^b u(s)\varphi'(s)\,\mathrm{d}s = \int_a^b u'(s)\varphi(s)\,\mathrm{d}s \qquad (\varphi \in C_c^\infty(a,b)).$$

Such a weak derivative is unique whenever it exists; see Exercise 14.3(a). We let

$$H^1(a,b;H) := \big\{u \in L_2(a,b;H)\,;\, u \text{ has a weak derivative } u' \text{ in } L_2(a,b;H)\big\}.$$

It is easy to see that $H^1(a,b;H)$ is a Hilbert space for the scalar product

$$(u\,|\,v)_{H^1} = \int_a^b \big((u(t)\,|\,v(t))_H + (u'(t)\,|\,v'(t))_H\big)\,\mathrm{d}t$$

(cf. Theorem 4.8 and Proposition 14.5).

The next result can be proved essentially as Proposition 4.6.

14.7 Proposition. *Let $-\infty < a < b < \infty$.*

(a) *Let $v \in L_2(a,b;H)$, $u_0 \in H$, $u(t) := u_0 + \int_a^t v(s)\,\mathrm{d}s$ $(t \in (a,b))$. Then $u \in H^1(a,b;H)$ and $u' = v$.*

(b) *Conversely, let $u \in H^1(a,b;H)$. Then there exists $u_0 \in H$ such that*

$$u(t) = u_0 + \int_a^t u'(s)\,\mathrm{d}s \qquad (\text{a.e. } t \in (a,b)).$$

In particular, each function $u \in H^1(a,b;H)$ has a continuous representative, and with this representative one has $\int_a^b u'(s)\,\mathrm{d}s = u(b) - u(a)$.

We will always use the continuous representative for functions $u \in H^1(a,b)$.

14.8 Remark. We note the following product rule for differentiation: if $u \in H^1(a,b;H)$ and $\varphi \in C^\infty[a,b]$, then $\varphi u \in H^1(a,b;H)$, and $(\varphi u)' = \varphi' u + \varphi u'$.

Indeed, for $\psi \in C_c^\infty(a,b)$ one has

$$\begin{aligned}\int_a^b (\varphi u)\psi' \,\mathrm{d}t &= \int_a^b (\varphi\psi)' u \,\mathrm{d}t - \int_a^b \varphi'\psi u \,\mathrm{d}t \\ &= -\int_a^b \varphi\psi u' \,\mathrm{d}t - \int_a^b \varphi' u\psi \,\mathrm{d}t = -\int_a^b (\varphi u' + \varphi' u)\psi \,\mathrm{d}t.\end{aligned}$$

Next we suppose that V is a separable Hilbert space such that $V \overset{\mathrm{d}}{\hookrightarrow} H$. Then we identify H with a dense subspace of V^* as in Section 14.1. The following mixed Sobolev space plays an important role for evolutionary problems. Let $-\infty < a < b < \infty$ and

$$MR(a,b) := H^1(a,b;V^*) \cap L_2(a,b;V).$$

Here the symbol "*MR*" stands for "maximal regularity". It is easy to see that $MR(a,b)$ is a Hilbert space for the norm

$$\|u\|_{MR(a,b)} := \big(\|u\|^2_{L_2(a,b;V)} + \|u'\|^2_{L_2(a,b;V^*)}\big)^{1/2}.$$

By Proposition 14.7 each $u \in MR(a,b)$ has a representative $u \in C([a,b];V^*)$. The following result shows that in fact the representative is even continuous with values in the smaller space H.

14.9 Proposition. (a) *One has* $MR(a,b) \hookrightarrow C([a,b];H)$.

(b) *If* $u \in MR(a,b)$, *then the function* $\|u(\cdot)\|_H^2$ *is in* $W_1^1(a,b)$ *and*

$$\big(\|u(\cdot)\|_H^2\big)' = 2\,\mathrm{Re}\,\langle u'(\cdot), u(\cdot)\rangle\,. \tag{14.1}$$

In the following remarks we explain the assertions of Proposition 14.9.

14.10 Remarks. (a) We define the scalar Sobolev space

$$W_1^1(a,b) := \big\{u \in L_1(a,b)\,;\, u' = \partial u \in L_1(a,b)\big\}$$

(cf. Subsection 4.1.2). Using Proposition 4.6 one can see that $W_1^1(a,b) \hookrightarrow C[a,b]$ and that $u(t) = u(a) + \int_a^t u'(s)\,\mathrm{d}s$ for all $u \in W_1^1(a,b)$. In particular, u is decreasing if $u' \leqslant 0$ a.e.

(b) From Section 14.1 recall the embeddings $V \hookrightarrow H \hookrightarrow V^*$ and the dual pairing $\langle\cdot,\cdot\rangle$.

(c) The measurability of $t \mapsto f(t) := \langle u'(t), u(t)\rangle$ in Proposition 14.9(b) is shown as in the proof of Proposition 14.5. Since $|f(t)| \leqslant \|u'(t)\|_{V^*}\|u(t)\|_V$ for all $t \in (a,b)$, the Schwarz inequality implies that $f \in L_1(a,b)$.

For the proof of Proposition 14.9 we need the following denseness property.

14.11 Proposition. *The space $C^\infty([a,b];V)$ is dense in $MR(a,b)$.*

Proof. In several arguments in the following proof it is used without mentioning that $L_2(\mathbb{R})\otimes V$ is dense in $L_2(\mathbb{R};V)$, and similarly for V^*.

(i) Let $u\in MR(a,b)$, assume that $u=0$ near b, and extend u by 0 to $\mathbb{R}$, keeping the notation u. For $\tau>0$ we define the left translate $u_\tau := u(\cdot+\tau)$; then $u_\tau\in MR(a-\tau,b)$ and $u_\tau|_{(a,b)}\to u$ in $MR(a,b)$ as $\tau\to 0$. (In order to see the latter, note that $u_\tau|_{(a,b)}\to u$ in $L_2(a,b;V)$ and $(u_\tau)' = (u')_\tau \to u'$ in $L_2(a,b;V^*)$.)

Now we fix $\tau>0$ and choose $\chi\in C_c^\infty(\mathbb{R})$ with $\operatorname{spt}\chi\subseteq(a-\tau,\infty)$ and $\chi=1$ on $[a,b]$; then $\chi u_\tau\in L_2(\mathbb{R};V)$ with $(\chi u_\tau)'\in L_2(\mathbb{R};V^*)$, by Remark 14.8. Let (ρ_k) be a δ-sequence in $C_c^\infty(\mathbb{R})$. Then $\rho_k*(\chi u_\tau)\in C_c^\infty(\mathbb{R};V)$ for $k\in\mathbb{N}$, $\rho_k*(\chi u_\tau)\to\chi u_\tau$ in $L_2(\mathbb{R};V)$, $\big(\rho_k*(\chi u_\tau)\big)' = \rho_k*(\chi u_\tau)'\to(\chi u_\tau)'$ in $L_2(\mathbb{R};V^*)$ as $k\to\infty$, by Proposition 4.3. In consequence, $\big(\rho_k*(\chi u_\tau)\big)|_{(a,b)}\to(\chi u_\tau)|_{(a,b)} = u_\tau|_{(a,b)}$ in $MR(a,b)$, as $k\to\infty$.

This double approximation procedure shows that u can be approximated as asserted.

(ii) The corresponding procedure yields the approximation if $u=0$ near a. For general $u\in MR(a,b)$ choose a function $\alpha\in C^\infty(\mathbb{R})$ with $\alpha=1$ in a neighbourhood of $(-\infty,a]$, $\alpha=0$ in a neighbourhood of $[b,\infty)$ and apply the result shown so far to αu and $(1-\alpha)u$. (Note that Remark 14.8 implies that $\alpha u,(1-\alpha)u\in MR(a,b)$.) □

We mention that the proof given above is analogous to the proof of Theorem 7.7.

Proof of Proposition 14.9. If $u\in C^1([a,b];V)$, then it is immediate that $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = \big(u'(t)\,|\,u(t)\big)_H + \big(u(t)\,|\,u'(t)\big)_H = 2\operatorname{Re}\langle u'(t),u(t)\rangle$.

(a) For $u\in C^1([a,b];V)$ we deduce (as in the proof of Theorem 4.9) that

$$\begin{aligned}\|u\|_{C([a,b];H)}^2 &\leqslant \inf_{t\in(a,b)}\|u(t)\|_H^2 + \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2\,\mathrm{d}t\\ &\leqslant \frac{1}{b-a}\int_a^b\|u(t)\|_H^2\,dt + 2\int_a^b\|u'(t)\|_{V^*}\|u(t)\|_V\,\mathrm{d}t\\ &\leqslant \frac{c}{b-a}\|u\|_{L_2(a,b;V)}^2 + 2\|u'\|_{L_2(a,b;V^*)}\|u\|_{L_2(a,b;V)},\end{aligned}$$

with the embedding constant $c\geqslant 0$ of $V\overset{\mathrm{d}}{\hookrightarrow}H$. As $C^1([a,b];V)$ is dense in $MR(a,b)$, by Proposition 14.11, this inequality shows that $MR(a,b)\hookrightarrow C([a,b];H)$, and the inequality carries over to all $u\in MR(a,b)$.

(b) Initially we have shown (14.1) for $u\in C^1([a,b];V)$. Let now $u\in MR(a,b)$. By Proposition 14.11 there exists a sequence (u_n) in $C^1([a,b];V)$ converging to u in $MR(a,b)$. Then

$$\big(\|u_n(\cdot)\|_H^2\big)' = 2\operatorname{Re}\langle u_n'(\cdot),u_n(\cdot)\rangle\to 2\operatorname{Re}\langle u'(\cdot),u(\cdot)\rangle$$

in $L_1(a,b)$. Moreover, $u_n\to u$ in $C([a,b];H)$ by part (a), and therefore $\|u_n(\cdot)\|_H^2\to\|u(\cdot)\|_H^2$ in $C[a,b]$. This implies that $2\operatorname{Re}\langle u'(\cdot),u(\cdot)\rangle$ is the distributional derivative of $\|u(\cdot)\|_H^2$. □

14.4 Lions' representation theorem

The great importance of Hilbert spaces is the representation theorem of Riesz-Fréchet, which so often gives us weak solutions of partial differential equations. We use it now in order to prove a much more general representation theorem.

14.12 Theorem. *(Lions' representation theorem) Let $\mathcal{V}$ be a Hilbert space, $\mathcal{W}$ a pre-Hilbert space such that $\mathcal{W} \hookrightarrow \mathcal{V}$. Let $E\colon \mathcal{V} \times \mathcal{W} \to \mathbb{K}$ be sesquilinear such that*

(i) *$E(\cdot, w) \in \mathcal{V}'$ for all $w \in \mathcal{W}$,*

(ii) *$|E(w,w)| \geqslant \alpha \|w\|_{\mathcal{W}}^2$ for all $w \in \mathcal{W}$, with some $\alpha > 0$.*

Let $L \in \mathcal{W}^*$. Then there exists $u \in \mathcal{V}$ and such that $L(w) = E(u,w)$ for all $w \in \mathcal{W}$.

The fact that $\mathcal{W}$ may not be complete makes Lions theorem more widely applicable than the Lax-Milgram lemma. To say that $\mathcal{W} \hookrightarrow \mathcal{V}$ means that

$$\|w\|_{\mathcal{V}} \leqslant c\|w\|_{\mathcal{W}} \qquad (w \in \mathcal{W}) \tag{14.2}$$

for some constant c. The larger the norm on $\mathcal{W}$ the less restrictive is the assumption on L to be continuous in the norm of $\mathcal{W}$. On the other hand the coercivity hypothesis (ii) becomes more restrictive if we take larger norms. Even if we choose as norm on $\mathcal{W}$ the norm of $\mathcal{V}$ the fact that E need not be defined on all of $\mathcal{V}$ in the second variable is an advantage; note that there is no continuity requirement on E with respect to the second variable.

Proof of Theorem 14.12. By the Riesz-Fréchet representation theorem there exists a linear operator $T\colon \mathcal{W} \to \mathcal{V}$ such that $E(v,w) = (v \,|\, Tw)_{\mathcal{V}}$ for all $v \in \mathcal{V}$, $w \in \mathcal{W}$.

It follows from property (ii) that

$$\alpha\|w\|_{\mathcal{W}}^2 \leqslant |E(w,w)| = |(w \,|\, Tw)_{\mathcal{V}}| \leqslant \|w\|_{\mathcal{V}}\|Tw\|_{\mathcal{V}} \leqslant c\|w\|_{\mathcal{W}}\|Tw\|_{\mathcal{V}},$$

where $c > 0$ is the embedding constant from (14.2). Thus $\|Tw\|_{\mathcal{V}} \geqslant \frac{\alpha}{c}\|w\|_{\mathcal{W}}$ for all $w \in \mathcal{W}$. This implies that T is injective and that $T^{-1}\colon T(\mathcal{W}) \to \mathcal{W}$ is linear and continuous, where $T(\mathcal{W})$ is provided with the norm of $\mathcal{V}$. Since $L \in \mathcal{W}^*$, the functional $L \circ T^{-1}$ extends to a continuous antilinear functional $\ell \in \mathcal{V}^*$, and by the Riesz-Fréchet theorem there exists $u \in \mathcal{V}$ such that $\ell(v) = (u \,|\, v)_{\mathcal{V}}$ for all $v \in \mathcal{V}$. In particular, for $w \in \mathcal{W}$ one has

$$L(w) = \ell(Tw) = (u \,|\, Tw)_{\mathcal{V}} = E(u,w). \qquad \square$$

14.13 Remark. (Uniqueness in Theorem 14.12) The vector $u \in \mathcal{V}$ is unique if and only if

$$v \in \mathcal{V},\ E(v,w) = 0 \text{ for all } w \in \mathcal{W} \ \text{ implies } \ v = 0.$$

This is the same as saying that $T(\mathcal{W})$ is dense in $\mathcal{V}$, with the operator T from the proof.

14.5 The non-autonomous equation

Now we come to the main result of this lecture. We study the non-autonomous inhomogeneous evolution equation

$$\begin{aligned} u'(t) + \mathcal{A}(t)u(t) &= f(t), \\ u(0) &= u_0. \end{aligned} \tag{14.3}$$

Our assumptions are as follows. Let V, H be separable Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$. With these spaces we form the Gelfand triple $V \overset{\mathrm{d}}{\hookrightarrow} H \overset{\mathrm{d}}{\hookrightarrow} V^*$ and use the notation introduced in Section 14.1.

Let $\tau > 0$ and let $a\colon [0,\tau] \times V \times V \to \mathbb{K}$ be a mapping such that

(i) $a(t,\cdot,\cdot)\colon V \times V \to \mathbb{K}$ is sesquilinear for all $t \in [0,\tau]$;

(ii) $|a(t,u,v)| \leqslant M\|u\|_V\|v\|_V$ for all $t \in [0,\tau]$, $u, v \in V$, with some $M \geqslant 0$;

(iii) $\operatorname{Re} a(t,u,u) \geqslant \alpha\|u\|_V^2$ for all $t \in [0,\tau]$, $u \in V$, with some $\alpha > 0$;

(iv) $a(\cdot,u,v)$ is measurable for all $u, v \in V$.

We point out that α and M are independent of t. Below we will repeatedly need that $t \mapsto a(t,u(t),v(t))$ is measurable for all $u, v \in L_2(0,\tau;V)$. This property follows from Lemma 14.6 and condition (ii).

For each $t \in [0,\tau]$ we denote by $\mathcal{A}(t) \in \mathcal{L}(V,V^*)$ the operator given by

$$\langle \mathcal{A}(t)u, v\rangle = a(t,u,v).$$

First we want to interpret $\mathcal{A}$ as an operator in the following way.

14.14 Proposition. *Defining*

$$(\mathcal{A}u)(t) := \mathcal{A}(t)u(t) \qquad (t \in (0,\tau))$$

for $u \in L_2(0,\tau;V)$, one obtains a bounded linear operator $\mathcal{A}\colon L_2(0,\tau;V) \to L_2(0,\tau;V^*)$.

Proof. Let $u \in L_2(0,\tau;V)$. Recalling that the function $t \mapsto \langle \mathcal{A}u(t), v\rangle = a(t,u(t),v)$ is measurable for all $v \in V$ one deduces that $\mathcal{A}u$ is measurable. The estimate

$$\|\mathcal{A}(t)u(t)\|_{V^*} \leqslant M\|u(t)\|_V \qquad (t \in [0,\tau]),$$

shows that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot) \in L_2(0,\tau;V^*)$.

Obviously the operator $\mathcal{A}$ is linear, and the previous inequality shows that $\mathcal{A}$ is bounded. □

In order to motivate our definition of E in the proof of Theorem 14.16 below we show the following proposition. We will use the notation $C_{\mathrm{c}}^\infty[0,\tau) := \{\varphi \in C^\infty[0,\tau)\,;\ \operatorname{spt}\varphi \text{ compact}\}$.

14.15 Proposition. *Let $u_0 \in H$, $f \in L_2(0,\tau;V^*)$, $u \in L_2(0,\tau;V)$. Then the following statements are equivalent.*

(i) *$u \in MR(0,\tau)$ and $u' + \mathcal{A}u = f$, $u(0) = u_0$.*

(ii) *For all $\varphi \in C_{\mathrm{c}}^\infty[0,\tau)$, $v \in V$ one has*

$$-\int_0^\tau (u(t)\,|\,\varphi'(t)v)_H\,\mathrm{d}t + \int_0^\tau a(t,u(t),\varphi(t)v)\,\mathrm{d}t = \int_0^\tau \langle f(t),\varphi(t)v\rangle\,\mathrm{d}t + (u_0\,|\,\varphi(0)v)_H\,.$$

Since $u \in MR(0,\tau) \subseteq C([0,\tau];H)$, the initial condition $u(0) = u_0$ in statement (i) makes sense.

Proof of Proposition 14.15. Note that condition (ii) is equivalent to

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)(\mathcal{A}u)(t)\,\mathrm{d}t = \int_0^\tau \varphi(t)f(t)\,\mathrm{d}t + \varphi(0)u_0 \tag{14.4}$$

for all $\varphi \in C_c^\infty[0,\tau)$. This equality holds in V^*; recall that $\mathcal{A}u = \mathcal{A}(\cdot)u(\cdot) \in L_2(0,\tau;V^*)$.

(i) $\Rightarrow$ (ii). Let $\varphi \in C_c^\infty[0,\tau)$. Using $\mathcal{A}u = f - u'$ and Remark 14.8 we obtain

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)(\mathcal{A}u)(t)\,\mathrm{d}t = -\int_0^\tau (\varphi u)'(t)\,\mathrm{d}t + \int_0^\tau \varphi(t)f(t)\,\mathrm{d}t.$$

Since $\varphi(\tau) = 0$ and $u(0) = u_0$, it follows from Proposition 14.7(b) that (14.4) holds.

(ii) $\Rightarrow$ (i). Using (14.4) with $\varphi \in C_c^\infty(0,\tau)$ we obtain

$$-\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t = \int_0^\tau \varphi(t)\big(f(t) - (\mathcal{A}u)(t)\big)\,\mathrm{d}t.$$

This implies that $u \in H^1(0,\tau;V^*)$ and $u' = f - \mathcal{A}u$.

In order to show that $u(0) = u_0$ we choose $\varphi \in C_c^\infty[0,\tau)$ with $\varphi(0) = 1$. From (i) $\Rightarrow$ (ii) it follows that (14.4) holds with $u(0)$ in place of u_0. Therefore $u(0) = u_0$. □

Now we can formulate and prove the main result of this lecture.

14.16 Theorem. *Let $u_0 \in H$, $f \in L_2(0,\tau;V^*)$. Then there exists a unique $u \in MR(0,\tau)$ such that*

$$\begin{aligned} u' + \mathcal{A}u &= f, \\ u(0) &= u_0. \end{aligned} \tag{14.5}$$

Note that both terms u', $\mathcal{A}u$ are in the same space $L_2(0,\tau;V^*)$ as f. For this reason we say that the problem has **maximal regularity in** V^*.

Proof of Theorem 14.16. To show existence we apply Lions' representation theorem 14.12 with $\mathcal{V} := L_2(0,\tau;V)$ and the pre-Hilbert space

$$\mathcal{W} := C_c^\infty[0,\tau) \otimes V = \operatorname{lin}\big\{\varphi v;\ \varphi \in C_c^\infty[0,\tau),\ v \in V\big\}$$

with norm $\|w\|_{\mathcal{W}} := \big(\|w\|_{\mathcal{V}}^2 + \|w(0)\|_H^2\big)^{1/2}$. We define $E\colon \mathcal{V} \times \mathcal{W} \to \mathbb{K}$ by

$$E(v,w) = -\int_0^\tau \big(v(t)\,|\,w'(t)\big)_H\,\mathrm{d}t + \int_0^\tau a(t,v(t),w(t))\,\mathrm{d}t.$$

For $w \in \mathcal{W}$, one has

$$|E(v,w)| \leqslant \|v\|_{\mathcal{V}}\|w'\|_{L_2(0,\tau;V^*)} + M\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}}$$

for all $v \in \mathcal{V}$. Thus condition (i) of Theorem 14.12 is satisfied.

In order to show (ii) let $w \in \mathcal{W}$. By Proposition 14.9(b) we have $\|w(\cdot)\|_H^2 \in W_1^1(0,\tau)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_H^2 = 2\operatorname{Re}(w'(t) \mid w(t))_H$. Thus

$$-\int_0^\tau \operatorname{Re}(w'(t) \mid w(t))_H \,\mathrm{d}t = -\frac{1}{2}\|w(\tau)\|_H^2 + \frac{1}{2}\|w(0)\|_H^2 = \frac{1}{2}\|w(0)\|_H^2.$$

By the coercivity condition (iii) it follows that

$$\operatorname{Re} E(w,w) \geqslant \frac{1}{2}\|w(0)\|_H^2 + \alpha\|w\|_{\mathcal{V}}^2 \geqslant \min\{1/2,\alpha\}\|w\|_{\mathcal{W}}^2.$$

Thus (ii) is satisfied.

Define $L \in \mathcal{W}^*$ by

$$Lw = \int_0^\tau \langle f(t), w(t)\rangle \,\mathrm{d}t + (u_0 \mid w(0))_H .$$

By Lions' representation theorem, there exists $u \in \mathcal{V}$ such that $E(u,w) = Lw$ for all $w \in \mathcal{W}$. This statement is equivalent to the validity of the statement (ii) of Proposition 14.15. Therefore Proposition 14.15 implies that $u \in MR(0,\tau)$ and that u is a solution of (14.5).

In order to show uniqueness, let $u \in MR(0,\tau)$ such that $u' + \mathcal{A}u = 0$ and $u(0) = 0$. Then by Proposition 14.9(b) we obtain

$$\big(\|u(\cdot)\|_H^2\big)'(t) = 2\operatorname{Re}\langle u'(t), u(t)\rangle = -2\operatorname{Re}\langle(\mathcal{A}u)(t), u(t)\rangle = -2\operatorname{Re} a(t,u(t),u(t)) \leqslant 0$$

for a.e. $t \in [0,\tau]$. Remark 14.10(a) implies that $\|u(\cdot)\|_H$ is decreasing, so $u(t) = 0$ for all $t \in [0,\tau]$. □

14.17 Remark. The notion of solution we use here is rather weak. This is reflected by the fact that our space of test functions $\mathcal{W}$ is small and consists of very regular functions. Of course, if we have a weak notion of solution, existence of solutions might become easy to prove (at the extreme case we might call every function a solution). But uniqueness may become very hard to prove. Here for existence we need the key equality (14.1) only for functions in $\mathcal{W}$, for which it is in fact trivial. With the help of Lions' representation theorem the proof of existence is indeed rather elegant and easy. In contrast, uniqueness is more technical, since (14.1) is needed for a rather general class of functions.

Notes

The results of Sections 14.4 and 14.5 go back to J. L. Lions and his school in the 1950s and 1960s. Lions' representation theorem, Theorem 14.12, is first proved in [Lio57]. The existence and uniqueness result Theorem 14.16 is contained in Lions' book ([Lio61; Théorème 1.1, p. 46]), with a slightly different formulation. Lions quotes I. M. Vishik [Vis56] and J. L. Lions [Lio59] for the first proof of existence. Uniqueness is due to J. L. Lions [Lio59]; this had been an open problem for a while. It is also possible to prove Theorem 14.16 by the Galerkin method; see [DL92]. This is of great interest for the numerical treatment but less elegant than via the representation theorem.

A revival of the subject is due to the interest in maximal regularity. We refer to the recent thesis [Die14] by D. Dier and to [ADLO14] for further information, in particular for the discussion of maximal regularity in H.

Exercises

14.1 Let V, H be separable Hilbert spaces such that $V \overset{\mathrm{d}}{\hookrightarrow} H$, $\dim H = \infty$. Assume that the injection of V into H is compact. Let $a\colon V \times V \to \mathbb{K}$ be continuous, coercive and symmetric.

(a) Show that there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$ such that

$$V = \Big\{ u \in H;\ \sum_{n=1}^{\infty} \lambda_n |(u \,|\, e_n)_H|^2 < \infty \Big\}$$

and $a(u,v) = \sum_{n=1}^{\infty} \lambda_n \, (u \,|\, e_n)_H \, (e_n \,|\, v)_H$ for all $u, v \in V$.

Hint: Use Theorem 6.17 and Corollary 6.18.

(b) Show that V^* can be identified with the space $\big\{(y_n) \in \mathbb{K}^{\mathbb{N}};\ \sum_{n=1}^{\infty} \lambda_n^{-1} |y_n|^2 < \infty\big\}$, with the canonical mapping $H \hookrightarrow V^*$ given by $u \mapsto ((u \,|\, e_n)_H)_{n\in\mathbb{N}}$.

(c) Show that the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ from Section 14.1, considered as an operator in V^*, is given by

$$\begin{aligned} \mathrm{dom}(\mathcal{A}) &= \{y \in V^*;\ (\lambda_n y_n)_{n\in\mathbb{N}} \in V^*\}, \\ \mathcal{A}y &= (\lambda_n y_n)_{n\in\mathbb{N}} \quad (y \in \mathrm{dom}(\mathcal{A})). \end{aligned}$$

Show that $-\mathcal{A}$ generates a C_0-semigroup on V^*, given by

$$\mathrm{e}^{-t\mathcal{A}} y = (\mathrm{e}^{-t\lambda_n} y_n)_{n\in\mathbb{N}}.$$

14.2 Prove Lemma 14.4.

14.3 Let $(a,b) \subseteq \mathbb{R}$ be an interval.

(a) Let H be a separable Hilbert space, $u \in L_1(a,b;H)$ with the property that

$$\int_a^b \varphi(t) u(t) \, \mathrm{d}t = 0$$

for all $\varphi \in C_c^\infty(a,b)$. Show that $u = 0$ a.e.

Deduce the uniqueness of the weak derivative defined in Section 14.3.

(b) Find a (non-separable) Hilbert space H and a function $u\colon (a,b) \to H$ with $\|u(t)\| = 1$ for all $t \in (a,b)$ and $(u(\cdot) \,|\, v) = 0$ a.e. for all $v \in H$. (Then $\int_a^b \varphi(t) u(t) \, \mathrm{d}t = 0$ for all $\varphi \in C_c^\infty(a,b)$, if the integral would be defined as in Lemma 14.4.)

14.4 Prove Theorem 14.16 in the case where the form a is not necessarily coercive but H-elliptic, i.e.,

$$\mathrm{Re}\, a(t,u,u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2 \qquad (t \in [0,\tau],\ u \in V)$$

holds for some $\omega \geqslant 0$, $\alpha > 0$.

Hint: Solve $v' + (\mathcal{A} + \omega) v = \mathrm{e}^{-\omega \cdot} f$.

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